A Teichmüller Cocycle for Finite Extensions

by

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1. Introduction

Let $k \subset K$ be a finite Galois extension of fields, with group G. The group G acts on the Brauer group $Br(K)$ and the invariant subgroup $Br(K)^G$ consists of classes of G-normal algebras, i.e. central simple K-algebras A with a map $G \to Aut_K(A)$ extending the action of G on K. In 1940 Teichmüller [T] constructed a map $d: Br(K)^G \longrightarrow H^3(G,K^*)$ such that the kernel consists of algebra classes which come from k by scalar extension.

In this paper we show how faithfully flat descent can be used to define a Teichmüller map for arbitrary finite extensions. Let R be a commutative ring and S an extension of R. The two different S-algebra structures of $S \otimes S$ (S acting on the first or on the second factor) induce two maps $Br(S) \longrightarrow Br(S \otimes S)$ of Brauer groups. Let $Br(S)^0$ be the kernel of the pair of maps

(1.1) $$0 \longrightarrow Br(S)^0 \longrightarrow Br(S) \rightrightarrows Br(S \otimes S)$$

The image of the map $Br(R) \longrightarrow Br(S)$ induced by scalar extension clearly lies in $Br(S)^0$. We shall define a group homomorphism d from $Br(S)^0$ into $H^3(S/R)$, the third Amitsur cohomology group of the extension $R \subset S$, such that the sequence

(1.2) $$Br(R) \longrightarrow Br(S)^0 \xrightarrow{d} H^3(S/R)$$

is exact, in the following cases:

I) The ring R is semi-local and S is finite faithful projective as R-module. If, furthermore, $R \subset S$ is Galois with group G, then we show that $Br(S)^0 = Br(S)^G$ and that our map corresponds to the classical Teichmüller map. Using ideas of Childs [C] it is probably possible to define a Teichmüller map for the non semilocal case. The group $H^3(S/R)$ then has to be replaced by a more complicated group.

II) The use of Amitsur cohomology also allows a Teichmüller cocycle map for purely inseparable field extensions. More generally the map d can be defined if S is finite faithful projective as R-module and the multiplication map $S \otimes S \to S$ has a nilpotent kernel. The ring R can be any commutative ring. In this case $\mathrm{Br}(S)^0$ coincides with $\mathrm{Br}(S)$. Consider for example a finite radical extension S of R, i.e. R has characteristic p, p a prime, and $S^q \subset R$ for some power q of p. By Berksons's theorem ([KO] p. 142) $H^3(S/R) = 0$ and one concludes that for such extensions the map $\mathrm{Br}(R) \to \mathrm{Br}(S)$ is surjective. This was first proved by Hochschild [H] for purely inseparable field extensions. As an application we show that the Brauer group of a domain of characteristic p, p a prime, is p-divisible.

§2 recalls some results on descent and cohomology. In §3 we define a notion of normal algebras for an arbitrary extension which generalizes the notion of a G-normal algebra for a Galois extension with group G. In §4 we introduce the Teichmüller cocycle map and analyse the kernel of the map in §5. §6 gives the relation with the classical map in the Galois case. In §7 we describe an explicit construction of the Chase-Rosenberg sequence. One of the homomorphisms turns out to be a Teichmüller map for the Picard group Pic. After completion of this work, Alex Rosenberg told us that the maps were already explicitely described by G. Garfinkel in his thesis (Cornell, 1968). Since apparently Garfinkel did not explicitly verify the exactness and since the verification with the methods of this paper is easy, we thought it worthwhile to keep these results here. §8 provides an exact sequence which describes the image of the Teichmüller map for the Brauer group of algebras split by a finite extension. This sequence contains various known sequences of Eilenberg-Mac Lane-Hochschild-Serre, Amitsur, Rosenberg-Zelinsky and Yuan ([EM], [HS], [A], [RZ], [Y]).

Rings and algebras are always commutative if nothing is mentioned. Unadorned tensor products are usually over R.

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2. Faithfully flat descent and cohomology.

We briefly recall some results. Pictures and proofs can be found in [KO] (for example!). Let S be an R-algebra and $A, B, \dots, U, A', B', \dots, U'$ R-modules. For any tensor product $T = A \otimes \dots \otimes U$ we denote the module obtained by putting a factor S in i-th position by T_i and the corresponding map $T \to T_i$ by ε_i. We write $\varepsilon_i(g) = g_i$ for $g \in T$ and for any homomorphism $f\colon T \to T' = A' \otimes \dots \otimes U'$ we denote the map $T_i \to T'_i$ obtained by tensoring f with the identity in i-th position by f_i. Finally we denote the iterated $(A_i)_j$ by A_{ij} (first i, then j) and shall systematically use the "cosemi-simplicial" identities $A_{ji} = A_{i,j+1}$ for $i \leq j$.

Let S be a faithfully flat R-algebra, M an S-module and $\phi\colon M_1 \to M_2$ an $S \otimes S$-isomorphism. We say that ϕ is a descent-datum from S to R for M if $\phi_2 = \phi_3 \phi_1$.

(2.1) Theorem. Let $\phi\colon M_1 \to M_2$ be a descent datum and let N be the R-module $\{x \in M \mid \phi\varepsilon_1 x = \varepsilon_2 x\}$. Then the homomorphism of S-modules $\eta\colon N_1 \to M$ defined by $\eta(s \otimes n) = sn$ is an isomorphism such that

(2.2) $\phi = \eta_3 \tau_1 \eta_1^{-1}$

where $\tau\colon S \otimes N \to N \otimes S$ is as usual the switch. If N' is an R-module and $\eta'\colon N'_1 \to M$ an isomorphism of S-modules such that $\phi = \eta'_3 \tau_1 (\eta'_1)^{-1}$, then there exists a unique isomorphism $\rho\colon N \to N'$ such that $\eta = \eta' \rho_1$.

(2.3) Remark. If M is an S-algebra and ϕ is an isomorphism of algebras, then the descended module N has a canonical structure of R-algebra. If for example M is an Azumaya S-algebra, i.e. a central separable S-algebra, then N is an Azumaya R-algebra.

(2.4) Example: Galois descent

Let S be a ring, G a finite group of automorphisms of S and $R = S^G$ the subring of invariant elements. Recall that $R \subset S$ is a Galois extension if the homomorphism of S-algebras

$$h\colon S \otimes S \longrightarrow \prod_{\sigma \in G} S$$

defined by $h(s \otimes t) = (\sigma(s)t)_\sigma$ is an isomorphism. Let M be an S-module. A <u>Galois descent-datum</u> on M is a family $\{\bar{\sigma}, \sigma \in G\}$ of R-automorphisms of M such that

(2.5)
i) $\bar{\sigma}$ is σ-semilinear, i.e. $\bar{\sigma}(sm) = \sigma(s)\bar{\sigma}(m)$, $s \in S$, $m \in M$
ii) $\overline{\sigma\tau} = \bar{\sigma}\bar{\tau}$ and $\bar{1} = 1$

Let $N = M^G = \{x \in M \mid \bar{\sigma}(x) = x,\ \sigma \in G\}$, then the map $\eta\colon N_1 \to M$ defined by $\eta(s \otimes n) = sn$ is an isomorphism such that $\bar{\sigma}\eta = \eta(\sigma \otimes 1_N)$ and the pair (N, η) is characterized (up to unique isomorphisms) by this property.

