QUADRATIC FORMS WITH VALUES IN LINE BUNDLES

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1. Introduction. Let X be a scheme such that $\frac{1}{2} \in \mathcal{O}_X$ and let $\mathcal{I}$ be a line bundle over X. A quadratic space over X with values in $\mathcal{I}$ is a triple $(\mathcal{F}, h, \mathcal{I})$, where $\mathcal{F}$ is a bundle over X and h is a selfdual isomorphism

$$h : \mathcal{F} \to \mathcal{F}^* \otimes_{\mathcal{O}_X} \mathcal{I},$$

$\mathcal{F}^* = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$ being the dual of $\mathcal{F}$. Such quadratic spaces appear naturally in connection with forms over smooth projective curves (see [GHKS]), in the theory of Azumaya algebras with involutions (see [KPSr], [PSr], [PS]), and also in the theory of composition of quadratic forms (see [B], [K] and [KOS]). They were already considered by T. Kanzaki ([Ka]) and, recently, by M. Rost ([R]). Some classical constructions for quadratic forms carry over to quadratic forms with values in line bundles. For example, it is possible to define the even Clifford algebra C_0. The odd part C_1 can be defined as a bimodule over C_0, but $C = C_0 \oplus C_1$ does not have a natural algebra structure. However there exists a map $C_1 \otimes C_1 \to C_0 \otimes \mathcal{I}$ such that $\widetilde{C}(\mathcal{F}, h, \mathcal{I}) = (C_0 \oplus C_1) \otimes L[\mathcal{I}]$, where $L[\mathcal{I}] = \oplus_{n \in \mathbb{Z}} \mathcal{I}^n$ and $\mathcal{I}^n = \mathcal{I} \otimes \cdots \otimes \mathcal{I}$ (n times), has a natural structure of a Z-graded algebra. In fact, as observed by F. van Oystaeyen, [vO], the $\mathcal{O}_X$-algebra $L[\mathcal{I}]$ is a natural splitting of $\mathcal{I}$ and $\widetilde{C}(\mathcal{F}, h, \mathcal{I})$ is the "classical" Clifford algebra of the quadratic space $(\mathcal{F} \otimes_{\mathcal{O}_X} L[\mathcal{I}], h \otimes 1)$.

For the sake of simplicity, we shall restrict in this paper to quadratic spaces over affine schemes $X = \mathrm{Spec}(R)$. But most of the constructions are functorial and can be easily globalized for bundles over schemes. On the other hand we shall not assume that $\frac{1}{2} \in R$. Unadorned tensor products are taken over R.

In Section 2, we recall some definitions and give as examples hyperbolic spaces, binary norms, quaternary norms and pfaffians. The Clifford algebra is introduced in Section 3 and some examples are computed in Section 4. Then we apply Clifford algebras to describe quadratic spaces of rank 4 and 6 with trivial Arf invariant. In Section 6, groups of similitudes for spaces of low rank are computed. With the Clifford algebra as a tool, most results of Sections 5 and 6 are straightforward generalizations of results of [KP], [KPS$_1$] and [KPS$_2$] and we shall in some cases only sketch the proofs. We conclude in the last section with some remarks on composition of quaternary spaces. In all this paper we assume that the quadratic form is nonsingular, i.e. that the polar $h : V \to V^* \otimes I$ is an isomorphism. Nondegenerate quadratic forms, i.e. such that h is only assumed to be injective, will be treated elsewhere.

Parts of this paper are inspired by the Ph.D. Thesis, [B], of the first author. In [B] the even algebra C_0 and the bimodule C_1 are constructed by descent, i.e. by splitting locally the line bundle $\mathcal{I}$ and patching. The main aim of [B] was the classification of quaternary spaces in relation with composition (see Section 7). Another application of the even Clifford algebra, to Azumaya algebras of rank 16 with involution, is in [PS], in the same volume.

We thank F. van Oystaeyen for his nice observation mentioned above. This observation lead us, in particular, to a much simpler proof of the structure theorem (3.7). Warm thanks are also due to R. Parimala, M. Ojanguren and M. Rost for remarks and discussions.

2. Quadratic modules. We begin with some definitions. Let V be a finitely generated projective R–module and let I be an invertible R-module, i.e. a line bundle over $X = \mathrm{Spec}(R)$. A map $q : V \to I$ such that

(1) $q(\lambda x) = \lambda^2 q(x),\ \lambda \in R,\ x \in V$

(2) $b_q(x, y) = q(x + y) - q(x) - q(y)$ is R–bilinear

is a *quadratic map* from V to I. We call the triple (V, q, I) a *quadratic module (with values in I)*. Let $V^* = \mathrm{Hom}_R(V, R)$ be the R-dual of V. Let

$$h : V \to \mathrm{Hom}_R(V, I) \simeq V^* \otimes I$$

by the *polar of* b_q, i.e. $h(x)(y) = b_q(x, y)$. We say that q is *nonsingular* and that (V, q, I) is a *quadratic space (with values in I)* if h is an isomorphism. We observe that the notion of a quadratic form with values in an R-module I of arbitrary rank makes sense, but that, to define nonsingularity, we need I to be invertible. For any submodule U of V, we put $U^\perp = \{x \in V \mid b_q(x, y) = 0, \forall y \in U\}$. By the rank of (V, q, I) we mean the rank of V as an R-module. Let (V, q, I) be a quadratic module with values in a free invertible R-module. Any choice of a basis element for I yields a quadratic module with values in R and different choices give forms which are proportional. Thus similitudes turn out to be the morphisms of quadratic modules with values in an invertible module. Two quadratic modules (V, q, I) and (V', q', I') are *similar* if there exist isomorphisms

$$\theta : V \xrightarrow{\sim} V' \ , \ \eta : I \xrightarrow{\sim} I'$$

such that $q'(\theta(x)) = \eta(q(x))$ for all $x \in V$ and the pair (θ, η) is a *similitude*. If $I = I', \eta$ is given by the multiplication by a unit of R, called the *multiplier* of the similitude.

