Algebras, their Invariants and K-forms. A tribute to [the work of] HEINRICH BRANDT on the 50th anniversary of his death

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Abstract

We discuss past and contemporary influence of the work of HEINRICH BRANDT on the structure theory and arithmetic of algebras. Invariants are taken up to isomorphism or antiisomorphism and not just up to equivalence.

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1 Introduction

Instead of a detailed appreciation of the work of Heinrich Brandt, we shall restrict ourselves here with hints on some of his distinguished discoveries in structural and arithmetic algebra.

In [18, 1928] Brandt proved that the structure of a maximal order in a rational quaternion algebra is determined uniquely by its norm form; further, the tetrahedral system of the bilinear substitutions due to G. Frobenius is established, and a conjecture in the general case is stated. This question has led to the work of the first author and K.W. Johnson (1992). Parallel side-notes of G. Frobenius (1897) and Brandt [18, 1928] were cleared up there, namely the so-called "remarkable sentences". This is treated by the first author in the present article. New research in this direction covers determinants and other invariants over non-commutative domains (J. Dieudonné, I.M. Gelfand et al.).

In [7, 1924] Brandt studied the existence of composition for integral quaternary quadratic forms. He introduced in this context the notion of a K-form and showed that primitive K-forms are exactly the integral quaternary forms which admit of a composition. In contrast to the theory of composition of binary forms, where, since Gauss' construction in his Disquisitiones arithmeticae, there have been many publications generalizing and simplifying the concept[1], not so many recent publications on composition of quaternary forms are available. A generalization in a purely algebraic setting (i.e., from forms over the integers to forms over arbitrary commutative rings) was given in joint publication of Kneser, Knus, Ojanguren, Parimala and Sridharan[2]. A description of this generalization is the main contribution of the second author to the present article.

2 Invariants of algebras and forms

2.1 The connection between the theories of forms and of ideals

Citation (B.L. van der Waerden[3]):

"Wenn wir die beiden grossen Zahlentheoretiker Dedekind und Kronecker miteinander vergleichen, so fällt auf, dass Dedekind viel mehr begrifflich denkt im Sinne der heutigen abstrakten Algebra, dass Kronecker dagegen viel mehr Wert auf explizite Rechenvorschriften legt. Z. B.: Dedekind definiert den Begriff Ideal so, wie wir es heute definieren. Kronecker degegen definiert seine Divisoren, die den Idealen gleichwertig sind, mit Hilfe von Formen in mehreren Unbestimmten u, u', [...] und Quotienten von solchen Formen, wobei er im Nenner sogenannte Einheitsformen zulässt. Er definiert das Produkt von Divisoren durch Multiplikation der Formen. [...]" S: 162: "Artin hat 1928 auf der Grundlage der Ideen von Brandt die Arithmetik der Algebren allgemein begründet. [...]" S: 164: "Emmy Noether hat in ihren Vorlesungen immer wieder betont: Die Idealtheorie ist eine Anwendung der Modultheorie.[...]"

The correspondence between splitting forms and ideals of an algebraic number field is well known in the commutative theory. However, there this transfer had a more methodical meaning. The analogous transfer of the arithmetics of quaternary quadratic forms to an ideal theory of algebras of quaternions (having rational center) has now produced the adequate fundamental notions. The related correspondence could be expressed in an aesthetically satisfactory way as the isomorphism between the groupoid of composable forms and the one-sided ideals (to be defined in terms of modules and discriminants) of maximal orders of quaternions. These orders then form the neutral elements of the groupoid. Cf. [18], [27]. In [18], p. 10, paragraph 22, Brandt points out the following:

"In the case of the reduced basis the constants of multiplication [= "structure constants"] [of a maximal order with respect to a minimal basis] is, strangely

[1] see for a survey M. Kneser, "Composition of binary quadratic forms" J. Number Theory **15** (1982) 406–413; "Komposition quadratischer Formen". (German) [Composition of quadratic forms] Algebra-Tagung Halle 1986, 161–173, Wissensch. Beitr., **3**3, Martin-Luther-Univ. Halle-Wittenberg, Halle, 1987.

[2] M. Kneser, M.-A. Knus, M. Ojanguren, R. Parimala, R. Sridharan "Composition of quaternary quadratic forms" Compositio Mathematica **60** (1986), 133-150.

[3] Die Algebra seit Galois. Jber. Deutsch. Math.-Verein. **68** (1966), 155(69)-165(79).

["merkwürdigerweise"], already completely determined by the coefficients of the norm form and its reciprocal form.[...] [In German]

This discovery was of extraordinary significance, and prompted Brandt to conjecture that, in analogy to the work of Gauss on the equivalence classes of binary quadratic forms[4], structure and arithmetic are already essentially determined by their norm forms not only for quaternion algebras, but also for arbitrary Dedekind (i.e. separable-semisimple) algebras. This holds in the commutative case. Therefore it is possible that Brandt had noticed that the exact counterpart of this discovery can be found in an article of Frobenius[5]. In the introduction (p.1), he likewise declared his results of §7 as "strange sentences on determinants of degree n, the elements of which are n^2 independent variables". While searching for a generalization of the results of his Diss. [D 5], the first author meets these strange Frobenius sentences, and found it possible to interpret his sentences I (p. 1011) and II (p. 1013) immediately as the structural statement that the algebra of matrices over a field is already uniquely determined up to an involutory antiisomorphism by the (reduced or generic) norm form. Thus the algebraic part of Brandt's invariance conjecture was confirmed.[6]

In fact, Brandt described his "dream" in a more specific fashion: He thought that the norm form of a Dedekind algebra might have an analogous influence on its arithmetic analogous to that of the potential function in Physics on the field quantities. That is, all relevant arithmetical data (e.g. the minimal basis of a maximal order and their structure constants) could be derived by suitable differentiations from the norm form.

The first author tried, in 1967 (see the preceding footnote), to realize this dream,.under the hypothesis that the reduced equation of the algebra has degree 3. But as in the general case, one would be able to arrive at final results only if one could squeeze out the coefficients of the norm forms by algebraically independent parameters. In the language of algebraic geometry, this problem is equivalent to the question as to whether a (not only rationally, but even) algebraically independent system of invariants classifies the group of regular linear transformations leaving invariant the norm form of an algebra and its identity element. For such an approach with a restriction on full rings of matrices, cf. C. Procesi[7] and the problems formulated there, which cohere closely with invariant-theoretical problems of Mechanics[8].

The algebraic part of Brandt's dream, the classification Dedekinds algebras by means of their norm forms presupposes its solution for the norm forms. Although this was not a problem which was originally considered by H.L. Schmid,

[4] C.F. Gauss "Disquisitiones arithmeticae" Leipzig 1801.

[5] "Über die Darstellung der endlichen Gruppen durch lineare Substitutionen.I" Sber. Preuss. Akad. Wiss. (**1897**), 994-1015.

[6] Cf. the first author: [D 5];
- - "Über Beziehungen zwischen Problemen von H. Brandt aus der Theorie der Algebren und den Automorphismen der Normenform", Math. Nachr. **34** (1967), 229-255;
- - "Invariants of algebras and finite groups", Acta Applicandae Math. **52** (1998), 277-283.

[7] Computing with matrices. Recent trends in Mathematics (Reinhardsbrunn 1982), pp. 27-250. Teubner-Texte zur Mathematik, Vol. 50. Leipzig 1982.

[8] A.J.M. Spencer "Theory of invariants" Continuum Physics. Vol.1, New York 1971.

for he posed the theme to investigate the structure of the ring of endomorphisms of a Jacobian manifold. But it finally led A. Bergmann and his pupils[9] to characterize algebras by their generic norms. However, in the current research of P. Gabriel's school, and in a very similar vein to the original task, the problem of classifying algebras is treated directly via their structure constants w_{ijk} , using methods from algebraic geometry such as deformation theory and combinatorics (Dynkin - diagrams, for example).

As concerns non-commutative arithmetic, there exists an interesting attempt to reduce it to a problem in abstract algebra: The complexity of the groupoids is enriched by a suitable continuation of the (partial) operation to a total one by means of a so-called sandwich-matrix $d_{ik} = (o_k \times o_i)^{-1}$ of distance ideals, where o_i, o_k are maximal orders. This gives a completely simple semigroup; the problem arises of classifying these semigroups[10].

