# TRIALITY AND ÉTALE ALGEBRAS

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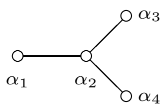
*Dedicated with great friendship to R. Parimala at the occasion of her* $60^{th}$ *birthday*

ABSTRACT. Trialitarian automorphisms are related to automorphisms of order 3 of the Dynkin diagram of type $D_4$. Octic étale algebras with trivial discriminant, containing quartic subalgebras, are classified by Galois cohomology with value in the Weyl group of type $D_4$. This paper discusses triality for such étale extensions.

## 1. INTRODUCTION

Among Dynkin diagrams, most of them admit at most automorphisms of order two, which are related to duality in algebra and geometry. The Dynkin diagram of $D_4$




is special, in the sense that it admits automorphisms of order 3. Algebraic and geometric objects related to $D_4$ are of particular interest as they also usually admit exceptional automorphisms of order 3, which are called trialitarian. For example the special projective orthogonal group $\mathrm{PGO}_8^+$ or the simply connected group $\mathrm{Spin}_8$ admit outer automorphisms of order 3. As already observed by E. Cartan, [5], the Weyl group $W(D_4) = \mathfrak{S}_2^3 \rtimes \mathfrak{S}_4$ of $\mathrm{Spin}_8$ or of $\mathrm{PGO}_8^+$ similarly admits trialitarian automorphisms. Let $F$ be a field and let $F_s$ be a separable closure of $F$. The Galois cohomology set $H^1\big(\Gamma, W(D_4)\big)$, where $\Gamma$ is the absolute Galois group $\mathrm{Gal}(F_s/F)$, classifies étale extensions $S/S_0$ where $S$ has dimension 8, $S_0$ dimension 4 and $S$ has trivial discriminant. Thus triality associates to such an extension $S/S_0$ two extensions $S'/S'_0$ and $S''/S''_0$, of the same kind, so that the triple $(S/S_0, S'/S'_0, S''/S''_0)$ is in triality. This paper is devoted to the study of such triples of étale algebras. It grew out of a study, in the spirit of [16], of Severi-Brauer varieties over the " field of one element", [13], which is in preparation (see also [17]).

In Part 2 and 3 we describe some basic constructions on étale algebras and finite $\Gamma$-sets. Some results are well-known, other were taken from [13], like the Clifford construction. In Part 4 we present different approaches for triality in connection with the Weyl group $W(D_4)$. We give in Table 1 a list of conjugacy classes of subgroups of $W(D_4)$ with a description of the triality action. We specially discuss subgroups of $W(D_4)$ which are fixed under triality. We then view triality as a way to create resolvents and give explicit formulae for polynomials. Finally we give in this section results of Serre on Witt invariants of $W(D_4)$. The paper [10] was

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a source of inspiration for optic fields. The last section sketches the definition of trialitarian étale algebras, in analogy with trialitarian central simple algebras, as introduced in [11].

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## 2. Étale algebras and Γ-sets

Throughout most of this work, $F$ is an arbitrary field. We denote by $F_s$ a separable closure of $F$ and by $\Gamma$ the absolute Galois group $\Gamma = \mathrm{Gal}(F_s/F)$, which is a profinite group.

A finite-dimensional commutative $F$-algebra $S$ is called *étale* (over $F$) if $S \otimes_F F_s$ is isomorphic to the $F_s$-algebra $F_s^n = F_s \times \cdots \times F_s$ ($n$ factors) for some $n \geq 1$. We refer to [11, §18.A] for various equivalent characterizations of étale $F$-algebras. In particular, étale $F$-algebras are the direct products of finite separable field extensions of $F$. These algebras (with $F$-algebra homomorphisms) form a category $\acute{\mathsf{E}}\mathsf{t}_F$ in which finite direct products and finite direct sums (= tensor products) are defined.