We now give examples of quadratic spaces with values in an invertible module.

(2.1) Hyperbolic spaces. Let P be a finitely generated projective R-module, let I be invertible and let $V = P \oplus P^* \otimes I$. We define

$$q : V = P \oplus P^* \otimes I \to I$$

by $q(x+f\otimes\xi) = f(x)\xi$ for $x \in P$, $f \in P^*$ and $\xi \in I$. The quadratic module (V, q, I) is obviously nonsingular and P is a direct summand of V such that $P^\perp = P$. Conversely, as in the "classical case", if, for a quadratic space (V, q, I), V has a direct summand P such that $P^\perp = P$, then $(V, q, I) \simeq P \oplus P^* \otimes I$. We call such a space *hyperbolic*.

(2.2) Norm forms. An R-algebra S is *quadratic* if it is projective of rank 2 as an R-module. A quadratic R-algebra has a unique R-linear involution $x \mapsto \bar{x}$ (i.e. an antiautomorphism of order ≤ 2) such that $x + \bar{x} \in R$ and $x\bar{x} \in R$ for all $x \in S$. Let next A be an R-algebra with A projective of rank 4 over R. We say that A is a *quaternion algebra* if A has an R-linear involution $x \mapsto \bar{x}$ such that $x + \bar{x} \in R$ and $x\bar{x} \in R$ for all $x \in A$. Such an involution then is unique. The quadratic map $n : x \mapsto x\bar{x}$ is the *norm* , resp. the *reduced norm* of the algebra. The algebra S, resp. A is *split* if $S \simeq R \times R$, resp. $A \simeq \mathrm{End}_R(P)$ for P projective of rank 2. We then have $n\left((x,y)\right) = xy$ for $(x,y) \in R \times R$, resp. $n(f) = \det(f)$ for $f \in \mathrm{End}_R(P)$. We call S, resp. A, *separable* if the norm n is nonsingular. Equivalently S, resp. A, is separable if and only if it is locally split for the etale topology. A separable quadratic algebra is a Galois algebra with group Z/2Z and a separable quaternion algebra is an Azumaya algebra of rank 4. In the following, we shall assume that A is either quadratic or a quaternion algebra. Let M be a right A-module and N an R-module. A map $q : M \to N$ is a *norm form* on M if

(1) $q(xa) = q(x)n(a),\ x \in M, a \in A,$
(2) $b_q : M \times M \to N$ defined by $b_q(x,y) = q(x+y) - q(x) - q(y)$ is R-bilinear.

One can construct a universal norm form on M:

Proposition 2.3. *Let M be a right A-module. There exists an R-module $J(M)$ and a norm form $j : M \to J(M)$ such that, for any norm form $q : M \to N$, there is a unique homomorphism of R-modules $\phi : J(M) \to N$ such that $\phi \circ j = q$.*

Proof. We construct $J(M)$ by generators and relations. Let Q be the submodule of the module $R^M \oplus M \otimes M$ generated by all elements

$$(e_{xa} - n(a)e_x, 0) \ \text{ and } (e_{x+y} - e_x - e_y, -x \otimes y),\ x, y \in M,\ a \in A.$$

We define
$$J(M) = (R^M \oplus M \otimes M)/Q$$
and the map j is induced by the embedding $M \to R^M, x \mapsto e_x$, composed with the canonical projection. (compare with Section 1.2 of [MR]).

We call the pair $(J(M), j)$ the *universal norm form* of M. By construction $(J(M), j)$ with the properties (1), (2) of (2.2) is unique (up to unique isomorphisms) and J commutes with scalar extensions. If $M = A$, then $J(A) \simeq R$ and j is the norm n, by uniqueness. Let P be a projective A-module of rank 1, i.e. $P_p \simeq A_p$ for all $p \in \mathrm{Spec}(R)$. Since J commutes with localization, $J(P)$ is an invertible R-module and $j : P \to J(P)$ is a quadratic map. Thus $(P, j, J(P))$ is a quadratic module with values in $J(P)$. Further j is nonsingular if n is so. We describe J in the split cases: if $A \simeq R \times R$, then $P \simeq P_1 \times P_2$ for P_1, P_2 invertible R-modules and $J(P) \simeq P_1 \otimes P_2$. If $A \simeq \mathrm{End}_R(P)$, then, by Morita theory, $P \simeq P_0 \otimes P^*$ with P_0 projective of rank 2 over R and $J(P) \simeq \wedge^2 P_0 \otimes \wedge^2 P^*$.

There is a cohomological description of the functor J. Let G_m be the functor "units" and $\mathrm{G}_m(A)$ the functor "units of A" i.e. $\mathrm{G}_m(A)(S) = \mathrm{G}_m(A \otimes S)$ for any commutative R-algebra S. The functors G_m and $\mathrm{G}_m(A)$ define sheaves for the

Zariski topology and the reduced norm n induces a morphism of sheaves $\mathrm{G}_m(A) \to \mathrm{G}_m$. Thus there is a map in cohomology

$$N : H^1_{\mathrm{Zar}}(X, \mathrm{G}_m(A)) \to H^1_{\mathrm{Zar}}(X, \mathrm{G}_m)$$

for $X = \mathrm{Spec}(R)$. The pointed set $H^1_{\mathrm{Zar}}(X, \mathrm{G}_m(A))$ classifies projective A-modules of rank one, $H^1_{\mathrm{Zar}}(X, \mathrm{G}_m)$ classifies invertible R-modules and the map N corresponds to the functor J. In the quaternary case, $(P, j, J(P))$ is called the *reduced norm* of P.