Related combinatorial problems were treated by Brandt in [16], [19], [20], in connection with counting the possible decompositions of an ideal into prime ideals. These problems are of the kind posed by J. Steiner[11]. One should add with respect to suitable problems of arithmetic equivalence (related to groupoid-congruences) that the factor groupoids arising become finite. There result several still open questions, in particular with groupoids which are "disturbed" (in German: "gestört"), cf. [27]. Each equivalence of this type and each integral rational norm m lead to a matrix $[m]$ of complexes $K_{\alpha\beta}(m)$ of equivalent integral ideals having a norm m with the following properties: If one replaces in $[m]$ the complex at position $\alpha\beta$ by the number of its elements (calling the matrix $\lambda(m)$ of the same shape emerging in this way the *Hecke matrix of the norm* m) then we have the formula

$$\lambda(m)\lambda(m') = \sum_t t\lambda(\tfrac{m\, m'}{t^2}); \qquad t \;/\; (m, m'); \quad (t, d) = 1,$$

where d designates the *basic number* (in German: "Grundzahl") of the rational algebra of quaternions in question. (One may represent $\lambda(m)$ if one chooses for $[m]$ the composition table of the natural ideal classes and also by putting in instead of an ideal class the number of reduced representations of m by a norm form representing the ideal class.) This is another generalization of the *Eulerian product formula*, proved by E. Hecke in special cases and by Brandt in general.

[9] A. Bergmann: "Hauptnorm und Struktur von Algebren" J. reine angew. Math. **222** (1966), 160-194.

- - "Reduzierte Normen und Theorie von Algebren" Algebra-Tagung Halle 1986, Tagungsband. Martin-Luther-Universität Halle-Wittenberg, Wissenschaftl. Beiträge 1987/33 (M 48), Halle (Saale) 1987. - pp. 29-57.

R. Schmähling: "Separable Algebren über kommutativen Ringen" J. reine angew. Math. **262/263** (1973), 307-322.

D. Ziplies: "A characterization of the norm of an Azumaya algebra of constant rank through the divided powers algebra of an algebra" Beitr. Algebra Geom. **22** (1986), 53-70.

- - "Generators for the divided powers algebra of an algebra and trace identities" Beitr. Algebra Geom. **24** (1987).

[10] Cf. E.A. Behrens: "A semigroup theoretical approach to the non-commutative arithmetic" Algebra Berichte Nr. 43, 9. Jahrg. 1982, 51 pp., Math. Inst. Univ. München.

- - "A tree-based arithmetic for semigroups and rings" Vienna Conf. 1984. Contributions to General Algebra, Vol. **3**, 33-73. Wien 1985.

[11] "Kombinatorische Aufgaben" J. reine angew. Math. **45** (1853), 181-182.

$\lambda(m)$ is sometimes called the *Brandt matrix.*

While so far we have considered only questions that do not go beyond the specific circle of our main tasks, we want to speak now, albeit briefly, on newer efforts applying the main ideas of *arithmetic algebra* (a term formed in analogy with "geometric algebra") to the classification of non-commutative rings. Its value was recognized especially in connection with the theory of modular representations of groups. The axiomatic foundation of non-commutative ideal theory created by the school of K. Asano proved to be valuable[12]. In a series of investigations the following concepts are defined for any ring A with 1 (in particular for a finite-dimensional semisimple k-algebra A over the quotient field k of a Dedekind ring g): orders (g-orders) in A as certain subrings $S \subseteq A$, S-lattices as certain S-left modules (those which are at the same time torsion free g-modules of finite type), and finally maximal orders (g-orders), studying their ideal theory.

2.2 Norm type forms

In 1958 the first author[13] obtained explicit formulae for the coefficients $s_4(x)$ and $s_5(x)$ of the generic polynomial (having a generic element x as zero) of a separable-semisimple algebra of dimension m over a field K, in terms of the first three coefficients $s_1(x), s_2(x), s_3(x)$. There is a general recursion that gives an analogous determination of any coefficient $s_i(x)$ $(i \geq 6)$ from $s_1(x), s_2(x), s_3(x)$. The recursion does not simply include these first three quantities as a whole, but also their coefficients as forms in the indeterminates $x_1, \ldots, x_m$ over K.

Suppose given a separable-semisimple algebra $A_K = K\omega_0 + \ldots + K\omega_{m-1}$ of finite dimension m over a field K. The reduced norm $N(x)$ (of a generic element x) of A_K is a form of a certain degree n in m indeterminates $x_0, \ldots, x_{m-1}$(the coordinates of x). Instead of the reduced characteristic (or "generic") polynomial $N(\lambda e - x)$ where e designates the identity element of A_K, we consider, more generally, the polynomial

$$\Phi(\lambda e - x) = \lambda^n - s_1(x)\lambda^{n-1} + \cdots + (-1)^n s_n(x) \tag{1}$$

belonging to any *composable* or *norm type form*[14] $\Phi(x)$ (its degree being again designated by n) of A_K. The elementary symmetric functions $s_i(x)$ defined by $\Phi(x)$ in this way are also called elementary invariants of A_K. Roughly speaking, the calculation of the invariants $s_i(x)$ for $i > 3$ from $s_1(x)$, $s_2(x)$, $s_3(x)$ is a certain procedure which eliminates the power sums $s_1(x^i)$ from Newton's formulae[15]. In particular one has:

[12]K. Asano and K. Murata: "Arithmetical theory in semigroups" J. Inst. Polyt. Osaka Univ., s. A, **4** (1953), 9-13.

[13]"Über komponierbare Formen und konkordante hyperkomplexe Größen" Math. Z. **70** (1958), 1-12.

[14]That means: $\Phi(xy) = \Phi(x)\Phi(y)$, and $\Phi(x)$ can be seen to be built up by some irreducible factors of $N(x)$.

[15]H.-J. Hoehnke "Algorithmical description of the calculation of the elementary symmetric functions which are defined by a composable form" Seminarber. a. d. Fachber. Math. d. Univ.-Gesamthochsch. Hagen **63** (3) (1998), 1-9 (307-315).

$$4s_4(x) = (n-2)s_2(x)^2 - 2s_1(x)s_3(x) - 9u_4(x), \tag{2}$$

$$\begin{aligned} 4\cdot 5s_5(x) &= -(2n-1)(n-2)s_1(x)s_2(x)^2 + 6s_1(x)^2 s_3(x) \\ &\quad +4(3n-7)s_2(x)s_3(x) + 5\cdot 9s_1(x)u_4(x) + 4\cdot 27u_5(x), \end{aligned} \tag{3}$$

where in general forms $u_i(x)$ $(i \geq 4)$ which are derivable from $s_3(x)$ are given by:

$$\begin{aligned} u_4(x) &= \sum_{i,k,l,p,q,r} c_{ikl} d^{lp} c_{pqr} x_i x_k x_q x_r, & (4)\\ u_5(x) &= \sum c_{ikl} d^{lp} c_{pqr} d^{rs} c_{stv} x_i x_k x_q x_t x_v, & (5)\\ &\cdots \text{ and so on.} \end{aligned}$$

2.3 Group characters

Given a finite group $\mathfrak{G}$ of order n its group matrix (x) is defined as follows. Let $x_e, x_{g_1}, \ldots, x_{g_{n-1}}$ be variables indexed by the elements of $\mathfrak{G}$. The $n \times n$ matrix $\Omega(x)$ is defined to be the matrix whose (i,j)th entry is $x_{g_i g_j^{-1}}$. The *group determinant* $\Theta_{\mathfrak{G}}$ of $\mathfrak{G}$ is $|\Omega(x)|$.

Although the group determinant first appeared in Frobenius' 1896 paper[16] in which he introduced characters for arbitrary groups, the roots of the work go back to the discussions of Gauss on the composition of equivalence classes of quadratic forms. There is a good discussion of the background to Frobenius' work in[17]. A natural question arises as to whether non-isomorphic groups necessarily have distinct group determinants. This was posed by K.W. Johnson in 1986 and was answered by Formanek and Sibley[18] in 1990, the positive answer being somewhat surprising. In fact there were already two early remarks in the literature, apart from the work of Gauss quoted above, that pointed in this direction. Both of these were described in quotes as strange ("merkwürdig") by their authors and were disregarded for a long time within the development of algebra. The first was by Frobenius[19] on matrix transformations and the second occurred in [18] on the subject of maximal orders of quaternion algebras.

[16] "Über die Primfaktoren der Gruppendeterminante" Sber.Akad. Wiss. Berlin (**1896**), 1343-1382.

[17] T. Hawkins: "The origins of the theory of group characters" Arch. Hist. Exact Sci. **7** (1971), 142-170.
- - "New light on Frobenius' creation of the theory of group characters" Arch. Hist. Exact Sci. **12** (1974), 217-243.

