On the discriminant of an involution

M.–A. Knus, R. Parimala and R. Sridharan

1. Introduction

In recent papers [KPS$_1$], [KPS$_2$] and [KPS$_3$] the authors gave some applications of the reduced pfaffian of a central simple algebra with involution or, more generally, of an Azumaya algebra. In [KPS$_3$], an invariant, called the pfaffian discriminant, was attached to any involution σ of a central simple 16–dimensional algebra A and it was shown that A has a σ–invariant quaternion subalgebra if and only if the discriminant of σ is trivial. In this note we use Galois descent to describe the discriminant and the pfaffian of a central simple algebra with involution of arbitrary dimension.

We thank J.–P. Tignol for useful discussions and for having shown us an early version of [CDTW], D. B. Shapiro for discovering a sign error in a formula of Proposition (3.1) and P. Hänsli for a careful reading of these notes.

2. The Pfaffian

Let K be a field and let A be a central simple K-algebra with an involution σ of the first kind. We recall that σ is an antiautomorphism of A of order ≤ 2. Typical examples are matrix algebras $M_n(K)$ with transpose $t : x \mapsto \overline{x}$ or quaternion algebras (i.e. central simple algebras of dimension 4) with conjugation $x \mapsto \overline{x}$. If n_H is the reduced norm of a quaternion algebra H, the conjugation map $x \mapsto \overline{x}$ is the unique map with $x\overline{x} = \overline{x}x = n_H(x)$.

For any central simple K–algebra A there exists a finite Galois extension $K \subset L$ such that

$$\alpha : A \otimes L \xrightarrow{\sim} M_n(L).$$

We call α (or L) a *Galois splitting* of A. The transport

$$\widetilde{\sigma} = \alpha(\sigma \otimes 1)\alpha^{-1}$$

of σ to $M_n(L)$ is, by the Skolem-Noether Theorem, of the form $\widetilde{\sigma} = \sigma_u = \operatorname{Int}(u) \circ t$, where $\operatorname{Int}(u)$ is the inner automorphism $x \mapsto uxu^{-1}, u \in GL_n(L)$, and t is, as above, the transpose. It follows from $\sigma^2 = 1$ that $u^t = \varepsilon u$ for some $\varepsilon = \pm 1$.

Let $\beta : A \otimes L' \simeq M_n(L')$ be another splitting. By replacing L and L' by a bigger Galois extension, we can assume that $L = L'$. If $\beta(\sigma \otimes 1)\beta^{-1} = \sigma_{u'}, \sigma_{u'}$ and σ_u, then

$\sigma_{u'} = \mathrm{Int}(v)\sigma_u \mathrm{Int}(v)^{-1}$ for some inner automorphism $\mathrm{Int}(v)$ and there is $\lambda \in L^\bullet$ such that

$$\lambda u' = vuv^t.$$

It follows that the element ε is independent of the splitting α. We call it the *type* of σ. In view of the formula $\lambda u' = vuv^t$ we can assume that u is either diagonal or of the form

$$u = \mathrm{diag}(S_2, \ldots, S_2) \ \text{ with } \ S_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Observe that this is also true in characteristic 2 (see Kaplansky [K] p. 23). We say that σ is of *orthogonal* type if u is diagonalizable and of *symplectic* type if u is alternating.

We call the set

$$\mathrm{Alt}^\sigma(A) = \{x - \varepsilon\sigma(x), x \in A\},$$

where ε is the type of σ, the *set of alternating elements* of (A, σ). Let $\mathrm{Alt}_n(K)$ be the set of alternating $(n \times n)$–matrices. If $\alpha : A \otimes L \xrightarrow{\sim} M_n(L)$ is a splitting of A such that $\widetilde{\sigma} = \alpha(\sigma \otimes 1)\alpha^{-1} = \mathrm{Int}(u) \circ t$, we have

$$u^{-1}\alpha(\mathrm{Alt}^\sigma(A) \otimes L) = \alpha(\mathrm{Alt}^\sigma(A) \otimes L)u = \mathrm{Alt}_n(L)$$

since

$$\begin{aligned} u^{-1}(x - \varepsilon u x^t u^{-1}) &= u^{-1}x - (u^{-1}x)^t \\ (x - \varepsilon u x^t u^{-1})u &= xu - (xu)^t . \end{aligned}$$