(2.4) Pfaffians. We refer to [KPS$_1$] or [Kn] for details. Let A be an Azumaya R-algebra of rank $4m^2$ whose class in the Brauer group is of order 2 and let $\varphi : A \otimes A \to \mathrm{End}_R(M)$, M projective of rank $4m^2$, be a fixed isomorphism. The switch $A \otimes A \to A \otimes A$ induces an automorphism τ of M of order 2 and the corresponding module of alternating elements $\mathrm{Alt}(M) = \{x - \tau(x), x \in M\}$ is projective of rank $m(2m-1)$. Let n be the reduced norm of A. There exists a universal homogeneous map of degree m

$$\mathrm{pf} : \mathrm{Alt}\ (M) \to \mathrm{Pf}(M)$$

such that $\mathrm{pf}(\varphi(a \otimes a)(x)) = n(a)pf(x)$ for $x \in \mathrm{Alt}(M)$, $a \in A$, and $\mathrm{Pf}(M)$ is invertible. In the split case $A \simeq \mathrm{End}_R(V)$ and $M \simeq V \otimes V$, we have $\mathrm{Alt}(M) \simeq \wedge^2 V$ and pf is the pfaffian $\wedge^2 V \to \wedge^{2m} V$. If $m = 2$ the map pf is quadratic and $(\mathrm{Alt}(M), \mathrm{pf}, \mathrm{Pf}(M))$ is a nonsingular quadratic space.

(2.5) Involutions. Let (V, q, I) be a quadratic space over R. The adjoint $h : V \to V^* \otimes I$ induces an R-linear involution σ of the algebra $\mathrm{End}_R(V)$, $\sigma(f) = h^{-1} f^* h$, for $f \in \mathrm{End}_R(V)$. Conversely, any R-linear involution of $\mathrm{End}_R(V)$, V a finitely generated projective R-module, is induced by an isomorphism $h : V \to V \otimes I$, I an invertible R-module, and $h = \varepsilon h^*$, $\varepsilon \in \mu_2(R) = \{x \in R \,|\, x^2 = 1\}$. This follows by Morita theory, observing that transpose identifies $\mathrm{End}(V)^{\mathrm{op}}$ with $\mathrm{End}(V^*)$. If $\varepsilon = 1$ and 2 is invertible in R, h is the adjoint of a nonsingular quadratic form with values in I.

Remark. Another descriptions of the reduced norm and of the pfaffian for modules can be found in [KOS] (for the reduced norm), [KPS$_1$] (for the pfaffian), [Kn] and in [PS] (this volume).

3. Clifford algebras. Let I be an invertible R-module and let $I^n = I \otimes \cdots \otimes I$, ($n$-times), $n = 1, 2, \ldots$, $I^{-1} = I^* = \mathrm{Hom}_R(I, R)$, $I^0 = R$ and $I^{-n} = (I^*)^n$. The tensor product and the canonical isomorphism

$$I \otimes I^* \xrightarrow{\sim} R,\ x \otimes f \mapsto f(x),\ x \in I,\ f \in I^*$$

define an R-algebra structure on $L[I] = \oplus_{n \in \mathbb{Z}} I^n$. We call $L[I]$ the *Laurent algebra* (in the literature $L[I]$ also appears as the *Rees algebra*) of I. Let $q : V \to I$ be a quadratic map. Let TV be the tensor algebra of V and $J(q)$ be the ideal of $TV \otimes L[I]$ generated by all elements $v \otimes v \otimes 1 - 1 \otimes q(v)$, $v \in V$. Defining a Z-grading on $TV \otimes L[I]$ by $\partial(v \otimes 1) = 1$ for $v \in V$ and $\partial(1 \otimes x) = 2$ for $x \in I$, we get that $J(q)$ is Z-graded. Thus the algebra

$$\widetilde{C} = \widetilde{C}(V, q, I) = TV \otimes L[I]/J(q)$$

is Z-graded. We call $\widetilde{C}$ the *Clifford algebra* of (V, q, I). Let C_n be the submodule of $\widetilde{C}$ of elements of degree n.

Lemma 3.1. *C_0 is a subalgebra of $\widetilde{C}$, C_1 is a C_0-bimodule and the multiplication in $\widetilde{C}$ induces an isomorphism*

$$(C_0 \oplus C_1) \otimes L[I] \xrightarrow{\sim} \widetilde{C}(V, q, I).$$

Proof. The first two claims are obvious. Since $I \otimes I^{-1} \xrightarrow{\sim} R$, we have isomorphisms $C_n \simeq C_0 \otimes I^m$ for $n = 2m$ and $C_n \simeq C_1 \otimes I^m$ for $n = 2m+1$.

Remark. In fact, for any Z-graded $L[I]$-module $\widetilde{M}$, we have $M_n = M_0 \otimes I^m$ for $n = 2m$ and $M_n = M_1 \otimes I^m$ for $n = 2m+1$. This will be used later.

In the following, we shall identify C_n with $C_0 \otimes I^m$, resp. $C_1 \otimes I^m$. We call $C_0(q) = C_0$ the *even Clifford algebra of* q and $C_1(q) = C_1$ the *Clifford module of* q. As observed by F. van Oystaeyen, the algebra $\widetilde{C} = \widetilde{C}(V, q, I)$ have the following nice interpretation:

Lemma 3.2. *The multiplication of the algebra $L[I]$ induces an isomorphism*

$$I \otimes L[I] \xrightarrow{\sim} L[I],$$

so that $(V, q, I) \otimes L[I]$ is a quadratic space with values in $L[I]$ and

$$\widetilde{C}(V, q, I) \simeq C(V \otimes L[I], q \otimes 1).$$

The next two results are immediate consequence of (3.2).

Lemma 3.3. *Any morphism $(\theta, \eta) : (V, q, I) \to (V', q', I')$ induces a morphism of graded algebras*

$$\widetilde{C}(\theta, \eta) : \widetilde{C}(V, q, I) \to \widetilde{C}(V', q', I')$$

such that $\widetilde{C}(\theta, \eta)(i(v)) = i(\theta(v))$ for all $v \in V$.