[18] E. Formanek and D. Sibley: "The group determinant determines the group" Proc. Amer. Math. Soc. **112** (1991), 649-656.

[19] G. Frobenius: "Über die Darstellung der endlichen Gruppen durch lineare Substitutionen.I" Sber. Akad. Wiss. Berlin (**1897**), 944-1015.

Today these phenomena are subsumed under the thesis "from norm invariance to constructive structure theory and (noncommutative) arithmetic".

A further look at Frobenius' early work reveals functions $\chi^{(k)}:\mathfrak{G}^k \to \mathbb{C}$ defined below in the case where $k \leq 3$ that appeared in his algorithm to calculate the factor of a group determinant that corresponds to an irreducible character χ. Some of the properties of these "k-characters" have been explored in Johnson[20] where it is shown that the k-characters corresponding to distinct irreducible characters χ_i, χ_j are orthogonal in the sense that

$$\sum_{\underline{g} \in \mathfrak{G}^k} \chi_i^{(k)}(\underline{g})\overline{\chi}_i^{(k)}(\underline{g}) = 0.$$

There is also given a 2-character table of a group $\mathfrak{G}$ that consists of the 2-characters corresponding to the irreducible representations of degrees greater than 1 together with "degenerate" characters. Basic for multicharacters is the book of G.D. James and A. Kerber[21].

One of the questions raised by Brauer[22] is that of determining what extra information may be added to the (ordinary) character table to completely determine a group. As a consequence of Frobenius' work and the Formanek-Sibley result, knowledge of the k-characters $\chi^{(k)}$ of a group for all k and all irreducible characters χ is sufficient to determine a group. Note that if $k > \deg(\chi)$ then $\chi^{(k)} = 0$. The k-characters thus provide an answer to Brauer's question, but since the amount of work involved in calculating the k-characters for large k is prohibitively large, it becomes interesting to examine the question of the extent to which the knowledge of the k-characters for small values of k determine a group. The final answer is: the 3-characters suffice.

2.4 An example

It is perhaps for the first time that the preceding formula (2) will be tested in a concrete case. This is of practical importance since the necessary calculations are rather involved and depend heavily on an optimal organization of all steps. Here we have used the program MAPLE.

Let us consider the symmetric group $\mathfrak{S}_3 = \{\omega_0, \omega_1, \omega_2, \omega_3, \omega_4, \omega_5\}$ of order 3! acting on the 3 elements $\{1, 2, 3\}$, where $\omega_0 = 1, \omega_1 = (23), \omega_2 = (31), \omega_3 = (12), \omega_4 = (123), \omega_5 = (132)$. One has

[20] "On the group determinant" Math. Proc. Cambridge Philos. Soc. **109** (1991), 299-311.

[21] "The Representation Theory of the Symmetric Group" ENCYCLOPEDIA OF MATHEMATICS and Its Applications, Gian-Carlo Rota (ed.), Vol. 16, 1981.

[22] R. Brauer: "Representations of finite groups" Lectures in Modern Mathematics (t.L. Saaty, ed.), Wiley, 1963, pp. 133-175.

$$
\begin{pmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \\ \omega_3 \\ \omega_4 \\ \omega_5 \end{pmatrix}
\begin{pmatrix} \omega_0 & \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 \end{pmatrix}
$$

$$
= \begin{pmatrix}
\omega_0\omega_0 & \omega_0\omega_1 & \omega_0\omega_2 & \omega_0\omega_3 & \omega_0\omega_4 & \omega_0\omega_5 \\
\omega_1\omega_0 & \omega_1\omega_1 & \omega_1\omega_2 & \omega_1\omega_3 & \omega_1\omega_4 & \omega_1\omega_5 \\
\omega_2\omega_0 & \omega_2\omega_1 & \omega_2\omega_2 & \omega_2\omega_3 & \omega_2\omega_4 & \omega_2\omega_5 \\
\omega_3\omega_0 & \omega_3\omega_1 & \omega_3\omega_2 & \omega_3\omega_3 & \omega_3\omega_4 & \omega_3\omega_5 \\
\omega_4\omega_0 & \omega_4\omega_1 & \omega_4\omega_2 & \omega_4\omega_3 & \omega_4\omega_4 & \omega_4\omega_5 \\
\omega_5\omega_0 & \omega_5\omega_1 & \omega_5\omega_2 & \omega_5\omega_3 & \omega_5\omega_4 & \omega_5\omega_5
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\omega_0 & \omega_1 & \omega_2 & \omega_3 & \omega_4 & \omega_5 \\
\omega_1 & \omega_0 & \omega_5 & \omega_4 & \omega_3 & \omega_2 \\
\omega_2 & \omega_4 & \omega_0 & \omega_5 & \omega_1 & \omega_3 \\
\omega_3 & \omega_5 & \omega_4 & \omega_0 & \omega_2 & \omega_1 \\
\omega_4 & \omega_2 & \omega_3 & \omega_1 & \omega_5 & \omega_0 \\
\omega_5 & \omega_3 & \omega_1 & \omega_2 & \omega_0 & \omega_4
\end{pmatrix}. \tag{6}
$$

Let $x = x_0\omega_0 + x_1\omega_1 + x_2\omega_2 + x_3\omega_3 + x_4\omega_4 + x_5\omega_5$ be a generic element of the groups ring $\mathfrak{O}_{K[x_0,x_1,x_2,x_3,x_4,x_5]}(\mathfrak{S}_3)$ of $\mathfrak{S}_3$ over the ring extension $K[x_0, x_1, x_2, x_3, x_4, x_5]$ of a basic field K of characteristic $\neq 2, 3$. With $y = y_0\omega_0 + y_1\omega_1 + y_2\omega_2 + y_3\omega_3 + y_4\omega_4 + y_5\omega_5$ one has

$$
x \cdot y = z = z_0\omega_0 + z_1\omega_1 + z_2\omega_2 + z_3\omega_3 + z_4\omega_4 + z_5\omega_5
$$

where

$$
\begin{aligned}
z_0 &= x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 + x_4y_5 + x_5y_4 \\
z_1 &= x_0y_1 + x_1y_0 + x_2y_4 + x_3y_5 + x_4y_3 + x_5y_2 \\
z_2 &= x_0y_2 + x_1y_5 + x_2y_0 + x_3y_4 + x_4y_1 + x_5y_3 \\
z_3 &= x_0y_3 + x_1y_4 + x_2y_5 + x_3y_0 + x_4y_2 + x_5y_1 \\
z_4 &= x_0y_4 + x_1y_3 + x_2y_1 + x_3y_2 + x_4y_0 + x_5y_5 \\
z_5 &= x_0y_5 + x_1y_2 + x_2y_3 + x_3y_1 + x_4y_4 + x_5y_0.
\end{aligned}
$$

The *group matrix* of $\mathfrak{S}_3$ is designated by $\Omega[x]$. It is given by

$$
\Omega[x] = \begin{pmatrix}
x_0 & x_4 & x_5 & x_2 & x_3 & x_1 \\
x_5 & x_0 & x_4 & x_3 & x_1 & x_2 \\
x_4 & x_5 & x_0 & x_1 & x_2 & x_3 \\
x_2 & x_3 & x_1 & x_0 & x_4 & x_5 \\
x_3 & x_1 & x_2 & x_5 & x_0 & x_4 \\
x_1 & x_2 & x_3 & x_4 & x_5 & x_0
\end{pmatrix} \tag{7}
$$

According to the decomposition of the group matrix into irreducible representations, each of them appearing as often as its degree indicates (i. e. $1, 1, 2 \times 2$), for the *group determinant* one has:

$|\Omega(x)| =$

$(x_4 + x_0 + x_5 - x_3 - x_2 - x_1) \cdot (x_4 + x_0 + x_5 + x_3 + x_2 + x_1) \cdot$

$\left(x_0^2 - x_4x_0 + x_4^2 - x_5x_0 - x_4x_5 + x_5^2 - x_2^2 + x_3x_2 - x_3^2 + x_1x_2 + x_1x_3 - x_1^2\right)^2$

.