It follows that $\dim_K \mathrm{Alt}^\sigma(A) = \frac{n(n-1)}{2}$. For n even the determinant of an alternating $(n \times n)$–matrix has a square root, the pfaffian. For example

$$pf(x) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$$

for $n = 4$ and $x = (a_{ij})$. For central simple algebras of dimension $n^2 = 4m^2$, we have, denoting the reduced norm by n_A :

Theorem (2.1) *Let A be a central simple K–algebra with involution σ. There exists an element $d_\sigma \in K^\bullet$ and a map $pf_\sigma : \mathrm{Alt}^\sigma(A) \to K$ such that $pf_\sigma(x)^2 = d_\sigma n_A(x)$ for all $x \in \mathrm{Alt}^\sigma(A)$. The map pf_σ is determined up to a unit of K and satisfies the identity*

$$pf_\sigma(ax\sigma(a)) = n_A(a)pf_\sigma(x) \ , \ x \in \mathrm{Alt}^\sigma(A), a \in A.$$

If $\varphi : (A, \sigma) \to (A', \sigma')$ is an isomorphism of algebras with involution and if pf_σ is fixed, then $pf_{\sigma'}$ can be chosen in such a way that $pf_{\sigma'}(\varphi(x)) = pf_\sigma(x), x \in \mathrm{Alt}^\sigma(A)$.

Proof Let $\alpha : A \otimes L \xrightarrow{\sim} M_n(L)$ be a Galois splitting and let G be the Galois group of the extension $K \subset L$. Let

$$\alpha(\sigma \otimes 1)\alpha^{-1} = \mathrm{Int}(u) \circ t$$

as above and let, for $g \in G$, $\tilde{g} : M_n(L) \to M_n(L)$ be the entrywise action, i.e. $\tilde{g}(a_{ij}) = (g(a_{ij}))$ for $a_{ij} \in M_n(L)$. The automorphism $\alpha(1 \otimes g)\alpha^{-1}\tilde{g}^{-1}$ of $M_n(L)$ is L–linear. Thus there exists $c(g) \in GL_n(L)$ such that

$$\alpha(1 \otimes g)\alpha^{-1} = \mathrm{Int}(c(g)) \circ \tilde{g}.$$

Replacing g by gh, $g, h \in G$ we get

$$c(gh) = c(g)\tilde{g}(c(h)) \cdot r(g,h)$$

for elements $r(g,h) \in L^\bullet$. Since $\alpha(1 \otimes g)\alpha^{-1}$ and $\alpha(\sigma \otimes 1)\alpha^{-1}$ commute, we have

$$\mathrm{Int}(c(g)^t u^{-1} c(g)\tilde{g}(u)) = 1.$$

Thus there exist elements $\lambda(g) \in L^\bullet$ such that

$$c(g)\tilde{g}(u)c(g)^t = \lambda(g)u$$

and

$$\lambda(gh) = \lambda(g)g(\lambda(h))r(g,h)^2.$$

It follows that the elements

$$\chi(g) = \det(c(g))\lambda(g)^{-n/2}$$

satisfy $\chi(gh) = \chi(g)g(\chi(h))$. By Hilbert theorem 90 there exists $a \in L^\bullet$ such that