Let now $e_\sigma \in S \otimes S$, $\sigma \in G$, be the inverse images of the canonical idempotents of $\prod_{\sigma \in G} S$. For any S-module M the canonical maps $M \to M_1 \to e_\sigma M_1$ and $M \to M_2 \to e_\sigma M_2$ are clearly isomophisms of S-modules. The following diagram

(2.6)
$$\begin{tikzcd}
e_\sigma M_1 \arrow[r, "e_\sigma \phi"] & e_\sigma M_2 \\
M \arrow[u, "s"] & M \arrow[u, "s"'] \arrow[l, "\bar{\sigma}"']
\end{tikzcd}$$

defines a one-to-one correspondence between $S \otimes S$-isomorphisms $\phi\colon M_1 \to M_2$ and families $\{\bar{\sigma}, \sigma \in G\}$ of semilinear automorphisms of M. An easy computation shows that ϕ is a descent-datum if and only if $\{\bar{\sigma}, \sigma \in G\}$ is a Galois descent-datum.

(2.7) We now recall the definition of Amitsur cohomology. Let S be an R-algebra and let $S^{(n)}$ be the tensor product of n copies of S. For any covariant functor F from the category of commutative rings to the category of abelian groups consider the sequence

(2.8) $$0 \to F(S) \xrightarrow{\Delta_o} F(S^{(2)}) \xrightarrow{\Delta_1} F(S^{(3)}) \xrightarrow{\Delta_2} \dots$$

where $\Delta_{n-1} = F(\varepsilon_1) - F(\varepsilon_2) + \dots + (-1)^n \ F(\varepsilon_{n+1})$. Using the cosemi-simplicial identities, one verifies easily that $\Delta_{n+1}\Delta_n = 0$ and one defines $H^n(S/R),F) = \ker \Delta_n / \mathrm{Im}\Delta_{n-1}$ to be the n-th Amitsur cohomology group of the extension S/R with value in F .

(2.9) Example. The group $Br(S)^o$ defined by (1.1) is just $H^o(S/R,Br)$.

(2.10) Example. If $R \subset S$ is Galois with group G, then it is well known that $H^n(S/R,F) \overset{\sim}{\to} H^n(G,F(S))$. We briefly recall how the map is defined. Let $E^n(S)$ be the S-algebra of functions of n variables defined on G with values in S . Then the isomorph~~a~~isms

(2.11) $$h_n : S \otimes .. \otimes S \longrightarrow E^n(S) \qquad \text{(n+1 factors S)}$$

defined by $h_n(s_1 \ .. \ s_{n+1})(\sigma_1,\dots,\sigma_n) = \sigma_1\sigma_2..\sigma_n(s_1)\sigma_2..\sigma_n(s_2),..\sigma_n(s_n)s_{n+1}$ induce isomorphisms of the Amitsur complex onto the standard non-homogeneous cochain complex of G with coefficients in F(S) . See [CHR] for details.

3. Normal Azumaya Algebras

(3.1) For a given extension $R \subset S$, a normal Azumaya S-algebra is an Azumaya S-algebra A together with an $S \otimes S$-isomorphism of algebras $\phi : A_1 \to A_2$. If A and B are normal then clearly $A \otimes_S B$ is normal. Any induced algebra $S \otimes C$, C Azumaya over R , is normal.

(3.2) Proposition. in both cases I and II (of the introduction) the subgroup $Br(S)^o$ of Br(S) consists of classes of normal algebras.

Proof. For any class $[A] \in Br(S)^o$, we can choose a representative A of constant rank over S . By definition of $Br(S)^o$, $[A_1] = [A_2]$, therefore there exist faithfully projective S-modules P and Q such that

(3.3) $$A_1 \otimes_{S\otimes S} \mathrm{End}_{S\otimes S}(P) \cong A_2 \otimes_{S\otimes S} \mathrm{End}_{S\otimes S}(Q)$$

Again we can assume that P and Q have constant ranks. In case I, $S\otimes S$ is semi-local, hence P and Q are free and the class $[A]$ clearly contains a representative which is normal. In case II, let us denote by $\bar{P}$ the S-module $P\otimes_{S\otimes S} S$ where S is viewed as $S\otimes S$-algebra via the multiplication map. Now (3.3) induces an isomorphism of S-algebras

$$A\otimes_S \mathrm{End}_S(\bar{P}) \cong A\otimes_S \mathrm{End}_S(\bar{Q}) \quad .$$

Tensoring over S with the opposite algebra A^o of A gives

$$\mathrm{End}_S(A)\otimes_S \mathrm{End}_S(\bar{P}) \cong \mathrm{End}_S(A)\otimes_S \mathrm{End}_S(\bar{Q}) \; .$$

Since A is faithfully projective over S, there exists an S-module M such that $M\otimes_S A \cong S^n$ for some $n \in N$ ([B] p.39), therefore $\mathrm{End}_S(\bar{P}^n) \cong \mathrm{End}_S(\bar{Q}^n)$. We now recall a result due to Rosenberg and Zelinsky $[R.Z]_1$.

(3.4) Lemma. Let P and Q be faithfully projective R-modules and let $\alpha: \mathrm{End}_R(P) \xrightarrow{\sim} \mathrm{End}_R(Q)$ be an isomorphism of R-algebras. Then α is induced by an isomorphism of R-modules $f: P\otimes I \to Q$ where I is an invertible R-module. If $g: P\otimes J \to Q$ is another isomorphism inducing α, then there is an isomorphism $\rho: I \to J$ and a unit $\lambda \in U(R)$ such that $\lambda f = g(1\otimes\rho)$.

Applying the lemma to $\mathrm{End}_S(\bar{P}^n) \xrightarrow{\sim} \mathrm{End}_S(\bar{Q}^n)$, we know that this isomorphism is induced by an isomorphism $f: \bar{P}^n\otimes_S J \xrightarrow{\sim} \bar{Q}^n$ with $(J) \in \mathrm{Pic}(S)$, the Picard group of S. Since the kernel of the multiplication map $\mu: S\otimes S \to S$ is nilpotent, the induced map $\mathrm{Pic}(S\otimes S) \to \mathrm{Pic}(S)$ is an isomorphism. Choose $(I) \in \mathrm{Pic}(S\otimes S)$ such that $\bar{I} \cong J$ and replace f by $\bar{g}: \bar{P}^n\otimes_S \bar{I} \longrightarrow \bar{Q}^n$. By Nakayama's lemma, $\bar{g}$ is induced by some $S\otimes S$-isomorphism $g: P^n\otimes_{S\otimes S} I \longrightarrow Q^n$ and this g defines an isomorphism

$$\mathrm{End}_{S\otimes S}(P)\otimes_{S\otimes S} M_n(S\otimes S) \xrightarrow{\sim} \mathrm{End}_{S\otimes S}(Q)\otimes_{S\otimes S} M_n(S\otimes S)$$

Putting everything together we obtain an isomorphism

$$A_1 \otimes_{S\otimes S} \mathrm{End}_{S\otimes S}(P) \otimes_{S\otimes S} M_n(S\otimes S) \cong A_2 \otimes_{S\otimes S} \mathrm{End}_{S\otimes S}(P) \otimes_{S\otimes S} M_n(S\otimes S) .$$

Choose finally P' such that $P' \otimes_{S\otimes S} P \cong (S\otimes S)^n$ and tensor with $\mathrm{End}_{S\otimes S}(P')$ for the result.