The *characteristic* and the *minimal polynomial* of $\Omega[x]$ are $|\lambda E - \Omega[x]|$ and

$m(\lambda; x) = \lambda^4 - m_1(x)\lambda^3 + m_2(x)\lambda^2 - m_3(x)\lambda^1 + m_4(x)$

respectively, where:

$m_1(x) = 4x_0 + x_4 + x_5$

$m_2(x) = 3x_0x_4 - 3x_4x_5 - 2x_2^2 - 2x_3^2 + 3x_0x_5 - x_1x_2 - x_2x_3 - x_3x_1 + 6x_0^2 - 2x_1^2$

$m_3(x) = -2x_0x_1x_2 - 2x_0x_3x_2 - x_5x_1^2 - x_2^2x_5 - x_5x_3^2 + 4x_3x_2x_4 + 4x_5x_3x_2 + 4x_1x_3x_5 + 4x_4x_1x_3 + 4x_1x_2x_5 + 4x_4x_1x_2$
$-6x_0x_5x_4 - 2x_0x_1x_3 + 4x_0^3 + x_5^3 + x_4^3 - 3x_4x_5^2 - x_4x_2^2 - x_4x_3^2 - x_4x_1^2 - 3x_4^2x_5 + 3x_0^2x_4 + 3x_0^2x_5 - 4x_0x_1^2 - 4x_0x_3^2 - 4x_0x_2^2$

$m_4(x) = (x_4 + x_0 + x_5 - x_3 - x_2 - x_1) \cdot (x_4 + x_0 + x_5 + x_3 + x_2 + x_1)$
$\cdot \left(x_0^2 - x_4x_0 + x_4^2 - x_5x_0 - x_4x_5 + x_5^2 - x_2^2 + x_3x_2 - x_3^2 + x_1x_2 + x_1x_3 - x_1^2\right)$
$= -3x_5x_0^2x_4 - x_1x_2x_4^2 - x_2x_3x_4^2 - x_3x_1x_4^2 - x_5x_2^2x_4 - x_5x_3^2x_4 - x_5x_1^2x_4 - x_5^2x_2x_3 - x_5^2x_3x_1 - x_5^2x_1x_2 + x_0x_4^3$
$-2x_2^2x_4^2 - 2x_3^2x_4^2 - 2x_1^2x_4^2 - 2x_5^2x_2^2 - 2x_5^2x_3^2 - 2x_5^2x_1^2 + x_0x_5^3 + 4x_0x_2x_3x_4 + 4x_0 x_3x_1x_4 + 4x_0x_1x_2x_4 + 4x_5x_2x_3x_4$
$+4x_5x_3x_1x_4 + 4x_5x_1x_2x_4 + 4x_5x_0x_2x_3 + 4x_5x_0x_3x_1 + 4x_5x_0x_1x_2 + x_4^4 - 3x_0x_5x_4^2 - x_0x_3^2x_4 - x_0x_2^2x_4 - x_0x_1^2x_4$
$-x_0^2x_2x_3 - x_0^2x_3x_1 - x_0^2x_1x_2 - x_0x_5x_2^2 - x_0x_5x_3^2 - x_0x_5x_1^2 + x_5^3x_4 + x_0^3x_4 + x_3^3x_2 + x_2^3x_3 + x_5^4 - 3x_1x_2^2x_3 - 3x_1^2x_2x_3$
$-3x_2x_3^2x_1 + x_2^4 + x_1^4 + x_3^4 + x_2^3x_1 + x_3^3x_1 + x_1^3x_2 + x_1^3x_3 + x_0^3x_5 - 2x_2^2x_0^2 - 2x_1^2x_0^2 - 2x_3^2x_0^2 + x_4^3x_5 + x_0^4 - 3x_0x_5^2x_4.$

The coefficients $m_1(x) = S(x)$ and $m_4(x) = N(x)$ are also called *reduced trace* and *reduced norm* of the group ring respectively. We also put $m_2(x) = T(x), m_3(x) = U(x)$.

The *discriminant* of the group ring is the determinant $|d|$ of the matrix

$$d = \begin{pmatrix} 4 & 0 & 0 & 0 & 1 & 1 \\ 0 & 4 & 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 1 & 1 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 4 \\ 1 & 0 & 0 & 0 & 4 & 1 \end{pmatrix}. \tag{8}$$

It is the matrix of coefficients of the symmetric bilinear form

$S(xy) = 4z_0 + z_4 + z_5 = \sum_{i,k=0}^{5} d_{ik}x_iy_k =$

$4(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3 + x_4y_5 + x_5y_4) + x_0y_4 + x_1y_3 + x_2y_1 + x_3y_2 + x_4y_0$

$$+x_5y_5 \quad + \quad x_0y_5 \quad + \quad x_1y_2 \quad + \quad x_2y_3 \quad + \quad x_3y_1 \quad + \quad x_4y_4 \quad + \quad x_5y_0.$$

(Here we remark that $S(xy)$ does not depend on the structure constants occurring in the product xy, since by (bilinearized) Newton's formulas one has: $S(xy) = S(x)S(y) - T(x,y)$ and $T(x,y) = \sum_{j,k=0}^{5}(c_{jk}+c_{kj})x_jy_k$ is the bilinearization of the 2^{nd} elementary symmetric polynomial $T(x) = \sum_{j,k=0}^{5} c_{jk}x_jx_k$.) Since by assumption on the characteristic of K, the group ring of $\mathfrak{S}_3$ is separable, which is equivalent to $|d| \neq 0$. One has indeed $|d| = -2916 = -2^2 \cdot 27^2$. The inverse

$$d^{-1} := \begin{pmatrix} d^{00} & d^{01} & d^{02} & d^{03} & d^{04} & d^{05} \\ d^{10} & d^{11} & d^{12} & d^{13} & d^{14} & d^{15} \\ d^{20} & d^{21} & d^{22} & d^{23} & d^{24} & d^{25} \\ d^{30} & d^{31} & d^{32} & d^{33} & d^{34} & d^{35} \\ d^{40} & d^{41} & d^{42} & d^{43} & d^{44} & d^{45} \\ d^{50} & d^{51} & d^{52} & d^{53} & d^{54} & d^{55} \end{pmatrix}$$

of d plays a striking role in our approach. It is given by

$$d^{-1} = \begin{pmatrix} \frac{5}{18} & 0 & 0 & 0 & -\frac{1}{18} & -\frac{1}{18} \\ 0 & \frac{5}{18} & -\frac{1}{18} & -\frac{1}{18} & 0 & 0 \\ 0 & -\frac{1}{18} & \frac{5}{18} & -\frac{1}{18} & 0 & 0 \\ 0 & -\frac{1}{18} & -\frac{1}{18} & \frac{5}{18} & 0 & 0 \\ -\frac{1}{18} & 0 & 0 & 0 & -\frac{1}{18} & \frac{5}{18} \\ -\frac{1}{18} & 0 & 0 & 0 & \frac{5}{18} & -\frac{1}{18} \end{pmatrix}. \tag{9}$$

The main goal is the calculation of the reduced norm $N(x)$ by means of the cubic form $U(x) = \sum_{i,j,k=1}^{6} c_{ijk}x_ix_jx_k$ according to the formula

$$4N(x) = 2T(x)^2 - 2S(x)U(x) - 9V(x), \tag{10}$$

where

$$V(x) = \sum_{i,k,l,p,q,r=0}^{5} c_{ikl}d^{lp}c_{pqr}x_ix_kx_qx_r. \tag{11}$$

Since $6 \neq 0$ in K, we are allowed to suppose the coefficients c_{ijk} as symmetric in all indices. Thus we get:

$$c_{0**} = \begin{pmatrix} 4 & 0 & 0 & 0 & 1 & 1 \\ 0 & -\frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{4}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},$$

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} c_{0**} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$\begin{aligned} &= (4x_0 + x_4 + x_5)\, x_0 + \left(-\tfrac{4}{3}x_1 - \tfrac{1}{3}x_2 - \tfrac{1}{3}x_3\right) x_1 + \left(-\tfrac{1}{3}x_1 - \tfrac{4}{3}x_2 - \tfrac{1}{3}x_3\right) x_2 \\ &\quad + \left(-\tfrac{1}{3}x_1 - \tfrac{1}{3}x_2 - \tfrac{4}{3}x_3\right) x_3 + (x_0 - x_5)\, x_4 + (x_0 - x_4)\, x_5 = \\ &\quad h_0 = 4x_0^2 + 2x_0x_4 + 2x_0x_5 - \tfrac{4}{3}x_1^2 - \tfrac{2}{3}x_1x_2 - \tfrac{2}{3}x_1x_3 - \tfrac{4}{3}x_2^2 - \tfrac{2}{3}x_2x_3 - \tfrac{4}{3}x_3^2 - 2x_4x_5, \end{aligned}$$