$$\chi(g) = g(a)a^{-1}$$

for all $g \in G$. Observe that a is determined up to a unit of K. We now define

$$pf_\sigma : \mathrm{Alt}^\sigma(A) \otimes L \to L$$

by $pf_\sigma(x \otimes \ell) = pf(\alpha(x \otimes \ell)u)a$, $x \in \mathrm{Alt}^\sigma(A)$, $\ell \in L$, where pf is the pfaffian $\mathrm{Alt}_n(L) \to L$. We claim that the restriction of pf_σ to $\mathrm{Alt}^\sigma(A)$ takes values in K. For this it suffices to check that

$$g(pf_\sigma(x \otimes 1)) = pf_\sigma(x \otimes 1) \text{ for all } g \in G.$$

We have

$$\begin{aligned}
&pf(\alpha(1\otimes g)(x\otimes 1)u)a = pf(c(g)\tilde{g}(\alpha(x\otimes 1))c(g)^{-1}u)a \\
&= pf(c(g)\tilde{g}(\alpha(x\otimes 1))\lambda(g)^{-1}\tilde{g}(u)c(g)^{t})a \\
&= \det(c(g))\lambda(g)^{-n/2}pf(\tilde{g}(\alpha(x\otimes 1))\tilde{g}(u))a \\
&= \chi(g)g(pf(\alpha(x\otimes 1)u))a = g(pf(\alpha(x\otimes 1)u)a) \\
&= g(pf_\sigma(x\otimes 1))
\end{aligned}$$

and on the other hand

$$pf(\alpha(1\otimes g)(x\otimes 1)u)a = pf_\sigma(x\otimes 1).$$

Thus, as claimed pf_σ restricts to a map $pf_\sigma : \mathrm{Alt}^\sigma(A) \to K$. We have

$$\begin{aligned}
(pf_\sigma(x\otimes 1))^2 &= det(\alpha(x\otimes 1))\det(u)a^2 \ . \\
&= n_A(x)\det(u)a^2
\end{aligned}$$

and obviously $d_\sigma = \det(u)a^2 \in K^\bullet$ is as wanted. We next check that the construction of pf_σ is (up to a unit of K) independent of the chosen splitting. Let

$$\beta : A\otimes L' \xrightarrow{\sim} M_n(L')$$

and let $\beta(\sigma\otimes 1)\beta^{-1} = \mathrm{Int}(u')\circ t$. By replacing L and L' by a common Galois extension, we can assume that $L = L'$. There exists $v \in GL_n(L)$ such that $\beta = \mathrm{Int}(v)\circ\alpha$. Since $\mathrm{Int}(v)$ is an isomorphism of algebras with involution there exists $\rho \in L^\bullet$ such that

$$u' = \rho v u v^t.$$

Let $c'(g)$, $\lambda'(g)$ and $\chi'(g)$ be the data induced by β. We have a relation

$$c'(g)\tilde{g}(v) = \nu v c(g)$$

for some $\nu \in L^\bullet$ and it follows from $\lambda'(g)u' = c'(g)\tilde{g}(u')c'(g)^t$ that

$$\lambda'(g) = \rho^{-1}\nu^2\lambda(g)g(\rho).$$

Furthermore we get

$$\det(c'(g)) = \nu^n \det(c(g))\det(v)\cdot g(\det(v))^{-1}.$$

Thus

$$\begin{aligned}
\chi'(g) &= \det(c'(g))\lambda'(g)^{-n/2} \\
&= \chi(g)\det(v)\rho^{n/2}g(\det(v)\rho^{n/2})^{-1} \ .
\end{aligned}$$

Putting $a' = a(\det(v)\rho^{n/2})^{-1}$ and defining $pf_{\sigma'}$ by

$$pf_{\sigma'}(x \otimes \ell) = pf(\beta(x \otimes \ell)u')a'$$

we get, as wanted, $pf_{\sigma'}(x \otimes 1) = pf_\sigma(x \otimes 1)$. The formula $pf_\sigma(ax\sigma(a)) = n_A(a)pf_\sigma(x)$ follows from the corresponding formula $pf(yxy^t) = \det(y)pf(x)$ for the classical pfaffian and the rest of the claims is an easy consequence of the above computations.