(3.5) Example. Let $R \subset S$ be a Galois extension with group G. We shall see in §6 that a normal algebra is a G-normal algebra in the classical sense [EM]. Notice that $Br(S)^0 = Br(S)^G$ follows from (2.9) and (2.10) since $Br(S)^G = H^0(G, Br(S))$.

(3.6) We now show that $Br(S)^0 = Br(S)$ in case II. Let $Br(S)^1 \subset Br(S)^0$ be the subgroup generated by classes of normal algebras A such that $\phi \otimes_{S\otimes S} 1_S : A \to A$ is the identity of A.

(3.7) Proposition. $Br(S)^1 = Br(S)^0 = Br(S)$ if the kernel of the multiplication map $\mu: S\otimes S \to S$ is nilpotent.

Proof. Let $\mu_*: Br(S\otimes S) \to Br(S)$ be the map induced by μ. By [RS] μ_* is injective and the proposition follows from $\mu_*[S\otimes A] = \mu_*[A\otimes S]$.

4. The Teichmüller cocycle.

If A is a normal Azumaya S-algebra, select an $S\otimes S$-isomorphism $\phi: A_1 \to A_2$. The composition $\psi = \phi_2^{-1}\phi_3\phi_1$ is an $S^{(3)}$-automorphism of A_{11} which is inner in both case I and II by Rosenberg and Zelinsky's generalisation of the Skolem-Nœther theorem. (See [KO] p.107) In any case ψ is of the form $\psi(x) = g x g^{-1}$ where g is a left A_{11}-isomorphism $A_{11} \otimes_{S^{(3)}} I \xrightarrow{\sim} A_{11}$ with $(I) \in \mathrm{Pic}(S^{(3)})$. In case I, $\mathrm{Pic}(S)^{(3)}) = 0$ since $S^{(3)}$ is semi-local. In case II, choose ϕ such that $\phi \otimes_{S\otimes S} 1_S$ is the identity of A, this can be done by (3.7). Then (I) lies in the kernel of the multiplication

$\mathrm{Pic}(S^{(3)}) \to \mathrm{Pic}(S)$ which is zero since the multiplication has nilpotent kernel. Let now $\psi(x) = g\,x\,g^{-1}$, $g \in U(A_{11})$. Using the cosemi-simplicial identities, one verifies that

insert ϕ_{11} /

$$(4.1)\quad \phi_{11}^{-1}(\phi_1^{-1}\phi_3^{-1}\phi_2)_4 \overset{\phi_{11}}{\vee} (\phi_1^{-1}\phi_3^{-1}\phi_2)_2(\phi_2^{-1}\phi_3\phi_1)_3(\phi_2^{-1}\phi_3\phi_1)_1 = 1\,,$$

hence $\phi_{11}^{-1}(g_4^{-1})g_2^{-1}g_3g_1$ lies in the center $S^{(4)}$ of A_{111}. We call this element a <u>Teichmüller cocycle</u> of A and denote it by

$$(4.2)\quad d(g) = \phi_{11}^{-1}(g_4^{-1})g_2^{-1}g_3g_1\,.$$

(4.3) <u>Proposition.</u> 1) $d(g)$ is a cocycle in Amitsur's complex with value in U (= "Units") and the class $[d(g)] \in H^3(S/R,U)$ is independent of the choice of g which induces ψ. 2) The class $[d(g)] \in H^3(S/R,U)$ is independent of the choice of ϕ and only depends on the class $[A]$ of A in $\mathrm{Br}(S)$.

<u>Proof.</u> The second assertion of 1) is easy. We prove the first, i.e. $\Delta_3 d(g) = 1$ in $U(S^{(4)})$. We shall use that any cyclic permutation of the factors of $d(g)$ does not modify $d(g)$. Then $\Delta_3 d(g) =$

$$(g_2^{-1}g_3g_1\phi_{11}^{-1}(g_4^{-1}))_1(\phi_{11}^{-1}(g_4^{-1})g_2^{-1}g_3g_1)_3(g_1^{-1}g_3^{-1}g_2\phi_{11}^{-1}(g_4))_4 g_{11}\phi_{111}^{-1}\{(\phi_{11}^{-1}(g_4^{-1})g_2^{-1}g_3g_1)\}_5 \cdot g_{1}$$

hence the product of the three first factors gives

$g_{11}\phi_{111}^{-1}(g_{41}^{-1})\phi_{113}^{-1}(g_{43}^{-1})g_{23}^{-1}g_{24}\phi_{114}^{-1}(g_{44})$. But now

$$g_{11}\phi_{111}^{-1}\{(\phi_{11}^{-1}(g_4^{-1})g_2^{-1}g_3g_1)_5\}g_{11}^{-1} = g_{11}g_{11}^{-1}\phi_{114}^{-1}(g_{44}^{-1})g_{11}\phi_{112}^{-1}(g_{42}^{-1})\phi_{113}^{-1}(g_{43})\phi_{111}^{-1}(g_{41})\cdot$$

$\cdot\, g_{11}^{-1}$ so that the product of the first four factors gives $g_{22}^{-1}g_{32}g_{12}\phi_{112}^{-1}(g_{42}^{-1})$ which is $d(g)_2$.

¶ For the proof of 2), let $\phi\colon A_1 \to A_2$ and $\psi\colon A_1 \to A_2$ be two $S\otimes S$-isomorphisms. As before, we can assume that there exists $g \in U(A_2)$ such that $\psi(x) = g\phi(x)g^{-1}$. Take now $f \in U(A_{11})$ inducing $\phi_2^{-1}\phi_3\phi_1$ by conjugation and $h \in U(A_{11})$ inducing $\psi_2^{-1}\psi_3\psi_1$. One has

$$h = \phi_2^{-1}(g_2^{-1}g_3)\phi_2^{-1}\phi_3(g_1)f\,u$$

for some $u \in U(S^{(3)})$. Since f and fu induce ϕ , we can suppose that $u = 1$. By definition of f , $\phi_2^{-1}\phi_3(g_1) = f\phi_1^{-1}(g_1)f^{-1}$, hence

$h = \phi_2^{-1}(g_2^{-1}g_3)f\phi_1^{-1}(g_1)$. We now compute $\psi_{11}^{-1}(h_4^{-1})h_2^{-1}h_3h_1 =$

$$= \phi_{11}^{-1}(g_{11}^{-1}h_4^{-1}g_{11})h_2^{-1}h_3h_1 = \phi_{11}^{-1}(g_{11}^{-1})\ \phi_{11}^{-1}\left\{\phi_{14}^{-1}(g_{14}^{-1})f_4^{-1}\phi_{24}^{-1}(g_{34}^{-1})\phi_{24}^{-1}(g_{24}\right\}\phi_{11}^{-1}(g_{11})\phi_{12}^{-1}(g_{12}^{-1})\ \cdot$$

$$\cdot\ f_2^{-1}\phi_{22}^{-1}(g_{32}^{-1})\phi_{22}^{-1}(g_{22})\phi_{23}^{-1}(g_{23}^{-1})\phi_{23}^{-1}(g_{33})f_3\phi_{13}^{-1}(g_{13})\phi_{21}^{-1}(g_{21}^{-1})\ \cdot\phi_{21}^{-1}(g_{31})f_1\phi_{11}^{-1}(g_{11}) =$$