Finite sets with a continuous left action of $\Gamma$ (for the discrete topology) are called (finite) $\Gamma$-sets. They form a category $\mathsf{Set}_\Gamma$ whose morphisms are the $\Gamma$-equivariant maps. Finite direct products and direct sums (= disjoint unions) are defined in this category. We denote by $|X|$ the cardinality of any finite set $X$.

For any étale $F$-algebra $S$ of dimension $n$, the set of $F$-algebra homomorphisms

$$\mathbf{X}(S) = \mathrm{Hom}_{F\text{-alg}}(S, F_s)$$

is a $\Gamma$-set of $n$ elements since $\Gamma$ acts on $F_s$. Conversely, if $X$ is a $\Gamma$-set of $n$ elements, the $F$-algebra $\mathbf{M}(X)$ of $\Gamma$-equivariant maps $X \to F_s$ is an étale $F$-algebra of dimension $n$,

$$\mathbf{M}(X) = \{f\colon X \to F_s \mid \gamma\big(f(x)\big) = f({}^\gamma x) \text{ for } \gamma \in \Gamma,\, x \in X\}.$$

As first observed by Grothendieck, there are canonical isomorphisms

$$\mathbf{M}\big(\mathbf{X}(S)\big) \cong S, \qquad \mathbf{X}\big(\mathbf{M}(X)\big) \cong X,$$

so that the functors $\mathbf{M}$ and $\mathbf{X}$ define an anti-equivalence of categories

$$\mathsf{Set}_\Gamma \equiv \acute{\mathsf{E}}\mathsf{t}_F \tag{2.1}$$

(see [6, Proposition (4.3), p.25] or [11, (18.19)]). Under this anti-equivalence, the cardinality of $\Gamma$-sets corresponds to the dimension of étale $F$-algebras. The disjoint union $\sqcup$ in $\mathsf{Set}_\Gamma$ corresponds to the product $\times$ in $\acute{\mathsf{E}}\mathsf{t}_F$ and the product $\times$ in $\mathsf{Set}_\Gamma$ to the tensor product $\otimes$ in $\acute{\mathsf{E}}\mathsf{t}_F$. For any integer $n \geq 1$, we let $\acute{\mathsf{E}}\mathsf{t}_F^n$ denote the groupoid[1] whose objects are $n$-dimensional étale $F$-algebras and whose morphisms are $F$-algebra isomorphisms, and $\mathsf{Set}_\Gamma^n$ the groupoid of $\Gamma$-sets with $n$ elements. The anti-equivalence (2.1) restricts to an anti-equivalence $\mathsf{Set}_\Gamma^n \equiv \acute{\mathsf{E}}\mathsf{t}_F^n$. The split étale

[1] A groupoid is a category in which all the morphisms are isomorphisms.

algebra $F^n$ corresponds to the $\Gamma$-set $\boldsymbol{n}$ of $n$ elements with trivial $\Gamma$-action. Étale algebras of dimension 2 are also called *quadratic* étale algebras.
A morphism of $\Gamma$-sets $Y \xleftarrow{\pi} Z$ is called a $\Gamma$-*covering* if the number of elements in each fiber $y^{\pi^{-1}} \subset Z$ does not depend on $y \in Y$. This number is called the *degree* of the covering. For $n$, $d \geq 1$ we let $\mathsf{Cov}_\Gamma^{d/n}$ denote the groupoid whose objects are coverings of degree $d$ of a $\Gamma$-set of $n$ elements and whose morphisms are isomorphisms of $\Gamma$-coverings.
A homomorphism $S \xrightarrow{\varepsilon} T$ of étale $F$-algebras is said to be an *extension of degree $d$ of étale algebras* if $\varepsilon$ endows $T$ with a structure of a free $S$-module of rank $d$. This corresponds under the anti-equivalence (2.1) to a *covering of degree $d$*:

$$\mathbf{X}(S) \xleftarrow{\mathbf{X}(i)} \mathbf{X}(T)$$

(see [13]). Let $\acute{\mathsf{E}}\mathsf{tex}_F^{d/n}$ denote the groupoid of étale extensions $S \xrightarrow{\varepsilon} T$ of degree $d$ of $F$-algebras with $\dim_F S = n$ (hence $\dim_F T = dn$). From (2.1) we obtain an anti-equivalence of groupoids

$$\acute{\mathsf{E}}\mathsf{tex}_F^{d/n} \equiv \mathsf{Cov}_\Gamma^{d/n}.$$

The $\Gamma$-covering with trivial $\Gamma$-action

$$\boldsymbol{d/n}: \quad \boldsymbol{n} \xleftarrow{p_1} \boldsymbol{n} \times \boldsymbol{d} \tag{2.2}$$

where $p_1$ is the first projection corresponds to the split extension $F^n \to (F^d)^n$.
Of particular importance in the sequel are coverings of degree 2, which are also called *double coverings*. Each such covering $Y \xleftarrow{\pi} Z$ defines a canonical automorphism $\sigma\colon Z \to Z$ of order 2, which interchanges the elements in each fiber of $\pi$. Clearly, this automorphism has no fixed point. Conversely, if $Z$ is any $\Gamma$-set and $\sigma\colon Z \to Z$ is an automorphism of order 2 without fixed point, the set of orbits

$$Z/\sigma = \bigl\{\{z, z^\sigma\} \mid z \in Z\bigr\}$$

is a $\Gamma$-set and the canonical map $(Z/\sigma) \leftarrow Z$ is a double covering. We call *involution* of a $\Gamma$-set with an even number of elements any automorphism of order 2 without fixed points.
Let $\sigma\colon S \to S$ be an automorphism of order 2 of an étale $F$-algebra $S$, and let $S^\sigma \subset S$ denote the $F$-sub-algebra of fixed elements, which is necessarily étale. The following conditions are equivalent (see [13]):

(a) the inclusion $S^\sigma \to S$ is a quadratic étale extension of $F$-algebras;
(b) the automorphism $\mathbf{X}(\sigma)$ is an involution on $\mathbf{X}(S)$.

We say under these equivalent conditions that the automorphism $\sigma$ is an *involution* of the étale $F$-algebra $S$.

**Basic constructions on $\Gamma$-sets.** We recall from [11, §18] and [12, §2.1] the construction of the discriminant $\Delta(X)$ of a $\Gamma$-set $X$ with $|X| = n \geq 2$: consider the set of arrays of the elements in $X$:

$$\Sigma_n(X) = \{(x_1, \ldots, x_n) \mid X = \{x_1, \ldots, x_n\}\}.$$

This $\Gamma$-set carries an obvious transitive (right) action of the symmetric group $\mathfrak{S}_n$. The *discriminant* $\Delta(X)$ is the set of orbits under the alternating group $\mathfrak{A}_n$:

$$\Delta(X) = \Sigma_n(X)/\mathfrak{A}_n.$$

It is a Γ-set of two elements, so Δ is a functor

$$\Delta\colon \mathsf{Set}_\Gamma^n \to \mathsf{Set}_\Gamma^2.$$

For any covering $Z_0 \xleftarrow{\pi} Z$ of degree 2 with $|Z_0| = n$, (hence $|Z| = 2n$), we consider the set of (not necessarily Γ-equivariant) sections of $\pi$:

$$C(Z/Z_0) = \big\{\{z_1, \dots, z_n\} \subset Z \mid \{z_1^\pi, \dots, z_n^\pi\} = Z_0\big\}.$$