Remark (2.2) The pfaffian can be constructed in a slightly different way (see Tamagawa [T] and [KS]). Using the same notations as above, we put $V = L^n$ and denote the action of G componentwise by gv. We define an action of G on $\wedge^m V$, $m = 2, 4, \ldots, n$, by

$$g^*(x_1 \wedge \cdots \wedge x_m) = \lambda(g)^{-m/2} c(g)gx_1 \wedge \cdots \wedge c(g)gx_m.$$

It is easy to check that $(gh)^* = g^*h^*$ and we put

$$A^{(m)} = (\wedge^m V)^G = \{\xi \in \wedge^m V | g^*(\xi) = \xi \ , \forall\, g \in G\}.$$

The space $A^{(2)}$ has dimension $\frac{n(n-1)}{2}$ and $A^{(n)}$ has dimension 1. We claim that there is a canonical isomorphism $A^{(2)} \simeq \mathrm{Alt}^\sigma(A)$ and that pf_σ can be viewed as a map $A^{(2)} \to A^{(n)}$. We write $\widetilde{\sigma}\,(x) = \alpha(\sigma \otimes 1)\alpha^{-1}(x) = ux^t u^{-1}$ with $u : V^* \xrightarrow{\sim} V$, $V^* = \mathrm{Hom}_L(V, L)$ and $x^t : V^* \to V^*$ the transpose. Identifying $\mathrm{End}_L(V)$ with $V \otimes_L V^*$ through the map $(x \otimes f)(y) = xf(y)$, $x, y \in V$, $f \in V^*$, we have

$$\widetilde{\sigma}\,(x \otimes f) = u(f) \otimes u^{-1}(x).$$

Through the composite

$$A \otimes L \xrightarrow{\alpha} \mathrm{End}_L(V) \xrightarrow{\sim} V \otimes V^* \xrightarrow{1 \otimes u} V \otimes V\,,$$

$\mathrm{Alt}^\sigma(A) \otimes L$ has image the subspace $\mathrm{Alt}(V \otimes V)$ generated by all tensors $x \otimes y - y \otimes x$. We further identify $\mathrm{Alt}(V \otimes V)$ with $\wedge^2 V$ by sending $x \otimes y - y \otimes x$ to $x \wedge y$, so that we get an isomorphism

$$\mathrm{Alt}^\sigma(A) \otimes L \xrightarrow{\sim} \wedge^2 V.$$

This map is G–equivariant for the action $1 \otimes g$ on $\mathrm{Alt}^\sigma(A) \otimes L$ and g^* on $\wedge^2 V$, thus induces an isomorphism $\mathrm{Alt}^\sigma(A) \xrightarrow{\sim} A^{(2)}$. To construct the pfaffian it now suffices to notice that the classical pfaffian can be viewed as a map $pf_V : \wedge^2 V \to \wedge^n V$ and that this map is G–equivariant. Let $x = (a_{ij})$ be an alternating matrix. We define

$$pf_V(\sum_{i<j} a_{ij} e_i \wedge e_j) = pf(x) e_1 \wedge \ldots \wedge e_n$$

for a fixed basis $\{e_1, \dots, e_n\}$ of V. Using the identity $pf(yxy^t) = \det(y)pf(x)$, it is easy to check that pf_V is independent of the choice of the basis and that it is G–equivariant.

Let d_σ be as given in (2.1). Its class $\delta(\sigma)$ in $K^\bullet/K^{\bullet 2}$ is independent of the choice of pf_σ. We call it the *discriminant* of σ. Observe that $\delta(\sigma)$ is the class of $n_A(x)$ in $K^\bullet/K^{\bullet 2}$ for any invertible $x \in \mathrm{Alt}^\sigma(A)$. This was already noticed in [CDTW].