Lemma 3.4. *For any commutative R-algebra S and any quadratic R-module (V, q, I), there exists a canonical isomorphism $\widetilde{C}(V, q, I) \otimes S \xrightarrow{\sim} \widetilde{C}((V, q, I) \otimes S)$.*

Let $i : V \to \widetilde{C}(V, q, I)$ be the map induced by the canonical map $V \to TV$. We have

$$i(x)^2 = 1 \otimes q(x) \in C_0 \otimes I = C_2 \quad \text{for all} \quad x \in V$$

and $\widetilde{C}$ is universal with respect to the following property. Let D_0 be an R-algebra, D_1 a D_0-bimodule such that $rd = d\,r$ for all $r \in R$ and $d \in D_1$, and let I be an invertible R-module. Assume that there exists an R-linear map

$$\mu : D_1 \otimes D_1 \to D_0 \otimes I$$

such that

$$\widetilde{D} = (D_0 \oplus D_1) \otimes L[I]$$

is, in a natural way, a Z-graded R-algebra, the gradation being defined as for $\widetilde{C}$. Then, for any quadratic module (V, q, I) and any R-linear map $\psi : V \to D_1$ such that

$$\mu(\psi(v) \otimes \psi(v)) = 1 \otimes q(v) \quad \text{for all} \quad v \in V,$$

there exists a unique homomorphism of graded R-algebras

$$\widetilde{\psi} : \widetilde{C}(V, q, I) \to \widetilde{D}$$

such that $\widetilde{\psi}|_V = \psi$ and $\widetilde{\psi}(i(v)^2) = 1 \otimes q(v)$. By the universal property of the Clifford algebra, the automorphism $x \mapsto -x$ of V extends to an involution σ of $\widetilde{C}(V, q, I)$. We call σ the *standard involution* of $\widetilde{C}(V, q, I)$. In particular, σ restricts to an involution of $C_0(q)$.

Proposition 3.5. *If V is finitely generated free with basis $\{e_1, \dots e_n\}$ and I is free with basis element t, then C_0 is a free R-module with basis*

$$\{1, e_{i_1} e_{i_2} \dots e_{i_{2m}} \otimes t^{-m}, 1 \le i_1 < i_2 < \dots < i_{2m} \le n\}$$

and C_1 is free with basis

$$\{e_{i_1} e_{i_2} \dots e_{i_{2m+1}} \otimes t^{-m}, 1 \le i_1 < i_2 < \dots < i_{2m+1} \le n\}.$$

Proof. A variation of the Poincaré-Birkhoff-Witt theorem!

Corollary 3.6. *For any quadratic module (V, q, I), the canonical map $i : V \to \widetilde{C}(V, q, I)$ and the map $\lambda : L[I] \to \widetilde{C}(V, q, I)$ induced by $L[I] \to TV \otimes L[I]$ are injective.*

Proof. By (3.4) and (3.5), since a finitely generated projective module is locally free.

Remark. Observe that for a quadratic map $q : V \to I$, with V not necessarily projective, the maps $i : V \to C_1 \subset \widetilde{C}$ and $\lambda : L[I] \to \widetilde{C}$ need not to be injective.

The centre $\widetilde{Z}$ of $\widetilde{C}$ is Z-graded and as in (3.1) we get

$$\widetilde{Z} = \oplus_{n \in \mathbb{Z}} Z_n = (Z_0 \oplus Z_1) \otimes L[I] = \widetilde{Z}_0 \oplus \widetilde{Z}_1$$

with $\widetilde{Z}_0 = Z_0 \otimes L[I]$ and $\widetilde{Z}_1 = Z_1 \otimes L[I]$. We now describe the structure of $\widetilde{C}$ and of C_0 in the nonsingular case. Observe that, in the odd rank case, nonsingularity implies that 2 is invertible. We use the notations $\widetilde{C}_0 = C_0 \otimes L[I] = C_0(V \otimes L[I], q \otimes 1)$ and $\widetilde{C}_1 = C_1 \otimes L[I] = C_1(V \otimes L[I], q \otimes 1)$.

Theorem 3.7. *Assume that (V, q, I) is nonsingular. Then*

(1) *If the rank of V is odd, then $Z_0 \simeq R$, Z_1 is an invertible module such that $Z_1^2 \xrightarrow{\sim} I$, C_0 is an Azumaya algebra over R and $\widetilde{C} \simeq C_0 \otimes L[Z_1]$.*
(2) *If the rank of V is even, then $Z_0 \simeq R$, $Z_1 = 0$ and $\widetilde{C}$ is an Azumaya algebra over $L[I]$. Furthermore the centre $Z(C_0)$ of C_0 is a separable quadratic algebra over R and C_0 is an Azumaya algebra over $Z(C_0)$.*

Proof. 1) By the classical case applied to $(V \otimes L[I], q \otimes 1)$ we get that (see for example [Kn]) $\widetilde{C} \simeq \widetilde{C}_0 \otimes \widetilde{Z}$, $\widetilde{C}_0$ is an Azumaya algebra over $L[I]$, $\widetilde{Z}_0 = L[I]$ and the multiplication in $C_0(q \otimes 1)$ induces an isomorphism $\widetilde{Z}_1 \otimes \widetilde{Z}_1 \simeq L[I]$. The claim then follows from $C_0(q \otimes 1) = C_0 \otimes L[I]$ and $\widetilde{Z}_1 = Z_1 \otimes L[I]$.

2) follows also from the classical case applied to $C(q \otimes 1)$. In particular $\widetilde{C}$ is an Azumaya algebra over $L[I]$ and the centre $Z(\widetilde{C}_0)$ of $\widetilde{C}_0$ is a separable quadratic algebra over $L[I]$. Let $Z(\widetilde{C}_0) = Z \otimes L[I]$. Since $L[I]$ is faithfully flat over R, Z is a separable quadratic R-algebra and $Z \subset Z(C_0)$. For the same reason, and since $\widetilde{C}_0 = C_0 \otimes L[I]$, C_0 is a separable R-algebra, i.e. is a projective $C_0 \otimes C_0^{\mathrm{op}}$-module. It follows that

$$\mathrm{End}_{C_0 \otimes C_0^{\mathrm{op}}}(C_0, C_0) \otimes L[I] \simeq \mathrm{End}_{\widetilde{C}_0 \otimes_{L[I]} \widetilde{C}_0^{\mathrm{op}}}(\widetilde{C}_0, \widetilde{C}_0).$$

Thus

$$Z(C_0) \otimes L[I] = \mathrm{End}_{C_0 \otimes C_0^{\mathrm{op}}}(C_0, C_0) \otimes L[I] \simeq Z(\widetilde{C}_0)$$

and, by faithfully flat descent, $Z = Z(C_0)$, so that, as claimed, $Z(C_0)$ is a separable quadratic R-algebra.