$$= \phi_{11}^{-1}\phi_{31}^{-1}(g_{31})\phi_{11}^{-1}(f_4^{-1})\phi_{12}^{-1}\phi_{32}^{-1}(g_{34}^{-1})\phi_{12}^{-1}\phi_{32}^{-1}(g_{24})\ \sqcup\ f_2^{-1}\phi_{22}^{-1}(g_{32}^{-1})\phi_{23}^{-1}(g_{33})f_3\phi_{21}^{-1}(g_{31})f_1 =$$

$$= f_1^{-1}\phi_{21}^{-1}(g_{31})f_1\phi_{11}^{-1}(f_4^{-1})f_2^{-1}\phi_{22}^{-1}(g_{34}^{-1})\ \sqcup\ f_2f_2^{-1}\phi_{22}^{-1}(g_{24})f_2f_2^{-1}\phi_{22}^{-1}(g_{32}^{-1})\ \phi_{23}^{-1}(g_{33})f_3\phi_{21}^{-1}(g_{31})f_1$$

$$= f_1\phi_{11}^{-1}(f_4^{-1})f_2^{-1}f_3 \quad .$$

(4.4) Proposition. 1) if A is a normal S-algebra with $[A] = 1 \in Br(S)$, then $[d(g)] = 1 \in H^3(S/R,U)$.

2) If A and A' are normal with maps ϕ and ϕ' and elements g and g' in $U(A_{11})$ inducing $\phi_2^{-1}\phi_3\phi_1$ and $\phi_2'^{-1}\phi_3'\phi'_1$, then $d(g\otimes_{S^{(3)}}g')$ is a Teichmüller cocycle for $A\otimes_S A'$ and $d(g\otimes_{S^{(3)}}g') = d(g)d(g')$.

3) If A is normal, then the opposite algebra A^o is normal, $d((g^{-1})^o)$ is a Teichmüller cocycle for A^o and $d((g^{-1})^o) = d(g)^{-1}$.

(4.5) Corollary. The map $A \mapsto d(g)$ induces a natural group homomorphism $d\colon Br(S)^o \longrightarrow H^3(S/R,U)$.

Proof. 1) Let $[A] = 1$ in $Br(S)$. We may assume that $A \cong End_S(P)$ has constant rank. Then P is free in case I, $A \cong S\otimes M_n(R)$ and the result follows from (4.3.2), choosing $\phi = \tau_1$. In case II, we may assume that $\phi\colon S\otimes A = End_{S\otimes S}(S\otimes P) \to End_{S\otimes S}(P\otimes S) = A\otimes S$ in induced by some $f\colon S\otimes P \to P\otimes S$ (as in the proof of 3.2) . The composition $\phi_2^{-1}\phi_3\phi_1$ then is induced by $g = f_2^{-1}f_3f_1$ and the result follows from the identity (4.1) with $f = \phi$.

2) is clear since $g \otimes_S 1_{A'}$ and $1_A \otimes_S g'$ commutes in $(A \otimes_S A')_{11}$. To prove 3), choose some $\phi: A_1 \to A_2$ and let $\psi = \phi_2^{-1}\phi_3\phi_1$ be induced by $g \in A_{11}$. The same ϕ defines an automorphism $\phi^o: A_1^o \to A_2^o$ but now $\psi^o(x^o) = g^{o^{-1}} x^o g^o$ since the multiplication of A_{11}^o is reversed. The formula $d\left(g^{o^{-1}}\right) = d(g)^{-1}$ follows for the same reason.

5. The kernel of the Teichmüller map.

If A is normal, choose an $S \otimes S$-isomorphism $\phi: A_1 \longrightarrow A_2$ and let $g \in U(A_{11})$ induce $\psi = \phi_2^{-1}\phi_3\phi_1$. Call N the left A-module $S \otimes A$ with the action $a \cdot n = \phi^{-1}(\varepsilon_2 a)n$ and define

(5.1) $$f(g): N_1 \to N_2$$

by $f(g)(x) = \tau_1 g x$ where τ, as usual, is the switch. This is clearly an $S \otimes S$-isomorphism.

(5.2) Proposition. 1) The map $f(g)$ is ϕ-linear, i.e. $f(g)\left\{(s \otimes a) \cdot (t \otimes n)\right\} = \phi(s \otimes a) \cdot f(g)(t \otimes n)$, where A acts on N as defined above.

2) The map $f(g)$ induces an $S \otimes S$-isomorphism of algebras

$$\phi(g): \mathrm{End}_{S \otimes A}(S \otimes N) \longrightarrow \mathrm{End}_{A \otimes S}(N \otimes S).$$

3) If $[d(g)] = 1 \in H^3(S/R,U)$, then $\phi(g)$ is a descent-datum.

Proof. 1) We show that $f(g)^{-1}$ is ϕ^{-1}-linear. Let $n = x \otimes b \in N$ and $\phi^{-1}(a \otimes 1) = \sum_i s_i \otimes a_i$, then $f(g)^{-1}\left\{(a \otimes s) \cdot (x \otimes b \otimes t)\right\} = f(g)^{-1}\left\{\left(\sum s_i \otimes a_i \otimes s\right)(x \otimes b \otimes t)\right\} =$

$= g^{-1}\left(\sum s_i \otimes s \otimes a_i)(x \otimes t \otimes b)\right) = g^{-1}\phi_2^{-1}(a \otimes s \otimes 1) g\, g^{-1}(x \otimes t \otimes b) = \phi_1^{-1}\phi_3^{-1}(a \otimes s \otimes 1)\, f(g)^{-1}(x \otimes b \otimes t) =$

$\phi_1^{-1}\left(\sum s_i \otimes s a_i \otimes 1\right) f(g)^{-1}(x \otimes b \otimes t) = \phi^{-1}(a \otimes s) \cdot f(g)^{-1}(x \otimes b \otimes t)$.

2) follows easily from 1). We now prove 3). Multiplying g by a unit of $(S)^3$ if necessary, we can assume that $\phi_{11}^{-1}(g_4^{-1})g_2^{-1}g_3g_1 = 1$ or $g_2^{-1}g_3g_1 = \phi_{11}^{-1}(g_4)$. Write $f(g) = f$ for the moment. Since $f_1 = \tau_{11}g_1$, $f_3 = \tau_{14}g_4 = \tau_{14}\tau_{11}g_3\tau_{11}$

and $f_2 = \tau_{14}\tau_{12}g_2$, we obtain that $f_2^{-1}f_3f_1$ is the left multiplication by $g_2^{-1}g_3g_1$, hence by $\phi_{11}^{-1}(g_4)$ which commutes with the elements of $\mathrm{End}_{S\otimes S\otimes A}(S\otimes S\otimes N)$ by definition of the $S\otimes S\otimes A$ -structure of $S\otimes S\otimes N$. Therefore $\phi(g)_2 = \phi(g)_3\phi(g)_1$.

(5.3) Corollary. The sequence

(1.2) $$\mathrm{Br}(R) \longrightarrow \mathrm{Br}(S)^{o} \xrightarrow{d} H^3(S/R,U)$$

is exact.

Proof. If C is an Azumaya R-algebra, then clearly $d(S\otimes C) = 1$. Choose $\phi\colon C_{11} \to C_{13}$ to be τ_1 , then $\phi_2^{-1}\phi_3\phi_1 = 1$. If $d(A) = 1$ in $H^3(S/R,U)$, let C be the descended algebra for the descent-datum $\phi(g)$. Then $[S\otimes C] = [\mathrm{End}_A(N)] = [A]$ in $\mathrm{Br}(S)$ by [B]p.109, since N is a faithfully projective A-module.