Remark (2.3) The construction of the $A^{(m)}$ given above could also be applied to symmetric powers of V or to a combination of exterior and symmetric powers. It was used by Jacobson in his construction of the even Clifford algebra of a central simple algebra with involution of orthogonal type [J_1] .

Remark (2.4) Theorem (2.1) can be generalized to the setting of 2–torsion data as introduced in [KPS_1]. We hope to come back to this at some other place.

Examples (2.5)

1. If $A = M_n(K)$ and $\sigma = \mathrm{Int}(u) \circ t$, we have
$$x - \varepsilon u x^t u^{-1} = u(u^{-1}x - (u^{-1}x)^t)$$
so that $\det(x - \varepsilon u x^t u^{-1}) = \det(u) pf(u^{-1}x - (u^{-1}x)^t)^2$ and $\delta(\sigma)$ is the class of $[\det(u)] \in K^\bullet/K^{\bullet 2}$.

2. Let σ be of even symplectic type, so that we can choose a splitting of A such that $u = \mathrm{diag}\{S_2, \dots, S_2\}$. With the notations of the proof of (2.1), we have $\tilde{g}(u) = u$ and taking pfaffians on both sides of $c(g)\tilde{g}(u)c(g)^t = \lambda(g)u$, we get $\det(c(g)) = \lambda(g)^{n/2}$ and $\chi(g) = 1$. Thus we can choose $a = 1$ and get $\delta(\sigma) = 1$. This case was considered by Fröhlich in [F].

3. If $A = A_1 \otimes A_2$ is the tensor product of two quaternion algebras and σ is the tensor product $\sigma_1 \otimes \sigma_2$ of two involutions, we can split both algebras A_1, A_2 separately and get $c(g) = c_1(g) \otimes c_2(g)$, $\lambda(g) = \lambda_1(g)\lambda_2(g)$. We have
$$c_i(g)\tilde{g}(u_i)c_i(g)^t = \lambda_i(g)u_i\,, i = 1, 2$$
and
$$\det(c_i(g))^2 = \lambda_i(g)^2 a_i g(a_i)^{-1}$$
with $a_i = \det(u_i)$. It follows that
$$\begin{aligned} \chi(g) &= \det(c(g))\lambda(g)^{-2} \\ &= \det(c_1(g))^2 \det(c_2(g))^2 \lambda_1(g)^{-2}\lambda_2(g)^{-2} \\ &= a_1 a_2 g(a_1 a_2)^{-1} = g(a)a^{-1} \end{aligned}$$

with $a = (a_1a_2)^{-1}$. Thus

$$\delta(\sigma) = [\det(u)a^2] = [\det(u_1)^2\det(u_2)^2a_1^{-2}a_2^{-2}] = 1.$$

In this case the pfaffian is the generic norm defined by Jacobson in [J_2] .

4. If K is an hilbertian field of characteristic not two and A is a central simple K–algebra of order two in the Brauer group $\mathrm{Br}(K)$, then, by a recent result of Saltman [S], there exists an involution on A with nontrivial discriminant.

3. The pfaffian adjoint

In this section we generalize the construction of the pfaffian adjoint to central simple algebras of arbitrary rank. It will follow that the discriminant defined in Section 2 and the discriminant of [KPS_3] are identical.

Let $n = 2m$. For any alternating $(n \times n)$–matrix $x = (x_{ij})$ let $p_o(x) = (X_{ij})$ be alternating with $X_{ij} = (-1)^{i+j+1}pf(\xi)$ for $i > j$ and ξ is the $((n-2) \times (n-2))$–matrix obtained from x by cancelling the i-th and j-th rows and columns. The map p_o is polynomial and homogeneous of degree $m-1$ satisfies $xp_o(x) = p_o(x)x = pf(x)$ [B, §5, N^o 3, Exercice 5] and $p_o^2(x) = (-1)^m pf(x)^{m-2}x$ for all $x \in \mathrm{Alt}_{2m}(K)$. For example, if $m = 2$, p_o is the K–linear automorphism $p_o : \mathrm{Alt}_4(K) \to \mathrm{Alt}_4(K)$ given by

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ & 0 & a_{23} & a_{24} \\ & & 0 & a_{34} \\ & & & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & -a_{34} & a_{24} & -a_{23} \\ & 0 & -a_{14} & a_{13} \\ & & 0 & -a_{12} \\ & & & 0 \end{pmatrix}$$

A similar map can be constructed for any central simple K–algebra with involution σ of dimension $2m$.