$$c_{1**} = \begin{pmatrix} 0 & -\frac{4}{3} & -\frac{1}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{4}{3} & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} c_{1**} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$\begin{aligned} &= \left(-\tfrac{4}{3}x_1 - \tfrac{1}{3}x_2 - \tfrac{1}{3}x_3\right) x_0 + \left(-\tfrac{4}{3}x_0 - \tfrac{1}{3}x_4 - \tfrac{1}{3}x_5\right) x_1 + \left(-\tfrac{1}{3}x_0 + \tfrac{2}{3}x_4 + \tfrac{2}{3}x_5\right) x_2 + \\ &\left(-\tfrac{1}{3}x_0 + \tfrac{2}{3}x_4 + \tfrac{2}{3}x_5\right) x_3 + \left(-\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 + \tfrac{2}{3}x_3\right) x_4 + \left(-\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 + \tfrac{2}{3}x_3\right) x_5 = \\ &h_1 = -\tfrac{8}{3}x_0x_1 - \tfrac{2}{3}x_0x_2 - \tfrac{2}{3}x_0x_3 - \tfrac{2}{3}x_1x_4 - \tfrac{2}{3}x_1x_5 + \tfrac{4}{3}x_2x_4 + \tfrac{4}{3}x_2x_5 + \tfrac{4}{3}x_3x_4 + \tfrac{4}{3}x_3x_5, \end{aligned}$$

$$c_{2**} = \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{4}{3} & -\frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ -\frac{4}{3} & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} c_{2**} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$\begin{aligned} &= \left(-\tfrac{1}{3}x_1 - \tfrac{4}{3}x_2 - \tfrac{1}{3}x_3\right) x_0 + \left(-\tfrac{1}{3}x_0 + \tfrac{2}{3}x_4 + \tfrac{2}{3}x_5\right) x_1 + \left(-\tfrac{4}{3}x_0 - \tfrac{1}{3}x_4 - \tfrac{1}{3}x_5\right) x_2 + \\ &\left(-\tfrac{1}{3}x_0 + \tfrac{2}{3}x_4 + \tfrac{2}{3}x_5\right) x_3 + \left(\tfrac{2}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3\right) x_4 + \left(\tfrac{2}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3\right) x_5 = h_2 = \\ &-\tfrac{2}{3}x_0x_1 - \tfrac{8}{3}x_0x_2 - \tfrac{2}{3}x_0x_3 + \tfrac{4}{3}x_1x_4 + \tfrac{4}{3}x_1x_5 - \tfrac{2}{3}x_2x_4 - \tfrac{2}{3}x_2x_5 + \tfrac{4}{3}x_3x_4 + \tfrac{4}{3}x_3x_5, \end{aligned}$$

$$c_{3**} = \begin{pmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{4}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ -\frac{1}{3} & 0 & 0 & 0 & \frac{2}{3} & \frac{2}{3} \\ -\frac{4}{3} & 0 & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} c_{3**} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= \left(-\tfrac{1}{3}x_1 - \tfrac{1}{3}x_2 - \tfrac{4}{3}x_3\right) x_0 + \left(-\tfrac{1}{3}x_0 + \tfrac{2}{3}x_4 + \tfrac{2}{3}x_5\right) x_1 + \left(-\tfrac{1}{3}x_0 + \tfrac{2}{3}x_4 + \tfrac{2}{3}x_5\right) x_2$$
$$+ \left(-\tfrac{4}{3}x_0 - \tfrac{1}{3}x_4 - \tfrac{1}{3}x_5\right) x_3 \ + \ \left(\tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_3\right) x_4 \ + \ \left(\tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_3\right) x_5$$
$= h_3 = -\frac{2}{3}x_0x_1 - \frac{2}{3}x_0x_2 - \frac{8}{3}x_0x_3 + \frac{4}{3}x_1x_4 + \frac{4}{3}x_1x_5 + \frac{4}{3}x_2x_4 + \frac{4}{3}x_2x_5 - \frac{2}{3}x_3x_4 - \frac{2}{3}x_3x_5,$

$$c_{4**} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix},$$

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} c_{4**} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= (x_0 - x_5)\, x_0 + \left(-\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 + \tfrac{2}{3}x_3\right) x_1 + \left(\tfrac{2}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3\right) x_2$$
$$+ \left(\tfrac{2}{3}x_1 + \tfrac{2}{3}x_2 - \tfrac{1}{3}x_3\right) x_3 + (x_4 - x_5)\, x_4 + (-x_0 - x_4 - x_5)\, x_5 =$$
$$h_4 = x_0^2 - 2x_0x_5 - \tfrac{1}{3}x_1^2 + \tfrac{4}{3}x_1x_2 + \tfrac{4}{3}x_1x_3 - \tfrac{1}{3}x_2^2 + \tfrac{4}{3}x_2x_3 - \tfrac{1}{3}x_3^2 + x_4^2 - 2x_4x_5 - x_5^2,$$

$$c_{5**} = \begin{pmatrix} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & -\frac{1}{3} & \frac{2}{3} & 0 & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{1}{3} & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} x_0 & x_1 & x_2 & x_3 & x_4 & x_5 \end{pmatrix} c_{5**} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$= (x_0 - x_4)\, x_0 + \left(-\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 + \tfrac{2}{3}x_3\right) x_1 + \left(\tfrac{2}{3}x_1 - \tfrac{1}{3}x_2 + \tfrac{2}{3}x_3\right) x_2$$

$+\left(\frac{2}{3}x_1+\frac{2}{3}x_2-\frac{1}{3}x_3\right)x_3+(-x_0-x_4-x_5)\,x_4+(-x_4+x_5)\,x_5=$
$h_5=x_0^2-2x_0x_4-\frac{1}{3}x_1^2+\frac{4}{3}x_1x_2+\frac{4}{3}x_1x_3-\frac{1}{3}x_2^2+\frac{4}{3}x_2x_3-\frac{1}{3}x_3^2-x_4^2-2x_4x_5+x_5^2.$

Using these expressions one has to calculate

$$d^{-1}\cdot\begin{pmatrix} h_0\\ h_1\\ h_2\\ h_3\\ h_4\\ h_5 \end{pmatrix} =: \begin{pmatrix} k_0\\ k_1\\ k_2\\ k_3\\ k_4\\ k_5 \end{pmatrix}$$

giving:
$k_0=x_0^2+\frac{2}{3}x_0x_4+\frac{2}{3}x_0x_5-\frac{1}{3}x_1^2-\frac{1}{3}x_1x_2-\frac{1}{3}x_1x_3-\frac{1}{3}x_2^2-\frac{1}{3}x_2x_3-\frac{1}{3}x_3^2-\frac{1}{3}x_4x_5,$
$k_1=-\frac{2}{3}x_0x_1-\frac{1}{3}x_1x_4-\frac{1}{3}x_1x_5+\frac{1}{3}x_2x_4+\frac{1}{3}x_2x_5+\frac{1}{3}x_3x_4+\frac{1}{3}x_3x_5,$
$k_2=-\frac{2}{3}x_0x_2+\frac{1}{3}x_1x_4+\frac{1}{3}x_1x_5-\frac{1}{3}x_2x_4-\frac{1}{3}x_2x_5+\frac{1}{3}x_3x_4+\frac{1}{3}x_3x_5,$
$k_3=-\frac{2}{3}x_0x_3+\frac{1}{3}x_1x_4+\frac{1}{3}x_1x_5+\frac{1}{3}x_2x_4+\frac{1}{3}x_2x_5-\frac{1}{3}x_3x_4-\frac{1}{3}x_3x_5,$
$k_4=-\frac{2}{3}x_0x_4+\frac{1}{3}x_1x_2+\frac{1}{3}x_1x_3+\frac{1}{3}x_2x_3-\frac{1}{3}x_4x_5-\frac{1}{3}x_4^2+\frac{1}{3}x_5^2,$
$k_5=-\frac{2}{3}x_0x_5+\frac{1}{3}x_1x_2+\frac{1}{3}x_1x_3+\frac{1}{3}x_2x_3-\frac{1}{3}x_4x_5+\frac{1}{3}x_4^2-\frac{1}{3}x_5^2.$

This leads to:

$$V(x)=\begin{pmatrix} h_0 & h_1 & h_2 & h_3 & h_4 & h_5 \end{pmatrix}\begin{pmatrix} k_0\\ k_1\\ k_2\\ k_3\\ k_4\\ k_5 \end{pmatrix}$$