(5.4) Remark. There is no hope in general to construct a descent-datum for the normal algebra A itself. See [EM] for an example in the Galois case.

(5.5) Theorem. Let R be a ring of characteristic p , p a prime, and S a finite radicial extension of R , i.e. S is finite faithful projective as R-module and $S^q \subset R$ for some power q of p . Then the map $\mathrm{Br}(R) \to \mathrm{Br}(S)$ induced by scalars extension is surjective.

Proof. By Berkson's theorem [KO], $H^3(S/R,U) = 0$ for such an extension, then apply (3.7).

(5.6) Remark. Another (more complicated) proof of (5.5) can be found in [KO] .

Out of date, result valid for any ring R of char. p

(5.7) Theorem. Let R be an integral domain of characteristic p . Then $\mathrm{Br}(R)$ is p-divisible.

Proof. Let K be the quotient field of R and $R^{1/p}$ all p-th roots of elements of R in $K^{1/p}$. By [KO] V.6.3, the p-th power $\Pi\colon R^{1/p} \to R$ induces an isomorphism $\Pi_*\colon \mathrm{Br}(R^{1/p}) \to \mathrm{Br}(R)$ such that the p-th power in the Brauer group $\cdot^p\colon \mathrm{Br}(R) \to \mathrm{Br}(R)$ factorizes:

$$\begin{tikzcd}
\mathrm{Br}(R) \arrow[rr, "\cdot p"] \arrow[dr] & & \mathrm{Br}(R) \\
& \mathrm{Br}(R^{1/p}) \arrow[ur, "\pi_*"'] &
\end{tikzcd}$$

Therefore it suffices to show that $\mathrm{Br}(R) \to \mathrm{Br}(R^{1/p})$ is surjective. Let A be an $R^{1/p}$ - Azumaya algebra.. Since $R^{1/p}$ is limit of rings of the form $R[a_1,\dots a_n]$, $a_i \in K^{1/p}$ with $a_i^p \in R$, and that A is finitely generated over $R^{1/p}$, there exists a ring $R_o = R[a_1,\dots,a_n]$ and an Azumaya R_o-algebra A_o such that $A_o \otimes_{R_o} R^{1/p} = A$. Let now $S = R[X_1,\dots,X_n]/(X_i^p - a_i^p)$. The kernel of the map $S \to R_o$ induced by $X_i \to a_i$ is clearly nilpotent. Hence by a recent result of DeMeyer [D], $\mathrm{Br}(S) \xrightarrow{\sim} \mathrm{Br}(R_o)$. Using (5.5) one finally obtains an Azumaya R-algebra C such that $[C \otimes R^{1/p}] = [C \otimes_R R_o \otimes_{R_o} R^{1/p}] = [A_o \otimes_{R_o} R^{1/p}] = [A]$.

(5.8) Remark. (5.6) was proved for very special rings in [KO].

(5.9) Remark. As noticed by Ojanguren, De Meyer's result $\mathrm{Br}(R) \xrightarrow{\sim} \mathrm{Br}(R/I)$ for I nilpotent can be proved by an easy cohomological argument due to Grothendieck, Groupe de Brauer I, Séminaire Bourbaki 64/65, Remarque 4.3.

6. The Galois case.

We show now that the Teichmüller cocycle (4.2) corresponds to the classical Teichmüller cocycle in Galois cohomology by the isomorphism (2.11). Let $R \subset S$ be a Galois extension with group G. If A is a G-normal algebra, select a family $\{\bar{\sigma}, \sigma \in G\}$ of semilinear automophisms of A. For each pair $(\sigma,\tau) \in G \times G$ the composition $\bar{\sigma}\bar{\tau}\overline{\sigma\tau}^{-1}$ is an S-automorphism of A, hence given by the conjugation with some unit $u_{\sigma,\tau} \in U(A)$ (in the semi-local case !). The classical Teichmüller cocycle, [T] or [EM], then is defined by

$$d(u)(\rho,\sigma,\tau) = \bar{\rho}(u_{\sigma,\tau})u_{\rho,\sigma\tau}u^{-1}_{\rho\sigma,\tau}u^{-1}_{\sigma,\tau}. \tag{6.1}$$

Let $E^n(A) = \Pi_{(\sigma_1,\dots,\sigma_n)}A$ be the algebra of functions defined on $G^n = G\times..\times G$ with values in A. The isomorphisms (2.11) extend to isomorphisms

$$h_n : S\otimes .. \otimes S \quad A \longrightarrow E^n(A) .$$

Recall that $h_n(s_1\otimes .. \otimes s_n \otimes a) = \sigma_1\sigma_2 .. \sigma_n(s_1)\sigma_2 .. \sigma_n(s_2) ... \sigma_n(s_n)a$. Recall furthermore that by (2.6) $S\otimes S$-isomorphisms $\phi : A_1 \to A_2$ are in one-to-one correspondence with families or semilinear automorphisms of A $\{\bar{\sigma}, \sigma \in G\}$. An easy computation shows that $h_3\phi_1^{-1}\phi_3^{-1}\phi_2 h_3^{-1}$ corresponds to $\{\bar{\sigma}\bar{\tau}\overline{\sigma\tau}^{-1}\}_{\sigma,\tau}$. Therefore if $g \in U(A_{11})$ induces $\phi_2^{-1}\phi_3\phi_1$ by conjugation, then $\bar{\sigma}\bar{\tau}\overline{\sigma\tau}^{-1}$ is induced by $u_{\sigma,\tau} = h_3 g^{-1}$. A straightforward computation then shows that $h_4 d(g) = h_4\phi_{11}^{-1}\left(g_4^{-1}\right)g_2^{-1}g_3g_1\Big) = d(u)(\rho,\sigma,\tau)$.

7. The Chase-Rosenberg sequence.

As usual denote the kernel of $\mathrm{Pic}(R) \to \mathrm{Pic}(S)$ for any R-algebra S by $\mathrm{Pic}(S/R)$ and the kernel of $\mathrm{Br}(R) \to \mathrm{Br}(S)$ by $\mathrm{Br}(S/R)$. Using spectral sequences Chase and Rosenberg [CHR] proved the existence of an exact sequence

$$(7.1) \qquad 0 \to H^1(S/R,U) \xrightarrow{\alpha} \mathrm{Pic}(R) \xrightarrow{\beta} H^0(S/R,\mathrm{Pic}) \xrightarrow{d} H^2(S/R,U) \xrightarrow{\gamma}$$
$$\longrightarrow \mathrm{Br}(S/R) \xrightarrow{\delta} H^1(S/R,\mathrm{Pic}) \xrightarrow{\rho} H^3(S/R,U)$$

for any finite projective extension $R \subset S$. We now give an explicit description** of natural homomorphisms to render the sequence (7.1) exact. We shall leave some easy computations to the reader. In particular we shall not verify that <u>all</u> the maps are well-defined. Also it will be an immediate consequence of the definition of the maps that the sequence is a complex.

(7.2) Definition of α .

It is well known that $H^1(S/R,U) \xrightarrow{\sim} \mathrm{Pic}(S/R)$. If $t \in U\left(S^{(2)}\right)$ is a cocycle, define $\phi : S\otimes S \longrightarrow S\otimes S$ by multiplication with t . Since $t_2 = t_3t_1$, ϕ is a descent datum which defines an invertible R-module I such that $S\otimes I \cong S$.