Proposition (3.1) *Let $pf_\sigma : \mathrm{Alt}^\sigma(A) \to K$ be a pfaffian on A such that $pf_\sigma(x)^2 = d_\sigma n_A(x)$ for all $x \in \mathrm{Alt}^\sigma(A)$. There exists a polynomial map*

$$p_\sigma : \mathrm{Alt}^\sigma(A) \to \mathrm{Alt}^\sigma(A),$$

homogeneous of degree $m-1$, such that $xp_\sigma(x) = p_\sigma(x)x = pf_\sigma(x)$ and

$$p_\sigma^2(x) = (-1)^m pf_\sigma(x)^{m-2} d_\sigma x.$$

Proof. Let $\alpha : A \otimes L \xrightarrow{\sim} M_{2m}(L)$ be a Galois splitting of A such that

$$\alpha(\sigma \otimes 1)\alpha^{-1} = \mathrm{Int}(u) \circ t$$

and, with the notations of Section 1, let $a \in K^\bullet$ with $\chi(g) = g(a)a^{-1}$, $g \in \mathrm{Gal}(L/K)$, so that $d_\sigma = \det(u)a^2$. We put

$$p_\sigma(x) = \alpha^{-1}(up_o(\alpha(x \otimes 1)u)a).$$

To prove that p_σ has image in $\mathrm{Alt}^\sigma(A)$, we shall need the formula

$$zp_o(z^t xz)z^t = \det(z)p_o(x)$$

for $x \in \mathrm{Alt}_{2m}(K)$ and $z \in M_{2m}(K)$. This formula is a simple consequence of the formulas $pf(z^t xz) = \det(z)pf(x)$ and $xp_o(x) = p_o(x)x = pf(x)$. We get

$$\begin{aligned}
(1 \otimes g)(p_\sigma(x)) &= \alpha^{-1}(c(g)\tilde{g}(up_o(\alpha(x \otimes 1)u)a)c(g)^{-1}) \\
&= \alpha^{-1}(\lambda(g)u(c(g)^{-1})^t p_o(\tilde{g}(\alpha(x \otimes 1))\tilde{g}(ua))c(g)^{-1}) \\
&= \lambda(g)g(a)\alpha^{-1}(u\det(c(g)^{-1})p_o(c(g)\tilde{g}(\alpha(x \otimes 1))\tilde{g}(u)c(g)^t)) \\
&= \det(c(g))^{-1}\lambda(g)^2 g(a)\alpha^{-1}(up_o(\alpha(x \otimes 1)u)) \\
&= p_\sigma(x)
\end{aligned}$$

so that $p_\sigma(x) \in \mathrm{Alt}^\sigma(A)$ for $x \in \mathrm{Alt}^\sigma(A)$. Further we have

$$\begin{aligned}
xp_\sigma(x) &= \alpha^{-1}(\alpha(x \otimes 1)u\, p_o(\alpha(x \otimes 1)u)a) \\
&= pf(\alpha(x \otimes 1)u)a = pf_\sigma(x) \\
p_\sigma(x)x &= \alpha^{-1}(u\, p_o(\alpha(x \otimes 1)u)a\alpha(x \otimes 1)uu^{-1}) \\
&= \alpha^{-1}(u\, pf(\alpha(x \otimes 1)u)au^{-1}) = pf_\sigma(x).
\end{aligned}$$