$=h_0k_0+h_1k_1+h_2k_2+h_3k_3+h_4k_4+h_5k_5$
$=-\frac{4}{9}x_1x_4^2x_3-\frac{4}{9}x_2x_4^2x_3-\frac{4}{9}x_1x_5^2x_3+\frac{32}{9}x_2^2x_4x_5+\frac{32}{9}x_3^2x_4x_5+\frac{8}{3}x_1x_2x_3^2+\frac{8}{3}x_1x_2^2$
$x_3-\frac{4}{9}x_0x_4x_3^2-\frac{4}{9}x_0^2x_1x_3-\frac{4}{9}x_2x_5^2x_3+\frac{4}{3}x_0x_4^2x_5-\frac{4}{9}x_1x_5^2x_2-\frac{4}{9}x_0x_5x_3^2+\frac{32}{9}x_1^2$
$x_4x_5-\frac{4}{9}x_0x_4x_1^2+\frac{4}{3}x_0^2x_4x_5-\frac{4}{9}x_1x_4^2x_2-\frac{2}{3}x_4^4-\frac{2}{3}x_5^4-\frac{4}{9}x_0x_4x_2^2+\frac{8}{3}x_1^2x_2x_3-\frac{4}{9}x_0$
$x_5x_2^2-\frac{4}{9}x_0x_5x_1^2+4x_0^3x_4+4x_0^3x_5-\frac{8}{9}x_0^2x_1^2-\frac{8}{9}x_0^2x_2^2-\frac{8}{9}x_0^2x_3^2+\frac{4}{3}x_0x_5^2x_4-\frac{4}{9}x_0^2$
$x_2x_3-\frac{4}{9}x_0^2x_1x_2+4x_0^4+\frac{4}{9}x_1^4+\frac{4}{3}x_0^2x_4^2+\frac{4}{3}x_0^2x_5^2+\frac{4}{9}x_1^3x_2+\frac{4}{9}x_1^3x_3+2x_1^2x_2^2+$
$2x_1^2x_3^2+\frac{4}{9}x_1x_2^3+\frac{4}{9}x_1x_3^3+\frac{4}{9}x_2^3x_3+2x_2^2x_3^2+\frac{4}{9}x_2x_3^3+\frac{10}{3}x_4^2x_5^2+\frac{10}{9}x_1^2x_4^2+\frac{10}{9}$
$x_1^2x_5^2+\frac{10}{9}x_2^2x_4^2+\frac{10}{9}x_2^2x_5^2+\frac{10}{9}x_3^2x_4^2+\frac{10}{9}x_3^2x_5^2-\frac{4}{3}x_0x_5^3-\frac{4}{3}x_4^3x_0-\frac{56}{9}x_0x_4x_1x_2-$
$\frac{56}{9}x_0x_4x_1x_3-\frac{56}{9}x_0x_4x_2x_3-\frac{56}{9}x_0x_5x_1x_2-\frac{56}{9}x_0x_5x_1x_3-\frac{56}{9}x_0x_5x_2x_3-\frac{20}{9}x_1$
$x_2x_4x_5-\frac{20}{9}x_1x_3x_4x_5+\frac{4}{9}x_2^4+\frac{4}{9}x_3^4-\frac{20}{9}x_2x_3x_4x_5,$

$9V(x)=$

$12x_0x_5x_4^2-12x_0x_4^3-12x_0x_5^3+36x_0^3x_4+4x_3^3x_2+4x_2^3x_3+4x_2^3x_1+4x_3^3x_1+$
$4x_1^3x_2+4x_1^3x_3+36x_0^3x_5+10x_2^2x_4^2+10x_3^2x_4^2+10x_1^2x_4^2+10x_5^2x_2^2+10x_5^2x_3^2+10x_5^2x_1^2-$
$8x_2^2x_0^2-8x_1^2x_0^2-8x_3^2x_0^2+12x_0^2x_4^2+30x_4^2x_5^2+12x_5^2x_0^2-6x_4^4-6x_5^4+4x_2^4+4x_1^4+4x_3^4+$
$36x_0^4+12x_5x_0^2x_4-4x_1x_2x_4^2-4x_2x_3x_4^2-4x_3x_1x_4^2+32x_5x_2^2x_4+32x_5x_3^2x_4+32x_5x_1^2$
$x_4-4x_5^2x_2x_3-4x_5^2x_3x_1-4x_5^2x_1x_2+24x_1x_2^2x_3+24x_2x_3^2x_1+12x_0x_5^2x_4-4x_0$
$x_3^2x_4-4x_0x_2^2x_4-4x_0x_1^2x_4-4x_0^2x_2x_3-4x_0^2x_3x_1-4x_0^2x_1x_2-4x_0x_5x_2^2-4x_0x_5$
$x_3^2-4x_0x_5x_1^2+24x_1^2x_2x_3-56x_0x_2x_3x_4-56x_0x_3x_1x_4-56x_0x_1x_2x_4-20x_5x_2$
$x_3x_4-20x_5x_3x_1x_4-20x_5x_1x_2x_4-56x_5x_0x_2x_3-56x_5x_0x_3x_1-56x_5x_0x_1x_2+18$
$x_2^2x_3^2+18x_2^2x_1^2+18x_3^2x_1^2.$

Now

$2T(x)^2 - 2S(x)U(x) - 9V(x)$

$=$

$2(3x_0x_4 - 3x_4x_5 - 2x_2^2 - 2x_3^2 + 3x_0x_5 - x_1x_2 - x_2x_3 - x_3x_1 + 6x_0^2 - 2x_1^2)^2 - 2(4x_0+x_4+x_5)(-2x_0x_1x_2 - 2x_0x_3x_2 - x_5x_1^2 - x_2^2x_5 - x_5x_3^2 + 4x_3x_2x_4 + 4x_5x_3x_2 + 4x_1x_3x_5 + 4x_4x_1x_3 + 4x_1x_2x_5 + 4x_4x_1x_2 - 6x_0x_5x_4 - 2x_0x_1x_3 + 4x_0^3 + x_5^3 + x_4^3 - 3x_4x_5^2 - x_4x_2^2 - x_4x_3^2 - x_4x_1^2 - 3x_4^2x_5 + 3x_0^2x_4 + 3x_0^2x_5 - 4x_0x_1^2 - 4x_0x_3^2 - 4x_0x_2^2) - (12x_0x_5x_4^2 - 12x_0x_4^3 - 12x_0x_5^3 + 36x_0^3x_4 + 4x_3^3x_2 + 4x_2^3x_3 + 4x_2^3x_1 + 4x_3^3x_1 + 4x_1^3x_2 + 4x_1^3x_3 + 36x_0^3x_5 + 10x_2^2x_4^2 + 10x_3^2x_4^2 + 10x_1^2x_4^2 + 10x_5^2x_2^2 + 10x_5^2x_3^2 + 10x_5^2x_1^2 - 8x_2^2x_0^2 - 8x_1^2x_0^2 - 8x_3^2x_0^2 + 12x_0^2x_4^2 + 30x_4^2x_5^2 + 12x_5^2x_0^2 - 6x_4^4 - 6x_5^4 + 4x_2^4 + 4x_1^4 + 4x_3^4 + 36x_0^4 + 12x_5x_0^2x_4 - 4x_1x_2x_4^2 - 4x_2x_3x_4^2 - 4x_3x_1x_4^2 + 32x_5x_2^2x_4 + 32x_5x_3^2x_4 + 32x_5x_1^2 x_4 - 4x_5^2x_2x_3 - 4x_5^2x_3x_1 - 4x_5^2x_1x_2 + 24x_1x_2^2x_3 + 24x_2x_3^2x_1 + 12x_0x_5^2x_4 - 4x_0 x_3^2x_4 - 4x_0x_2^2x_4 - 4x_0x_1^2x_4 - 4x_0^2x_2x_3 - 4x_0^2x_3x_1 - 4x_0^2x_1x_2 - 4x_0x_5x_2^2 - 4x_0x_5 x_3^2 - 4x_0x_5x_1^2 + 24x_1^2x_2x_3 - 56x_0x_2x_3x_4 - 56x_0x_3x_1x_4 - 56x_0x_1x_2x_4 - 20x_5x_2 x_3x_4 - 20x_5x_3x_1x_4 - 20x_5x_1x_2x_4 - 56x_5x_0x_2x_3 - 56x_5x_0x_3x_1 - 56x_5x_0x_1x_2 + 18 x_2^2x_3^2 + 18x_2^2x_1^2 + 18x_3^2x_1^2)$