** See the introduction.

Conversely, if $f: S\otimes I \xrightarrow{\sim} S$ is an S-isomorphism the map $\phi = f_3\tau_1 f_1^{-1}: S\otimes S \to S\otimes S$ is the multiplication with a cocycle $t \in U\left(S^{(2)}\right)$. The map α is the composition $H^1(S/R,U) \xrightarrow{\sim} \mathrm{Pic}(S/R) \to \mathrm{Pic}(R)$. It is easy to verify that α is an isomorphism.

(7.3) Definition of β and exactness at Pic(R).

The map β is induced by scalars extension $(I) \in \mathrm{Pic}(R) \longmapsto (S\otimes I)$. The exactness is clear.

(7.4) Definition of d and exactness at $H^0(S/R,\mathrm{Pic})$.

Let I be an invertible $S\otimes S$-module such that $[I] \in H^0(S/R,\mathrm{Pic})$. It means that there exists an $S\otimes S$-isomorphism $\phi: I_1 \to I_2$. Then $\phi_2^{-1}\phi_3\phi_1$ is an $S^{(3)}$- automorphism of I_{11} and hence given by a unit $u \in U\left(S^{(3)}\right)$. One verifies that u is a cocycle and defines $d(I) = [u] \in H^2(S/R,U)$. The map d clearly is the analogous for Pic of the Teichmüller map. The exactness is easy (modify §5!).

(7.5) Definition of γ.

The map γ is the map defined by Rosenberg and Zelinsky in $[RZ]_2$,p.339. We shall give here a description of γ using descent. The construction then is similar to the construction of α or of (5.1). Actually it is the translation for Amitsur cohomology of the crossed product construction for Galois cohomology.

Let P be the S-module $S\otimes S$ with S acting on the first factor. For any $t \in U\left(S^{(3)}\right)$ define an $S\otimes S$-isomorphism $f(t): P_1 = S\otimes S\otimes S \longrightarrow S\otimes S\otimes S = P_2$ by multiplying with t and switching the last two factors S, $f(t)(y) = \tau_1 ty$. For any $s \in U\left(S^{(2)}\right)$ denote the map $P \to P$ induced by left multiplication with s by $l(s)$.

(7.6) Proposition. 1) Let $f = f(t)$. The composition $f_2^{-1}f_3f_1: P_{11} \to P_{11}$ is the multiplication with $t_1t_2^{-1}t_3$.

2) If t is a cocycle, $f(t)$ induces a descent-datum

$$\phi(t)\colon S \otimes \mathrm{End}_S(P) \longrightarrow \mathrm{End}_S(P) \otimes S$$

and hence defines an Azumaya R-algebra $A(t) = \{x \in \mathrm{End}_S(P) \mid \phi(t)\varepsilon_1 x = \varepsilon_2 x\}$ such that $\eta\colon S \otimes A(T) \xrightarrow{\sim} \mathrm{End}_S(P)$. Moreover $A(t)$ contains S as maximal commutative subalgebra. If $t = 1$, then $A(1) = \mathrm{End}_R(S)$.

3) If t is a cocycle and $s \in U(S^{(2)})$, then $l(s)A(t\Delta_2 s)l(s^{-1}) = A(t)$.

4) If s and t are cocycles, then $l(t)A(ts) \otimes A(1)l(t^{-1}) = A(t) \otimes A(s)$.

5) If t is a cocycle, then $A(t)^o = A(t^{-1})$.

(7.7) <u>Corollary.</u> The map $t \to A(t)$ induces a well defined homomorphism

$$\gamma\colon H^2(S/R,U) \longrightarrow \mathrm{Br}(S/R) .$$

<u>Proof of the proposition.</u> 1) Since $f_1 = \tau_{11}t_1, f_3 = \tau_{14}t_4 = \tau_{14}\tau_{11}t_3\tau_{11}$ and $f_2 = \tau_{14}\tau_{12}t_2$, the result is evident.

2) The first part is clear since $t_1t_2^{-1}t_3 = t_4$ lies in the center of $\mathrm{End}_{S^{(3)}}(S \otimes S \otimes P)$. The algebra $S \otimes 1$ clearly lies in $A(t)$ and is maximal since $S \otimes S$ is maximal in $\mathrm{End}_S(P)$. The last assertion is obvious.

3) An element $x \in \mathrm{End}_S(P)$ lies in $l(s)^{-1}A(t)l(s)$ if and only if $l(s) \times l(s)^{-1} \in A(t)$, i.e. $\tau_1 t s_1 x_1 s_1^{-1} t^{-1} \tau_1 = s_3 x_2 s_3^{-1}$. Using that s_3 lies in the center of $\mathrm{End}_{S \otimes S}(S \otimes P)$ and that $\tau_1(s_2) = s_3$, one obtains $x_2 = \tau_1 t s_1 s_2^{-1} x_1 s_2 s_1^{-1} t^{-1} \tau_1 = \tau_1 t s_1 s_2^{-1} s_3 x_1 s_3^{-1} s_2 s_1^{-1} t^{-1} \tau_1$, hence $x \in A(t\Delta_2 s)$.

4) The proof of 4) is similar to the preceeding one. Let $f(s)$ and $f(t)\colon P_1 \to P_2$ be the maps induced by s and t . Then $f(s) \otimes_{S \otimes S} f(t)$ induces a descent-datum $\phi(s) \otimes_{S \otimes S} \phi(t)$ for $A(s) \otimes A(t)$. Explicitely $f(s) \otimes_{S \otimes S} f(t)$ is given by the composition $\tau_{12}t_2\tau_{14}s_4 = \tau_{12}\tau_{14}t_3s_4$. Similarly $f(st) \otimes_{S \otimes S} f(1)$ is given by $\tau_{12}\tau_{14}s_4t_4$. Let now $x \in \mathrm{End}_S(P \otimes_S P)$ be in $l(t^{-1})A(s) \otimes A(t)l(t)$. Then $l(t) \times l(t^{-1}) \in A(s) \otimes A(t)$ or $\tau_{12}\,\tau_{14}t_3s_4t_1x_1t_1^{-1}s_4^{-1}t_3^{-1}\tau_{14}\tau_{12} = t_4x_2t_4^{-1}$. Using that $\tau_{12}\tau_{14}(t_2) = t_4$ and that $t_1t_2^{-1}t_3 = t_4$ one obtains $x_2 = \tau_{12}\tau_{14}t_4s_4x_1s_4^{-1}t_4^{-1}\tau_{14}\tau_{12}$, i.e. $x \in A(st) \otimes A(1)$.

5) is easy.

(7.8) Exactness at $H^2(S/R,U)$.