It is a Γ-set with $2^n$ elements, so $C$ is a functor

$$C\colon \mathsf{Cov}_\Gamma^{2/n} \to \mathsf{Set}_\Gamma^{2^n},$$

called the *Clifford functor* (see [13]). The Γ-set $C(Z/Z_0)$ is equipped with a canonical surjective morphism

$$\Delta(Z) \xleftarrow{\delta} C(Z/Z_0), \tag{2.3}$$

which is defined in [12, §2.2] as follows: let $\sigma\colon Z \to Z$ be the involution canonically associated to the double covering $Z_0 \xleftarrow{\pi} Z$, so the fiber of $z^\pi$ is $\{z, z^\sigma\}$ for each $z \in Z$; then $\delta$ maps each section $\{z_1, \dots, z_n\}$ to the $\mathfrak{A}_{2n}$-orbit of the $2n$-tuple $(z_1, \dots, z_n, z_1^\sigma, \dots, z_n^\sigma)$,

$$\{z_1, \dots, z_n\}^\delta = (z_1, \dots, z_n, z_1^\sigma, \dots, z_n^\sigma)^{\mathfrak{A}_{2n}}.$$

Note that the canonical involution $\sigma$ induces an involution $\underline{\sigma}$ on $C(Z/Z_0)$, which maps each section $\omega$ to its complement $Z\backslash\omega$. We may view $C(Z/Z_0)$ as a covering of degree 2 of the set of orbits $C(Z/Z_0)/\underline{\sigma}$, and thus consider the Clifford construction as a functor

$$C\colon \mathsf{Cov}_\Gamma^{2/n} \to \mathsf{Cov}_\Gamma^{2/2^{n-1}}. \tag{2.4}$$

**Proposition 2.5.** *For sections $\omega$, $\omega' \in C(Z/Z_0)$, we have $\omega^\delta = (\omega')^\delta$ if and only if $|\omega\cap\omega'| \equiv n \bmod 2$. Moreover, denoting by $\iota$ the nontrivial automorphism of $\Delta(Z)$, we have*

$$\underline{\sigma}\circ\delta = \begin{cases}\delta & \text{if } n \text{ is even,}\\ \delta\circ\iota & \text{if } n \text{ is odd.}\end{cases}$$

*Proof.* Let $\omega = \{z_1, \dots, z_n\}$ and $\omega' = \{z_1, \dots, z_r, z_{r+1}^\sigma, \dots, z_n^\sigma\}$, so $r = |\omega\cap\omega'|$,

$$\omega^\delta = (z_1, \dots, z_n, z_1^\sigma, \dots, z_n^\sigma)^{\mathfrak{A}_{2n}} \qquad \text{and}$$
$$(\omega')^\delta = (z_1, \dots, z_r, z_{r+1}^\sigma, \dots, z_n^\sigma, z_1^\sigma, \dots, z_r^\sigma, z_{r+1}, \dots, z_n)^{\mathfrak{A}_{2n}}.$$

The permutation $\sigma'$ that interchanges $z_i$ and $z_i^\sigma$ for $i = r+1, \dots, n$ satisfies

$$(z_1, \dots, z_n, z_1^\sigma, \dots, z_n^\sigma)^{\sigma'} = (z_1, \dots, z_r, z_{r+1}^\sigma, \dots, z_n^\sigma, z_1^\sigma, \dots, z_r^\sigma, z_{r+1}, \dots, z_n);$$

it is in $\mathfrak{A}_{2n}$ if and only if $n-r$ is even, which means $|\omega\cap\omega'| \equiv n \bmod 2$. For $\omega' = \omega^{\underline{\sigma}}$ the complement of $\omega$ we have $|\omega\cap\omega^{\underline{\sigma}}| = 0$, hence $\omega^{\underline{\sigma}\delta} = \omega^\delta$ if and only if $n \equiv equiv.0 \bmod 2$. □