Remark. As already observed in the introduction, the even Clifford algebra C_0 and the Clifford module C_1 can also be constructed by trivializing locally the line bundle I and by patching the corresponding C_0 and C_1. This was done in [B] and (for C_0) in [PS].

4. Examples. In this section we compute the Clifford algebras of the spaces described in Section 2.

(4.1) Hyperbolic spaces. Let $V = P \oplus P^* \otimes I$ be hyperbolic and let $\wedge P$ be the exterior algebra of P. Let $\lambda_x : \wedge P \otimes L[I] \to \wedge P \otimes L[I]$ be the left exterior multiplication with $x, x \in P$ and let $d_f : \wedge P \otimes L[I] \to \wedge P \otimes L[I], f \in P^* \otimes I = \mathrm{Hom}_R(P, I)$ be the derivation which extends the linear form $f : P \to I$. The map $(x, f) \mapsto \lambda_x + d_f \in \mathrm{End}_R(\wedge P) \otimes L[I]$ induces, as in the classical case (see [Kn] or [Ba]), an isomorphism

$$\widetilde{C}(V, q, I) \xrightarrow{\sim} \mathrm{End}_R(\wedge P) \otimes L[I].$$

(4.2) Binary norms. Let S be a separable quadratic R-algebra with norm n and let M be a projective right S-module of rank 1. Let $j : M \to J(M)$ be the universal norm on M. We have $C_0(j) = S$, $C_1(j) = M$ and $\mu : C_1 \otimes C_1 \to C_0 \otimes J(M)$ is locally given by $\mu(ex \otimes ey) = j(e)\bar{x}y$, where $x \mapsto \bar{x}$ is the involution of S and e is (locally) a basis of M as an S-module. Conversely, any quadratic space (V, q, I) of rank 2 is similar to the norm form $(M, j, J(M))$ for some rank one module over a separable quadratic R-algebra S. We take $M = C_1(q) = V$ and $S = C_0(q)$.

(4.3) Quaternary norms. Let A be a separable quaternion algebra and let P be a projective right A-module of rank one. To compute the Clifford algebra of the reduced norm of P, we shall use another description of the space $(P, j, J(P))$. Let $P^{(*)} = \mathrm{Hom}_A(P, A)$ be the A-dual of P. The algebra A has an involution

$\sigma_A : a \mapsto \bar{a}$ given by $\bar{a} = \mathrm{tr}(a) - a$, where tr is the reduced trace of A. We view $P^{(*)}$, which has a natural structure of left A-module, as a right A-module through σ_A. Similarly $B = \mathrm{End}_A(P)$, which is also an Azumaya R-algebra of rank 4, has an involution σ_B and we view $P^{(*)}$ as a left B-module through σ_B. Thus P and $P^{(*)}$ are A-B-bimodules. The R-module

$$I(P) = \mathrm{Hom}_{A\text{-}B}(P^{(*)}, P)$$

is invertible and the evaluation induces an isomorphism of A-B-bimodules

$$\theta : P^{(*)} \otimes I(P) \xrightarrow{\sim} P.$$

Let $\pi = \theta^{-1}$. We define a quadratic map $q : P \to A \otimes I(P)$ by $q(x) = \pi(x)(x)$. The map q is such that $q(xa) = q(x)n(a),\ x \in P,\ a \in A$. We claim that q has values in $I(P) = R \otimes I(P)$. Localizing, we may assume that P is free over A with generator e, $P = eA$. Then $P^* = Af$, with $f(e) = 1$, and $I(P)$ is free with generator $u, u(f) = e$. The map θ is given by $\theta(f \otimes u) = e$ and $q(ea) = \bar{a}a \otimes u = 1 \otimes n(a)u$. Similarly, we have a quadratic map

$$q' : P \to B \otimes I(P)$$

given by

$$q'(x) = x\pi(x).$$

The map q' also has values in I and the above computation for P free shows that $q'(x) = q(x), x \in P$. By the universal property of $J(P)$, there exists a linear map $\gamma : J(P) \to I(P)$ such that $q = \gamma \circ j$. Localizing, we see that $(1, \gamma) : (P, j, J(P)) \to (P, q, I(P))$ is an isomorphism. Similarly, there is an isomorphism

$$(1, \gamma') : (P, j, J(P)) \xrightarrow{\sim} (P, q', I(P)),$$

where the quadratic space on the right is obtained by considering P as a B-module. We shall use this description of the quaternary norm to compute its even Clifford algebra and its Clifford module. Let D_0 be the algebra $A \times B$, with $B = \mathrm{End}_A(P)$ as above, let D_1 be the D_0-bimodule $P \oplus P^{(*)} \otimes I(P)$ with the operation

$$(a, b)(x, f \otimes \xi)(a', b') = (bxa', afb' \otimes \xi) \tag{4.4}$$

for $a, a' \in A$, $b, b' \in B$, $x \in P, f \in P^{(*)}$ and $\xi \in I(P)$. It is convenient to use matrix notations:

$$D_0 = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad D_1 = \begin{pmatrix} 0 & P^{(*)} \otimes I(P) \\ P & 0 \end{pmatrix}.$$