$= -12x_0x_5x_4^2 + 4x_0x_4^3 + 4x_0x_5^3 + 4x_0^3x_4 + 4x_3^3x_2 + 4x_2^3x_3 + 4x_2^3x_1 + 4x_3^3x_1 + 4x_1^3x_2 + 4x_1^3x_3 + 4x_0^3x_5 - 8x_2^2x_4^2 - 8x_3^2x_4^2 - 8x_1^2x_4^2 - 8x_5^2x_2^2 - 8x_5^2x_3^2 - 8x_5^2x_1^2 - 8x_2^2x_0^2 - 8x_1^2x_0^2 - 8x_3^2x_0^2 + 4x_4^4 + 4x_5^4 + 4x_2^4 + 4x_1^4 + 4x_3^4 + 4x_0^4 - 12x_5x_0^2x_4 - 4 x_1x_2x_4^2 - 4x_2x_3x_4^2 - 4x_3x_1x_4^2 - 4x_5x_2^2x_4 - 4x_5x_3^2x_4 - 4x_5x_1^2x_4 - 4x_5^2x_2x_3 - 4 x_5^2x_3x_1 - 4x_5^2x_1x_2 - 12x_1x_2^2x_3 - 12x_2x_3^2x_1 - 12x_0x_5^2x_4 - 4x_0x_3^2x_4 - 4x_0x_2^2x_4 - 4x_0x_1^2x_4 - 4x_0^2x_2x_3 - 4x_0^2x_3x_1 - 4x_0^2x_1x_2 - 4x_0x_5x_2^2 - 4x_0x_5x_3^2 - 4x_0x_5x_1^2 - 12x_1^2x_2x_3 + 16x_0x_2x_3x_4 + 16x_0x_3x_1x_4 + 16x_0x_1x_2x_4 + 16x_5x_2x_3x_4 + 16x_5x_3x_1x_4 + 16x_5x_1x_2x_4 + 16x_5x_0x_2x_3 + 16x_5x_0x_3x_1 + 16x_5x_0x_1x_2 + 4x_5^3x_4 + 4x_4^3x_5.$

εὕρηκα! This is indeed equal to $4N(x)$.

3 K-forms

3.1 Introduction

The study of compositions of binary quadratic forms has a long history. Gauss' composition of integral binary quadratic forms is the first example in mathematics of a group composition law which arrows not from numbers, even in a broad sense. We refer to the papers of Kneser quoted in footnote 1 for a recent survey, as well as for a presentation of the binary case in the context of arbitrary commutative rings.

In contrast to the binary case, not every integral quaternary form admits of a composition. The main differences between the binary and the quaternary case can be summarized as follows:

1) In the binary case two forms with the same discriminant can always be composed in a unique way. This is not the case for four variables. A composable integral form must have a square discriminant $d(M) = D^2$ and the adjoint form has to be divisible by D. Brandt calls such a form a K-form in [7, 1924].

2) Two composable forms (K-forms) cannot necessarily be composed together. If, for example q_1 and q_2 are normforms of two ideals M_1 and M_2 in a quaternion algebra, then q_1 and q_2 are only composable if the right order of M_1 is

isomorphic to the left order of M_2.
3) For two forms which can be composed together, the composition is not necessarily unique.
Fact 2) lent Brandt to the notion of a groupoid (see Section 3.5 for more details).

3.2 Composition of quadratic forms

A modern account of composition of quaternary forms in the framework of arbitrary commutative rings is given in the 1986 joint article Kneser *et al.*, (see footnote 2). In particular Brandt's conditions for the existence of a composition of integral quaternary forms are formulated for arbitrary commutative rings and are shown to be necessary (but in general not sufficient). However it is proved that under some restrictions on the ring (which cover the classical case) these conditions are sufficient. To adapt Brandt's conditions in full generality is still an open problem. As in the binary case, the main tool used is the Clifford algebra of the quaternary quadratic form. An exact condition for the form to admit a composition can be derived in terms of the Clifford algebra.
In this section we give a summary of the main results of that paper, mostly without proofs, for forms over domains.
Let R be a domain with quotient field K. For simplicity we assume that K has characteristic different from 2. Let M be a finitely generated projective R-module of rank 4 and q a quadratic form on M with associated polar form $b(x,y) = q(x+y) - q(x) - q(y)$. We call the pair (M,q) a *quaternary quadratic module* and say that (M,q) is *regular* if the *adjoint homomorphism* $\varphi : M \to M^* = \mathrm{Hom}_R(M,R)$ defined by $\varphi(x)(y) = b(x,y)$ is injective. We call (M,q) *primitive* if $Rq(M) = R$. For $x_1, x_2, \ldots, x_n \in M$, we set $d(x_1, x_2, \ldots, x_n) = \det(b(x_i, x_j))$ and define the *discriminant ideal* $d(M)$ as the ideal of R generated by the elements $d(x_1, x_2, \ldots, x_n)$ for all choices of $x_i \in M$. The ideal $d(M)$ is locally principal and is invertible if (M,q) is regular.

Let (M_1, q_1), (M_2, q_2) and (M,q) be quaternary quadratic modules. A *composition*

$$\mu : M_1 \times M_2 \to M$$

is an R-bilinear map $\mu : M_1 \times M_2 \to M$ such that $q(\mu(x_1, x_2)) = q_1(x_1) q_2(x_2)$, $x_i \in M_i$. The composition ic called *proper* if q_1, q_2 are regular and $d(M) = d(M_1) = d(M_2)$.

An associative R-algebra A is called a *quaternion algebra* if
(1) A is a projective module of rank 4,
(2) A has an involution (i.e., an R-algebra antiautomorphism of order 2) $x \to \bar{x}$ such that the *trace* $t(x) = x + \bar{x}$ and the *norm* $n(x) = x\bar{x}$ take values in R.

For quaternion algebras A and B, we say that a quaternary quadratic module (M,q) is of *type* (A,B), A operating for on the left, B on the right, if
(1) M is projective of rank 1 as a left A-module and as a right B-module,
(2) $q(axb) = n(a)q(x)n(b)$ for $A \in A$, $b \in B$ and $x \in M$.

If M is a projective right B-module of rank 1 and $q(xb) = q(x)n(b)$ for $x \in M$ and $b \in B$, then M is of type (A,B) for $A \xrightarrow{\sim} \mathrm{End}_B(M)$. We call such a module of *quaternionic type*.

There is a natural way to compose modules of quaternionic type:

Proposition 1 *If (M_1, q_1) is of type (A, B) and (M_2, q_2) is of type (B, C), then there exists a unique quadratic form $q = q_1 \otimes q_2$ on $M = M_1 \otimes_B M_2$ such that the map*

$$\alpha : M_1 \times M_2 \to M, \ \alpha(x_1, x_2) = x_1 \otimes x_2$$

is a composition. Further α is proper if M_1 and M_2 are regular and primitive.

Example 2 Given a module M of type (A, B), we have obvious compositions $A \times M \to M$, $M \times B \to M$. Furthermore any module M of type (A, B) can be transformed into a module $\overline{M}$ of type (B, A) through the involution of quaternion algebras. Thus we also get compositions $M \times \overline{M} \to A$ and $\overline{M} \times M \to B$.

Conversely any composition occurs as in Proposition 1:

Theorem 3 *(Kneser et al., Theorem 2.10) Let $\mu : M_1 \times M_2 \to M$ be a proper composition of quaternary quadratic modules.*
1) There exists up to isomorphisms a unique quaternion algebra B operating on M_1 on the right and on M_2 on the left, making M_1, M_2 into modules of quaternionic type, such that $\mu(x_1 b, x_2) = \mu(x_1, b x_2)$.
2) There is an isomorphism $\nu : M_1 \otimes_B M_2 \xrightarrow{\sim} M$ such that $\nu(x_1 \otimes x_2) = \mu(x_1, x_2)$.
3) If M_1 is of type (A, B) and M_2 is of type (B, C), then M is of type (A, C).

The algebra B in Theorem 3 can be explicitly described as the set of pairs (s_1, s_2) of similitudes of M_1 and M_2 respectively, satisfying

$$\mu(s_1 x_1, x_2) = \mu(x_1, s_2 x_2)$$

for all $x_i \in M_i$.