If $\gamma(t) = 1$, choose an R-isomorphism $\sigma\colon A(t) \xrightarrow{\sim} \mathrm{End}_R(Q)$ and denote the composition $(1\otimes\sigma)\eta^{-1}\colon \mathrm{End}_S(P) \to \mathrm{End}_S(S\otimes Q)$ by ψ. Recall that $P = S\otimes S$ where S acts on the first factor and that η is the canonical splitting $S\otimes A(t) \to \mathrm{End}_S(P)$. By (3.4) ψ is induced by an S-isomorphism $h\colon P\otimes_S I \to S\otimes Q$, where $(I) \in \mathrm{Pic}(S)$. Consider the $S\otimes S$-isomorphism g defined by

$$\begin{tikzcd}
S\otimes S\otimes Q \arrow[r, "h_1"] \arrow[d, "\tau_1"'] & (S\otimes P\otimes_{S\otimes S} I_1 = S\otimes I\otimes S \arrow[d, "g"] \\
S\otimes Q\otimes S \arrow[r, "h_3"'] & (P\otimes S)\otimes_{S\otimes S} I_2 = I\otimes S\otimes S
\end{tikzcd}$$

The map g clearly verifies the descent condition (the descended module has to be Q!) which looks like

$$g_4 g_1 = \tau_{14} g_2 \colon S\otimes S\otimes I\otimes S \to I\otimes S\otimes S\otimes S .$$

Moreover g induces by conjugation the map $\psi_3^{-1}\tau_1\psi_1 = \eta_3\sigma_{13}\tau_1\sigma_{11}^{-1}\eta_1 = \eta_3\tau_1\eta_1^{-1} = \phi(t)$. Hence

$$g\colon (S\otimes P)\otimes_{S\otimes S} I_1 \longrightarrow (P\otimes S)\otimes_{S\otimes S} I_2$$

and

$$f(t)\colon S\otimes P \longrightarrow P\otimes S$$

induce the same map $\phi(t)$. By (3.4), there exists an isomorphism of $S\otimes S$-modules $\phi\colon I_1 \xrightarrow{\sim} I_2$ such that $(f(t)\otimes_{S\otimes S}\phi)g^{-1}$ belongs to the center of $\mathrm{End}_{S\otimes S}(P\otimes S)$. Let $(f(t)\otimes_{S\otimes S}\phi)g^{-1} = s \in U(S^{(2)})$. Replacing ϕ by ϕs^{-1}, one obtains that $g = f(t)\otimes_{S\otimes S}\phi$. Writing the descent condition for g and using (7.6) one obtains that $t_1 t_2^{-1} t_3 \phi_2^{-1}\phi_3\phi_1$ is the identity of $S\otimes S\otimes I\otimes S$. Since $t_1 t_2^{-1} t_3$ is the multiplication by t_4, $\phi_2^{-1}\phi_3\phi_1$ is just the multiplication by t^{-1}, hence $d(I^{-1}) = [t]$.

(7.9) Definition of δ and exactness at $\mathrm{Br}(S/R)$.

Let $[A] \in \mathrm{Br}(S/R)$ and let $\sigma\colon S\otimes A \to \mathrm{End}_S(Q)$ be a splitting of A. Denote the composition $\sigma_3\tau_1\sigma_1^{-1}\colon \mathrm{End}_{S\otimes S}(S\otimes Q) \to \mathrm{End}_{S\otimes S}(Q\otimes S)$ by ϕ. Use (3.4) again:

ϕ is induced by some $f: (S\otimes Q)\otimes_{S\otimes S} I \to Q\otimes S$, $(I) \in \mathrm{Pic}(S\otimes S)$. Since $\phi_2 = \phi_3\phi_1$, $I_2 \cong I_3 \otimes_{S(3)} I_1$, and one can put $\delta(A) = [I] \in H^1(S/R, \mathrm{Pic})$. If $\delta(A) = [I] = 1$, then $I \cong (S\otimes J)\otimes_{S\otimes S}(J\otimes S)^{-1}$ and one can replace f by a map $g: Q_1 \to Q_2$ to induce ϕ . Then $g_2^{-1}g_3g_1$ is a unit $t \in U\left(S^{(3)}\right)$. Consider now

$$\sigma \otimes_S \eta(t^{-1}): S\otimes A\otimes A(t^{-1}) \xrightarrow{\sim} \mathrm{End}_S(Q\otimes_S P)$$

where $\eta(t^{-1})$ denotes the canonical splitting of $A(t^{-1})$ and $P = S\otimes S$ as usual. Then $\phi\otimes_{S\otimes S}\phi(t^{-1})$ is induced by $h = g\otimes_{S\otimes S} f(t^{-1})$ and $h_2^{-1}h_3h_1$ is the multiplication by the element $t\cdot t^{-1} = 1$ of the center of $\mathrm{End}_{S(3)}(S\otimes S\otimes Q\otimes_S P)$. The map h is therefore a descent-datum for $Q\otimes_S P$. Let M be the descended module. Using uniqueness of descent it follows easily that $A\otimes A(t^{-1}) \cong \mathrm{End}_R(M)$, or $[A] = [A(t)]$ in $\mathrm{Br}(S/R)$.

(7.10) Definition of ρ and exactness at $H^1(S/R,\mathrm{Pic})$.

If $[I] \in H^1(S/R,\mathrm{Pic})$, choose $(I) \in \mathrm{Pic}(S\otimes S)$ and an $S^{(3)}$-isomorphism $f: I_1\otimes_{S(3)} I_3 \xrightarrow{\sim} I_2$. Using the cosemi-simplicial identities, one can consider the composition $f_4^{-1}f_2^{-1}f_3f_1$ which gives an automorphism of $I_{11}\otimes_{S(4)} I_{31}\otimes_{S(4)} I_{33}$. Therefore $f_4^{-1}f_2^{-1}f_3f_1$ is a unit u of $S^{(4)}$ and one defines $\rho(I) = [u] \in H^3(S/R,U)$. The reader should verify that u is really a cocycle ! If $\rho(I) = [u] = 1$, then $u = s_1s_2^{-1}s_3s_4^{-1}$. Replacing f by fs^{-1} , one can suppose that $f_2^{-1}f_1f_3 = f_4$. Denote the S-module I with S acting on the first factor of $S\otimes S$ by Q and consider the composition

$$g: (S\otimes Q)\otimes_{S\otimes S} I = (S\otimes Q)\otimes_{S(3)} I_3 \xrightarrow{f} I_2 \xrightarrow{\tau_1} Q\otimes S .$$

As it should be, we obtain $g_2^{-1}g_3g_1 = f_2^{-1}f_3f_1$ since $g_1 = \tau_{11}f_1$, $g_3 = \tau_{14}f_4 = \tau_{14}\tau_{11}f_3\tau_{11}$ and $g_2 = \tau_{14}\tau_{12}f_2$. But

$$f_4: (S\otimes S\otimes Q)\otimes_{S(4)} I_{14}\otimes_{S(4)} I_{34} \longrightarrow (S\otimes S\otimes Q\otimes_{S(4)} I_{24}$$

induces the identity of $\mathrm{End}_{S(3)}(S\otimes S\otimes Q)$. Therefore the conjugation with g gives a descent-datum which defines an Azumaya algebra A. Clearly $\delta(A) = [I]$.

8. The image of the Teichmüller map.

Let $R \subset S$ be a finite projective extension satisfying condition I or II of the introduction and let T be a finite projective extension of S. Consider $S \otimes T$ as S-algebra with action on the first factor and denote the subgroup of $Br(S)$ consisting of classes of normal S-algebras split by $S \otimes T$ by $Br(S\otimes T/S)^o$.

(8.1) Theorem. The sequence

$$0 \to Br(S/R) \xrightarrow{i} Br(T/R) \xrightarrow{\alpha} Br(S\otimes T/S)^o \xrightarrow{d} H^3(S/R,U) \xrightarrow{\beta} H^3(T/R,U) ,$$

where α is obtained by scalars extension, d is the Teichmüller map and β is induced by the injection $S \to T$, is exact.