The map

$$\mu : D_1 \otimes D_1 \to D_0 \otimes I(P)$$

given by

$$((x, f \otimes \xi), (u, g \otimes \eta)) \mapsto (f(u), 0) \otimes \xi + (0, xg) \otimes \eta$$

for $u, x \in P$, $f, g \in P^*$ and $\xi, \eta \in I(P)$, induces naturally a graded algebra structure on $\widetilde{D} = (D_0 \oplus D_1) \otimes L[I(P)]$. The map $\iota : P \to D_1$, $x \mapsto (x, \pi(x))$ is such that

$$\mu(\iota(x) \otimes \iota(x)) = (\pi(x)(x), x\pi(x)) = (1,1)q(x).$$

By the universal property of the Clifford algebra of $\widetilde{C}(P, q, I(P))$, there exists a homomorphism

$$\tilde{\iota} : \widetilde{C}(P, q, I(P)) \to \widetilde{D}$$

such that $\tilde{\iota}|_P = \iota$. It follows from (3.7) that $\tilde{\iota}$ is an isomorphism. Thus, identifying $(P, j, J(P))$ with $(P, q, I(P))$ as above, we get

Proposition 4.5. *Let A be an Azumaya R-algebra of rank 4 and let P be a projective A-module of rank one. The even Clifford algebra and the Clifford module of the reduced norm $(P, j, J(P))$ are given by*

$$C_0 \simeq A \times \mathrm{End}_A(P) \quad \textit{and} \quad C_1 \simeq P \oplus P^{(*)} \otimes I(P)$$

with the bimodule operation as in (4.4). In particular, the centre of C_0 is a split quadratic algebra.

We shall identify $\widetilde{C}$ with $\widetilde{D}$ and use the matrix notation for $D_0 \oplus D_1$. The standard involution of $\widetilde{C}$ then is induced by

$$\sigma : \begin{pmatrix} a & f \otimes \xi \\ p & b \end{pmatrix} \mapsto \begin{pmatrix} \sigma_A(a) & -\pi(p) \\ -\pi^{-1}(f \otimes \xi) & \sigma_B(b) \end{pmatrix} \tag{4.6}$$

(4.7) Pfaffians. Let A be an Azumaya algebra of rank 16 with a fixed isomorphism $\varphi : A \otimes A \xrightarrow{\sim} \mathrm{End}_R(M)$. We now compute the Clifford algebra of the quadratic space $(\mathrm{Alt}(M), \mathrm{pf}, \mathrm{Pf}(M))$ as defined in (2.4). We regard M as a left A-module (or as an right A^{op}-module) through the action $ax = \varphi(a \otimes 1)x$, $x \in M$, $a \in A$ and denote by $M^{(*)}$ the A-dual of M. Let

$$\varphi^* : A^{\mathrm{op}} \otimes A^{\mathrm{op}} \xrightarrow{\sim} \mathrm{End}_R(M^{(*)})$$

be given by $\varphi^*(a \otimes a')(f)(x) = f(\varphi(1 \otimes a')x)a$ and let $\mathrm{Alt}(M^{(*)})$ be the corresponding set of alternating elements. In view of [KPS$_1$, Proposition 4.2] there exists a unique R-linear isomorphism

$$\pi : \ \mathrm{Alt}(M) \xrightarrow{\sim} \mathrm{Alt}(M^{(*)}) \otimes \mathrm{Pf}(M)$$

such that $\pi(x)(x) = \mathrm{pf}(x)$ for all $x \in \mathrm{Alt}(M)$. Furthermore we have $x\pi(x) = 1 \otimes \mathrm{pf}(x)$ in $M \otimes_{A^{\mathrm{op}}} \otimes M^{(*)} \otimes \mathrm{Pf}(M) = \mathrm{End}_{A^{\mathrm{op}}}(M) \otimes \mathrm{Pf}(M)$. The construction of the Clifford algebra now is very similar to the construction for quaternary forms given in (4.3). We put $B = \mathrm{End}_{A^{\mathrm{op}}}(M)$ and define

$$D_0 = \begin{pmatrix} A^{\mathrm{op}} & 0 \\ 0 & B \end{pmatrix} \quad \text{and} \quad D_1 = \begin{pmatrix} 0 & M^{(*)} \otimes \mathrm{Pf}(M) \\ M & 0 \end{pmatrix}.$$

As above, $D_0 \oplus D_1$ extends to a graded algebra $\widetilde{D}$. The map $\iota : \mathrm{Alt}(M) \to D_1$, $x \mapsto (x, \pi(x))$ is such that $\mu(\iota(x) \otimes \iota(x)) = (\pi(x)(x), x\pi(x)) = (1,1) \cdot \mathrm{pf}(x)$. By the universal property of the Clifford algebra of $\widetilde{C}(\mathrm{Alt}(M), \mathrm{pf}, \mathrm{Pf}(M))$, there exists a homomorphism

$$\tilde{\iota} : \widetilde{C}(\mathrm{Alt}(M), \mathrm{pf}, \mathrm{Pf}(M)) \to \widetilde{D}$$

such that $\tilde{\iota}|_{\mathrm{Alt}(M)} = \iota$. It follows from (3.7) that $\tilde{\iota}$ is an isomorphism. Thus

Proposition 4.8. *Let A be a Azumaya R-algebra of rank 16 with a fixed isomorphism $\varphi : A \otimes A \tilde{\to} \mathrm{End}_R(M)$. The even Clifford algebra and the Clifford module of the pfaffian* $(\mathrm{Alt}(M), \mathrm{pf}, \mathrm{Pf}(M))$ *are given by*

$$C_0 = A^{\mathrm{op}} \times \mathrm{End}_{A^{\mathrm{op}}}(M) \quad \textit{and} \quad C_1 = M \oplus M^{(*)} \otimes_R \mathrm{Pf}(M)$$

In particular, the centre of C_0 is a split quadratic algebra.