Finally we get

$$\begin{aligned}
p_\sigma^2(x) &= \alpha^{-1}(u\, p_o(\alpha(\alpha^{-1}(u\, p_o(\alpha(x \otimes 1)u)a))u)a) \\
&= \alpha^{-1}(a^m u\, p_o(u\, p_o(\alpha(x \otimes 1)u)u)) \\
&= \alpha^{-1}(a^m u\, p_o(u^t p_o(\alpha(x \otimes 1)u)u)u^t u^{-1} \\
&= \alpha^{-1}(a^2 \det(u)a^{m-2}p_o^2(\alpha(x \otimes 1)u)u^{-1}) \\
&= (-1)^m pf_\sigma(x)^{m-2}d_\sigma x
\end{aligned}$$

using that $p_o^2(x) = (-1)^m pf(x)^{m-2}x$. For algebras of dimension 16, the map p_σ is a K–linear automorphism of $\mathrm{Alt}^\sigma(A)$ and $p_\sigma^2(x) = d_\sigma x$. The class $\delta(\sigma)$ of the element d_σ with the property that $p_\sigma^2(x) = d_\sigma x$ was defined in [KPS$_3$] as the discriminant of σ. Thus both discriminants coincide. We finally mention the main result of [KPS$_3$], which relies on (3.1) for $n = 16$:

Theorem 3.2 *Let A be a central simple algebra of dimension 16 with an involution σ of the first kind. There exists a σ–invariant quaternion subalgebra A_1 of A if and only if the discriminant $\delta(\sigma)$ of σ is trivial.*

In characteristic not 2, the set A_1' of pure quaternions of A_1 can be chosen as

$$A_1' = \{x \in \mathrm{Alt}^\sigma(A) \mid p_\sigma(x) = x\},$$

where p_σ is taken such that $p_\sigma^2(x) = x$. For this choice of A_1 the restriction of σ to A_1 is the conjugation map of A_1. Details are in [KPS$_3$].

References

[B] Bourbaki, N.: Algèbre, Chapitre 9, Paris: Hermann 1959.

[CDTW] Chacron, M., Dherte, H., Tignol, J.-P., Wadsworth, A.: Stable decompositions of division algebras with involution, in preparation.

[F] Fröhlich, A.: Class groups and hermitian modules. Progress in Mathematics 48. Boston, Basel, Stuttgart: Birkhäuser 1984.

[J_1] Jacobson, N.: Clifford algebras for algebras with involution of type D, J. Algebra 1, 288-300 (1964).

[J_2] Jacobson, N.: Some applications of Jordan norms to involutorial simple associative algebras, Adv. in Math 48, 1-15 (1983).

[K] Kaplansky, I.: Linear algebra and geometry. A second course. Allyn and Bacon, Inc., Boston 1969.

[KPS_1] Knus, M.-A., Parimala R., Sridharan, R.: A classification of rank 6 quadratic spaces via pfaffians, J. Reine Angew. Math. 398, 187-218 (1989).

[KPS_2] Knus, M.-A., Parimala, R., Sridharan, R.: Pfaffians, central simple algebras and similitudes., Math. Z. , to appear 1990.

[KPS_3] Knus, M.-A., Parimala, R., Sridharan, R.: Involutions on rank 16 central simple algebras, J. Indian Math. Soc., to appear.

[KS] Knus, M.-A., Schofield, A.: Aspects of the pfaffian, Commun. Algebra, to appear.

[S] Saltman, D.: A note on generic division algebras, Preprint, Austin 1990.

[T] Tamagawa, T.: On quadratic forms and pfaffians. J. Fac. Sci. Tokyo, Sect. I, 24, 213-219 (1977).

Mathematik
ETH–Zentrum
CH–8092–Zürich
Switzerland

knus@math.ethz.ch

Tata Institute of Fundamental Research
School of Mathematics
Bombay 400005
India

SRIDHAR@TIFRVAX.bitnet