3.3 The Clifford algebra

Next we give a necessary and sufficient condition for a quaternary quadratic module (M, q) to admit a composition in terms of its Clifford algebra. Let

$$C(M) = C_0(M) \oplus C_1(M)$$

be the Clifford algebra of (M, q). We recall that $C(M, q) = TM/I$, where TM is the tensor algebra of M and I is the ideal of TM generated by all elements $(x \otimes x - q(x)), x \in M$. There is a canonical R-module map $j_M : M \to C(M)$ and the pair $(C(M), j_M)$ is universal among pairs $(C, j : M \to C)$, C an associative R-algebra, with respect to the property $j(x)^2 = q(x)$ in C for all $x \in M$. Let M be a quadratic module of type (A, B) and let $\overline{M}$ be as in Example 2. Using the compositions defined in Example 2 we have a multiplication on the set of matrices

$$\begin{pmatrix} A & M \\ \overline{M} & B \end{pmatrix}$$

which is an associative algebra. The map

$$i_M : M \to \begin{pmatrix} A & M \\ \overline{M} & B \end{pmatrix}, \quad x \mapsto \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix}$$

satisfies $[i_M(x)]^2 = q(x)$ and hence extends to a homomorphism

$$i_M : C(M) \to \begin{pmatrix} A & M \\ \overline{M} & B \end{pmatrix}.$$

Remark 4 It follows from the fact that (M, q) is regular that i_M is injective and that i_M is an isomorphism over K. In particular we get $C_0(M) \otimes K = A \otimes K \times B \otimes K$ and the discriminant $d = d(M)$ of M is a square in K.

Theorem 5 *(Kneser et al., Theorem 4.1) Let (M, q) be a regular primitive quaternary quadratic module over a domain R and let K be the quotient field of R. Then M is of quaternionic type if and only if there exists an idempotent e in the center Z of $C_0(M) \otimes K$ which generates Z and such that $Me = C_1(M)e$. if M is of type (A, B), then the idempotent e can be chosen such that $B \xrightarrow{\sim} C_0(M)e$ and $A \xrightarrow{\sim} C_0(M)(1 - e)$.*

Since we assume that K has characteristic different from 2 the existence of the idempotent e generating the center of Z is equivalent with the fact that the discriminant d of M is a square in K. However the condition $Me = C_1(M)e$ (which then is satisfied over K) does not hold in general over R. In some cases this condition, which is not easy to check, can be replaced by another condition, which is more intrinsically related to the given quadratic form, and which generalizes the condition considered by Brandt.

3.4 K-forms

Let (M, q) be a regular quadratic module over R. The adjoint $\varphi : M \to M^*$ is an isomorphism over K and we have a quadratic form

$$q^{-1} : M^* \otimes K \to K$$

defined by $q^{-1}(f) = (q \otimes 1_K)((\varphi \otimes 1_K)^{-1}(f)), f \in M^* \otimes K$. The *adjoint form* form dq^{-1}, where $d = d(M)$ is the discriminant, is defined over R. Following Brandt ([7, 1924]) we say that (M, q) is a *K-form* if its discriminant d is a square in K, $d = D^2$, $D \in K$, and if Dq^{-1} defines a quadratic form on M^* over R, i.e., if the adjoint form dq^{-1} is divisible by D. One of the main results of [7, 1924] is that an integral primitive quaternary form admits of a composition if and only if it is a K-form. Over arbitrary domains we have:

Theorem 6 *(Kneser et al., Proposition 5.1 and Corollary 5.7) Let R be a domain with a quotient field K of characteristic not equal to 2.*
1) A primitive regular quaternary quadratic module (M, q) which admits a composition is a K-form.
2) Conversely, a primitive regular quaternary quadratic module which is a K-form admits a composition if R is integrally closed in $R[1/2]$.

Example 7 Assume that there exists $c \in K$, $c \notin R$, such that $2c \in R$ and $c^2 = \alpha c + \beta$, α, $\beta \in R$ (for example $R = \mathbb{Z}[\sqrt{5}]$, $c = (1+\sqrt{5})/2$). Let (M, q) be the quadratic R-submodule of $(M_2(K), \det)$ with basis

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; e_2 = \begin{pmatrix} c & 2c^2 \\ 0 & \bar{c} \end{pmatrix}, \; e_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; e_4 = \begin{pmatrix} 2c^2 & 0 \\ 0 & 0 \end{pmatrix},$$

where $\bar{c} = a - c$. Then q is a K-form but (M, q) does not admit a composition.

Example 7 shows that Theorem 6 does not necessarily hold if R is not integrally closed in $R[1/2]$. Exact conditions are not known.

3.5 Groupoids: some historical comments

From the point of view of category theory a groupoid is simply a small category in which every morphism is invertible. Thus as an algebraic structure a groupoid is a set G with a partially defined binary operation $(f, g \mapsto fg$. That is fg is an element of G or may be undefined. There is also a everywhere defined function $f \mapsto f^{-1}$. The two operations satisfy the following rules:
1) Whenever fg and gh are defined, then $(fg)h$ and $f(gh)$ are also defined and are equal;
2) $f^{-1}f$ and ff^{-1} are always defined;
3) When ever fg is defined, then $fgg^{-1} = f$ and $f^{-1}fg = g$.

Examples 8 1) Given a topological space X, the *fundamental groupoid* of X consists of equivalence classes of homotopic paths.
2) If X is a set and $\sim$ is an equivalence relation on X, there is a groupoid representing the equivalence relations as follows (using the language of category theory): The objects are the elements of X and for any two elements x, y of X there is a morphism from x to y if and only if $x \sim y$.
3) The *group action* of a group G on a set X can be represented by a groupoid: the objects are the elements of X and for any two elements x and y in X there is a morphism from x to y for every element g of G such that $gx = y$. Thus a groupoid can be viewed as a common generalization of the concepts of space and group.
4) One can also associate different groupoids to the composition of quaternary forms. We use the notation of Theorem 3. For any quaternion algebra A over a commutative ring R one can define in a natural manner a groupoid $G(A)$: let $\mathcal{B}(A)$ be the class of quaternion algebras B such that there exists a quaternary quadratic module of type (A, B). For each isomorphism class of such algebras we

pick a representative A_i. Let G_{ij} be the set of equivalence classes of quadratic modules of type (A_i, A_j), equivalence being isometry preserving the bimodule structure. The set $G(A) = \cup_{i,j} G(A_i, A_j)$ is a groupoid with tensor product as multiplication. Brandt' groupoids are different specialisations of $G(A)$. We refer to Kneser *et al.*, Section 6, for more details.

Brandt's work on composition of quaternary forms was a starting point for his contributions to the arithmetic of non-commutative algebras. Brandt used the notion of a groupoid to extend the ideal class group in rings of algebraic integers to the non-commutative case. His theory was later generalised to orders in non-commutative rings (see Section 2.1). In contrast to the theory of ideals, there were few attempts to generalize Brandt's composition of quaternary forms. Brandt himself never returned to such a study. Kaplansky (1968) gave a generalization in the set-up of Bezout, resp. Prüfer rings (i.e., domains such that finitely generated ideals are principal, resp. invertible)[23]. As far as we know, the only presentation valid for arbitrary commutative rings is the one given by Kneser *et al.*.
It is believed that Brandt's axioms influenced Eilenberg and Maclane in their definition[24] of a category, however they did not mention groupoids as an example. The fundamental groupoid of a topological space appeared in Reidemeister's book on topology[25]. The spread of groupoids in many areas of topology however really started in the fifties with the work of Ehresmann[26]. Applications of the fundamental groupoid to algebraic topology can be found in the book[27] of Higgins (1971), which is also a good general introduction to groupoids. Groupoids were used by Grothendieck[28] in algebraic geometry in relation with the construction of moduli spaces ("the idea of making systematic use of groupoids, however evident it may look today, is to be seen as a significant conceptual advance, which has spread in many areas of mathematics. In my own work in algebraic geometry, I have made extensive use of groupoids" . In analysis Mackey used groupoids under the name of *virtual groups* to treat ergodic actions on groups. They also have a significant place in Connes' book[29] on Noncommutative Geometry (1994).

Even if the concept of a groupoid can be considered as very natural, it is far from being universally accepted within the mathematical community. Connes, in his book mentioned above makes the observation p. 6 that "it is fashionable among mathematicians to despise groupoids and to consider that only groups have authentic mathematical status, probably because of of the pejorative suffix oid". Brown (1987, 1999) and Weinstein (1996) published nice expository arti-

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cles to promote groupoids[30]. Many informations can also be found in an article[31] of Corfield (2001). The paper gives an interesting, historically-minded, practice-oriented philosophical analysis of the mathematical importance and acceptance of mathematical concepts via the case study analysis of groupoids. There is a *Groupoid Homepage* on the web (http://pompeiu.imar.ro/ ramazan/groupoid/) and a list of papers on groupoids (1900-present) can be found at the webmail address http://faculty.colostate-pueblo.edu/karla.oty/groupoids/list.htm).

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[A 4] Aufg. **328** (**54**(1951), S. *1*; Brandt) Lösungen: **55**(1952), *42*: Brandt; *42*: A. Stöhr, *43*: H, Salié. (Weitere Lösungen eingesandt von: W. Maier, H. Lenz)

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