Proof. It is evident that $\alpha i = 1$ and $d\alpha = 1$ is a consequence of (5.3). Exactness at $Br(T/R)$ is clear. Let $[A] \in Br(S\otimes T/S)^o$ with $d(A) = 1$ in $H^3(S/R,U)$. By (5.3), $[A] = [S\otimes B]$ for some $[B] \in Br(R)$. Since $[A\otimes_S(S\otimes T)] = [B\otimes S\otimes T]$, we have $[B\otimes S\otimes T] = 1$ in $Br(S\otimes T)$, hence $[B\otimes T] = 1$ in $B(T)$ by the multiplication map $S\otimes T \to T$. Therefore $[B] \in Br(T/R)$ and $\alpha[B] = [A]$. Finally we prove the exactness at $H^3(S/R,U)$. Denote the injection $S \to S\otimes T$ of S in the first factor by ρ and by ρ_* all possible induced maps. Since $\alpha\rho_* = \rho_* d$, the map $\rho_* d$ sends $Br(S\otimes T/S)^o$ to zero of $H^3(S\otimes T/R,U)$ by (4.3). Composing with the map $H^3(S\otimes T/R) \to H^3(T/R)$ induced by the multiplication, shows that $\beta d = 0$. Let now $[u] \in H^3(S/R,U)$ with $\beta[u] = 1$. Choose $v \in U\left(S^{(3)}\right)$ such that $u = v_1 v_2^{-1} v_3$ in $T^{(4)}$ (we identify $S^{(4)}$ with its image in $T^{(4)}$). Let P be the $S\otimes T$-module $S\otimes T\otimes T$ with action on the first two factors and define an $S\otimes T\otimes T$-isomorphism

$$f\colon (S\otimes T\otimes_S P = S\otimes T\otimes T\otimes T \to S\otimes T\otimes T\otimes T = P\otimes_S(S\otimes T)$$

by $f(x) = \tau_{11} u v_1^{-1} x$. The map f is not a descent-datum from $S\otimes T$ to S for P but the obstruction is given by $f_3^{-1} f_4 f_2$. A by now routine calculation shows that $f_3^{-1} f_4 f_2$ is the multiplication by

$(uv_1^{-1})_3^{-1}(uv_1^{-1})_4(uv_1^{-1})_2 = u_3^{-1}u_4u_2v_{13}v_{14}^{-1}v_{12}^{-1} = u_5u_1(v_1^{-1}v_2v_3^{-1})_1 = u_5$ which belongs to the center of $\mathrm{End}_{S\otimes T^{(3)}}\left(S\otimes T^{(4)}\right)$. Hence f induces a descent-datum on $\mathrm{End}_{S\otimes T}(P)$. The descended algebra is an Azumaya S-algebra A split by $S\otimes T$. Using the notation of (7.6) one can write $A = A(uv_1^{-1})$. We now show that A is normal. Consider $S\otimes A = A(u_1v_{11}^{-1})$ and $A\otimes S = A(u_2v_{12}^{-1})$ as subalgebras of $\mathrm{End}_{S\otimes S\otimes T}(S\otimes S\otimes T\otimes T)$. By (7.6.3) (for the extension $S\otimes S\otimes T/S\otimes S$) one has $l(u)A(u_1v_{11}^{-1}\Delta_2u)l(u)^{-1} = A(u_1v_{11}^{-1})$. But in our context $\Delta_2u = u_3u_4^{-1}u_5$, so that $u_1v_{11}^{-1}\Delta_2u = u_2v_{11}^{-1} = u_2v_{12}^{-1}$ and finally $S\otimes A = l(u)(A\otimes S)l(u)^{-1}$ in $\mathrm{End}_{S\otimes S\otimes T}(S\otimes S\otimes T\otimes T)$. Therefore an $S\otimes S$-isomorphism $\phi\colon A_1 \to A_2$ is given by the conjugation with u. The composition $\phi_2^{-1}\phi_3\phi_1$ then is induced by the conjugation with $u_1u_2^{-1}u_3 = u_4u_5^{-1}$. Since u_5 is a central element of $\mathrm{End}_{S^{(3)}\otimes T}\left(S^{(3)}\otimes T\otimes T\right)$, $\phi_2^{-1}\phi_3\phi_1$ is already induced by u_4 and we can choose $g = u_4$ for the construction of the Teichmüller cocycle. Then $d(g) = \phi_{11}^{-1}(g_4^{-1})g_2^{-1}g_3g_1 = u_{11}u_{44}^{-1}u_{11}^{-1}u_{42}^{-1}u_{43}u_{41} = (u_1u_2^{-1}u_3u_4^{-1})_5 = u_{55}$. But u_{55} as element of the center of $\mathrm{End}_{S^{(4)}\otimes T}\left(S^{(4)}\otimes T\otimes T\right)$ comes already from the center of $S^{(4)}\otimes A$, namely from u, hence $[u] = d(A)$.

(8.2) Proposition. The multiplication $S\otimes T \to T$ in T induces an equality $Br(S\otimes T/S) = Br(T/S)$ in the following cases:

i) the extension S/R verifies condition II of the introduction

ii) the extensions S/R and T/S are finite field extensions, S/R is separable and T/S is normal.

Proof. i) Let $[A] \in Br(T/S)$, i.e. $[A\otimes_S T] = 1$ in $Br(T)$. Then $[A\otimes_S(S\otimes T)] = [A\otimes T]$ lies in the kernel of $Br(S\otimes T) \to Br(T)$ which is zero by [RS] since $S\otimes T \to T$ has nilpotent kernel. It then follows that $[A] \in Br(S\otimes T/S)$.

ii) It follows from (7.1) that $Br(T/S) \cong H^2(T/S)$ and $Br(S\otimes T/S) \cong H^2(S\otimes T/S)$ since everything is semilocal. The result then is a consequence of [A], Th.5.12.

(8.3) In the cases where (8.2) applies, there is an exact sequence

$0 \to \mathrm{Br}(S/R) \to \mathrm{Br}(T/R) \to \mathrm{Br}(T/S)^{o} \to H^{3}(S/R) \to H^{3}(T/R)$.

Proof. If $\mathrm{Br}(S \otimes T/S) \xrightarrow{\simeq} \mathrm{Br}(T/S)$, then $\mathrm{Br}(S \otimes T/S)^{o} \cong \mathrm{Br}(T/S)$.

(8.4) Remark. In the Galois case, (8.3) corresponds to the Eilenberg-MacLane results [EM]. In case II, i.e. if the kernel of the multiplication $S \otimes S \to S$ is nilpotent, $\mathrm{Br}(T/S)^{o} = \mathrm{Br}(T/S)$ by (3.7) and we obtain the sequence

(8.5) $0 \to \mathrm{Br}(S/R) \to \mathrm{Br}(T/R) \to \mathrm{Br}(T/S) \xrightarrow{d} H^{3}(S/R) \to H^{3}(T/R)$

which gives an explicit description of various exact sequences of Amitsur, Rosenberg-Zelinsky and Yuan [A], $[RZ]_3$, [Y] . If Berkson's theorem applies, for example if R has characteristic p , p a prime and $S^{q} \subset R$ for some power q of p , then

(8.6) $0 \to \mathrm{Br}(S/R) \to \mathrm{Br}(T/R) \to \mathrm{Br}(T/S) \to 0$

is exact. This result was proved by Yuan for purely unseparable ring extensions of exponent one (see [Y]) .

(8.7) Remark. The sequence (8.6) is not equivalent to (5.5) since it is not known if an Azumaya algebra can always be split by a finite projective extension.

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