Let $\psi : A^{\mathrm{op}} \tilde{\to} \mathrm{End}_{A^{\mathrm{op}}}(M)$ be given by $a' \mapsto \varphi(1 \otimes a')$. Identifying $\widetilde{C}$ with $\widetilde{D}$, the standard involution of $\widetilde{C}$ is given on $D_0 \oplus D_1$ by

$$\begin{pmatrix} a & n \otimes \xi \\ m & b \end{pmatrix} \mapsto \begin{pmatrix} \psi^{-1}(b) & \tau_0(n) \otimes \xi \\ \tau(m) & \psi(a) \end{pmatrix}, \tag{4.9}$$

where τ is the automorphism of order 2 of M defined in (2.4) and τ_0 is the corresponding automorphism of $M^{(*)}$ (with respect to φ^*).

(4.10) Involutions. Let A be an Azumaya R-algebra with an R-linear involution σ of orthogonal type, i.e A is locally isomorphic (for the etale topology) to an algebra $\mathrm{End}_R(V)$ and σ is induced by a quadratic form (V, q, I) (we assume that 2 is invertible in R). Then $C_0(V, q, I)$ glues to an R-algebra $C_0(A, \sigma)$, the *even Clifford algebra* of (A, σ). This algebra was introduced over fields by N. Jacobson, [J] (see also [T]). Its generalization to Azumaya algebras over algebraic schemes is described in [PS]. The Clifford module $C_1(V, q, I)$ cannot be globalized, but, as observed by M. Rost ([R]), $V \otimes C_1(V, q, I)$ can be globalized.

5. Spaces with trivial Arf invariant.

In this section, we prove converses to (4.5) and (4.8). We refer to [KP] and [KPS$_1$] for details (in the "classical" case).

Theorem (5.1). *Let (V, q, I) be a quadratic space of rank 4 with trivial Arf invariant. There exists an Azumaya R-algebra A of rank 4 and a projective A-module P of rank one such that (V, q, I) is similar to the reduced norm $(P, j, J(P))$.*

Proof. Let e be an idempotent generating the centre $Z \simeq R \times R$ of C_0. We put $A = C_0 e$ (as Re-algebra) and $P = C_1 e$. Then (5.1) follows as in the proof of (6.4) of [KP].

Remark. The algebra A and the module P are clearly not uniquely determined in (5.1) since, for example, A can be replaced by $B = \mathrm{End}_A(P)$ and (independently) P can be replaced by $P^* \otimes_R J(P)$. More generally, if (A', P') is another pair satisfying (5.1), one can show that there is a decomposition $R = R_1 \times R_2$ of R such that, for the induced decompositions

$$A = A_1 \times A_2, A' = A'_1 \times A'_2, P = P_1 \times P_2 \text{ and } P' = P'_1 \times P'_2$$

there exist algebra isomorphisms $A'_1 \simeq A_1, A'_2 \simeq \mathrm{End}_{A_2}(P_2)$ and corresponding semilinear isomorphisms $P'_1 \simeq P_1, P'_2 \simeq P^*_2 \otimes J(P_2)$.

Theorem (5.2). *Let (V, q, I) be a quadratic space of rank 6 with trivial Arf invariant. There exist an Azumaya R-algebra A of rank 16, a projective R-module M of rank 16 and an isomorphism $\varphi : A \otimes A \xrightarrow{\sim} \operatorname{End}_R(M)$ such that (V, q, I) is similar to the pfaffian $(\operatorname{Alt}(M), \operatorname{pf}, \operatorname{Pf}(M))$.*

Proof. Let e be an idempotent generating the centre $Z \simeq R \times R$ of C_0 and let $f = 1 - e$. We put $A = C_0 e$ (as Re-algebra), $M = C_1 f$ and define $\varphi : C_0 e \otimes C_0 e \to \operatorname{End}_R(C_1 f)$ by $\varphi(ae \otimes be)(xf) = aexf\sigma(be) = ax\sigma(b)f$, where σ is the standard involution of C_0. The involutary map τ of M is the restriction of the standard involution to $C_1 f$. We have $M^{(*)} = C_1 e \otimes I^{-1}$ and

$$\pi : \operatorname{Alt}(M) = \operatorname{Alt}(C_1 f) \xrightarrow{\sim} \operatorname{Alt}(M^{(*)}) \otimes \operatorname{Pf}(M) = \operatorname{Alt}(C_1 e)$$

is given by $xf \mapsto xe, \ x \in \operatorname{Alt}(C_1 f)$.

Remark. As in (5.1), the algebra A and the module M are not uniquely determined. A corresponding discussion (for the "classical" case) is in [KPS$_1$].

6. Groups of similitudes.

Let (V, q, I) be a quadratic space of even rank and $GO(q)$ be the group of similitudes of (V, q, I). Any similitude $u \in GO(q)$ induces an automorphism of $C_0(q)$, hence an automorphism $\gamma(u)$ of its centre $Z = Z(C_0(q))$. We call u a *direct similitude* if $\gamma(u) = 1$. Thus we have an exact sequence

$$1 \longrightarrow GO_+(q) \longrightarrow GO(q) \longrightarrow \operatorname{Aut}(Z)$$

denoting by $GO_+(q)$ the group of direct similitudes of (V, q, I). Observe that $\operatorname{Aut}(Z) = \mathrm{Z}/2\mathrm{Z}$ if R is connected. We now describe $GO_+(q)$ for V of rank 2, 4 and 6.

(6.1) Binary forms. We may assume that a rank 2 quadratic space (V, q, I) is a norm form $(M, j, J(M))$ for a rank one projective module M over a separable quadratic R-algebra S (see (4.2)). Let $u \in GO_+(q)$. Since u is the identity on $Z(C_0) = C_0 = S$, the map u is an S-linear automorphism of M. Thus

$$GO_+(q) = \mathrm{G}_m(S).$$

(6.2) Quaternary forms. If (V, q, I) is a quadratic space of rank 4 we get as in [KPS$_2$] an exact sequence

$$1 \longrightarrow \{(n_0(z), z^{-1}) \mid z \in \mathrm{G}_m Z\} \longrightarrow \mathrm{G}_m(R) \times \mathrm{G}_m(C_0) \xrightarrow{\alpha} GO_+(q) \xrightarrow{\beta} \operatorname{Pic}(Z),$$

where n_0 is the norm on Z. The map α is defined as $\alpha(\lambda, c)(x) = \lambda c x \sigma(c)$, where σ is the standard involution of C_0 and β is given by

$$\beta(u) = \{y \in C_0 \mid u(y)c = cy \ \text{ for all } c \in C_0\}.$$

If the Arf invariant of (V, q, I) is trivial, we can assume by (5.1) that the quadratic space is a norm form on a rank one module P over a separable quaternion algebra A. Then $C_0 \simeq A \times B$ with $B = \operatorname{End}_A(P)$ (see (4.5)), the exact sequence reduces to

$$1 \longrightarrow \{(\lambda, \lambda^{-1}) \mid \lambda \in \mathrm{G}_m(R)\} \longrightarrow \mathrm{G}_m(B) \times \mathrm{G}_m(A) \xrightarrow{\alpha} GO_+(q) \xrightarrow{\beta} \operatorname{Pic}(R \times R)$$

and the map α is given by $\alpha(b, a)(x) = bx\sigma_A(a)$.

(6.3) Pfaffians. For a quadratic space (V, q, I) of rank 6 we define

$$GU(C_0) = \{c \in C_0 \mid \sigma(c)c \in \mathrm{G}_m(Z)\}.$$

Let now

$$H = \{(z, c) \in \mathrm{G}_m(Z) \times GU(C_0) \mid \sigma_0(z)z^{-1} = n_{C_0}(c)(\sigma(c)c)^{-2}\},$$

where σ_0 is the involution of Z and n_{C_0} is the reduced norm of C_0. We get, as in [KPS$_2$], an exact sequence

$$1 \longrightarrow \{(z^2, z^{-1}) \mid z \in \mathrm{G}_m(Z)\} \longrightarrow H \xrightarrow{\alpha} GO_+(q) \xrightarrow{\beta} \mathrm{Pic}(Z),$$

where $\alpha(z, c)(x) = zcx\sigma(c)$. If the Arf invariant of the quadratic space is trivial, we can assume, by (5.2) that q is the pfaffian $(\mathrm{Alt}(M), \mathrm{pf}, \mathrm{Pf}(M))$ for some Azumaya algebra A of rank 16 over R and, with the notations of (4.7), the sequence reduces to

$$1 \longrightarrow \{(\lambda^2, \lambda^{-1}) \mid \lambda \in \mathrm{G}_m(R)\} \longrightarrow \mathrm{G}_m(R) \times \mathrm{G}_m(A) \xrightarrow{\alpha} GO_+(q) \xrightarrow{\beta} \mathrm{Pic}(R \times R)$$

where $\alpha(\rho, a)(x) = \rho\varphi(a \otimes a)(x)$.

7. Composition of quaternary forms. Let A be an Azumaya R-algebra of rank 4 and let P be a projective right A-module of rank one. Let $B = \mathrm{End}_A(P)$, so that P is a B-A-bimodule. Further let Q be a projective left A-module of rank one, hence a projective right C-module of rank one with $C = \mathrm{End}_A(Q)^{\mathrm{op}} \simeq \mathrm{End}_A(Q)$. The tensor product $M = P \otimes_A Q$ is a projective right B-, left C-module of rank one. Let $q_1 : P \to I(P),\ q_2 : Q \to I(Q)$ be fixed reduced norms. Then

$$q_3 : P \otimes_A Q \to I(P) \otimes_R I(Q),$$

given by

$$q_3(x \otimes y) = q_1(x) \otimes q_2(y),$$

is the reduced norm of $P \otimes_A Q$ and $I(P \otimes Q) \simeq I(P) \otimes I(Q)$.

Let $(P_i, q_i, I_i),\ i = 1, 2, 3$, be quadratic spaces of rank 4 with values in invertible ideals. A *composition* $P_1 \times P_2 \to P_3$ is an R-bilinear map $\mu : P_1 \times P_2 \to P_3$ together with an isomorphism

$$\mu' : I_1 \otimes_R I_2 \xrightarrow{\sim} I_3$$

such that

$$q_3(\mu(x_1, x_2)) = \mu'(q_1(x_1) \otimes q_2(x_2)),\ x_1 \in P_1,\ x_2 \in P_2.$$

Thus the canonical map $P \times Q \to P \otimes_A Q$ is a composition of reduced norms. We say that (P_1, q_1, I_1) is *composable* if there exist quadratic spaces $(P_2, q_2, I_2), (P_3,\ q_3, I_3)$ and a composition $\mu : P_1 \times P_2 \to P_3$.

Theorem (7.1). *Let (P_1, q_1, I_1) be a quadratic space of rank 4. The following conditions are equivalent :*

(1) *(P_1, q_1, I_1) has trivial Arf invariant.*
(2) *There exists an Azumaya R-algebra A of rank 4 such that P_1 is a projective A-module of rank one and $q_1 : P_1 \to I_1$ is (similar to) the reduced norm of P_1 with respect to A.*
(3) *(P_1, q_1, I_1) is composable.*

Proof. (1)$\Leftrightarrow$ (2) is (4.5) and (5.1). To show (2) $\Rightarrow$ (3), we may take, for example, $(P_2, q_2, I_2) = (A, n, R)$. To check (3) $\Rightarrow$ (2), we use the idea of the proof of Theorem (2.10) of [K+]. Let (P_2, q_2, I_2) be such that there exists a composition $\mu : P_1 \times P_2 \to P_3$. Let A be the set of pairs (s_1, s_2) of morphisms s_1 of (P_1, q_1, I_1), resp. s_2 of (P_2, q_2, I_2) satisfying $\mu(s_1x_1, x_2) = \mu(x_1, s_2x_2)$. Then A is an R-algebra with the multiplication $(s_1, s_2)(s_1', s_2') = (s_1' \circ s_1, s_2 \circ s_2')$. One can check as in [K+] that A is an Azumaya algebra of rank 4, that P_1 is a projective right A-module of rank one and that q_1 is the reduced norm of P_1.

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