**Oriented $\Gamma$-sets.** An *oriented $\Gamma$-set* is a pair $(Z, \partial_Z)$ where $Z$ is a $\Gamma$-set and $\partial_Z$ is a fixed isomorphism of $\Gamma$-sets $\Delta(Z) \xrightarrow{\sim} \mathbf{2}$. In particular the $\Gamma$-action on $\Delta(Z)$ is trivial. There are two possible choices for $\partial_Z$. A choice is an *orientation of $Z$*. Oriented $\Gamma$-sets with $n$ elements form a groupoid $(\mathsf{Set}_\Gamma^n)^+$ whose morphisms are isomorphisms $f\colon Z_1 \xrightarrow{\sim} Z_2$ such that $\Delta(f)\circ\partial_{Z_1} = \partial_{Z_2}$. Similarly *oriented coverings* are pairs $(Z_0 \leftarrow Z, \partial_Z)$ where $Z_0 \leftarrow Z$ is a $\Gamma$-covering and $\partial_Z$ is an orientation of $Z$. We denote by $(\mathsf{Cov}_\Gamma^{d/n})^+$ the groupoid of oriented coverings of degree $d$ of $\Gamma$-sets with $n$ elements. Changing the orientation through the twist $\iota\colon \mathbf{2} \to \mathbf{2}$ defines an involute functor

$$\kappa\colon (\mathsf{Cov}_\Gamma^{d/n})^+ \to (\mathsf{Cov}_\Gamma^{d/n})^+.$$

**Proposition 2.6.** *If $n$ is even the functor $C\colon \mathsf{Cov}_\Gamma^{2/n} \to \mathsf{Cov}_\Gamma^{2/2^{n-1}}$ of* (2.4) *restricts to a pair of functors*

$$C_1,\, C_2\colon (\mathsf{Cov}_\Gamma^{2/n})^+ \to \mathsf{Cov}_\Gamma^{2/2^{n-2}}.$$

*Moreover two sections $\omega$ and $\omega'$ of the covering $Z/Z_0$ lie in the same set $C_1(Z/Z_0)$ or $C_2(Z/Z_0)$ if and only if $|\omega\cap\omega'| \equiv 0 \mod 2$.*

*Proof.* Let $Z/Z_0$ be a $2/n$-covering. Proposition 2.5 implies that the covering $\delta\colon \Delta(Z) \leftarrow C(Z/Z_0)$ factors through $C(Z/Z_0)/\underline{\sigma}$, where $\underline{\sigma}$ is the canonical involution of $C(Z/Z_0)$:

$$\Delta(Z) \leftarrow C(Z/Z_0)/\underline{\sigma} \leftarrow C(Z/Z_0).$$

Thus the choice of an isomorphism $\Delta(Z) \xrightarrow{\sim} \mathbf{2}$ determines a unique decomposition

$$C(Z/Z_0) = C_1(Z/Z_0) \sqcup C_2(Z/Z_0)$$

as $\Gamma$-sets and $\underline{\sigma}$ restricts to involutions of $C_1(Z/Z_0)$ and $C_2(Z/Z_0)$. The last claim follows from the proof of Proposition 2.5. □

We call the two functors $C_1$ and $C_2$ the *spinor functors*.

**Basic constructions on étale algebras.** We now consider analogues of the functors $\Delta$ and $C$ for étale algebras and étale extensions.

For $S$ an étale $F$-algebra of dimension $n \geq 2$, the discriminant $\Delta(S)$ is a quadratic étale $F$-algebra such that

$$\mathbf{X}\big(\Delta(S)\big) = \Delta\big(\mathbf{X}(S)\big).$$

We thus have a functor

$$\Delta\colon \mathsf{\acute{E}t}_F^n \to \mathsf{\acute{E}t}_F^2 \qquad \text{for } n \geq 2.$$

If the field $F$ has characteristic different from 4, it is usual to represent $\Delta(S)$ as $F[x]/\big(x^2 - \mathrm{Disc}(S)\big)$, $\mathrm{Disc}(S) \in F^\times$, and the class of $\mathrm{Disc}(S)$ in $F^\times/(F^\times)^2$ is the usual discriminant. We refer to [11] and [12] for details.

Let $S \xrightarrow{\varepsilon} T$ be an étale extension of degree 2 of (étale) $F$-algebras, with $\dim_F S = n$, $\dim_F T = 2n$. In [12][2] we define an étale $F$-algebra $C(T/S)$ such that

$$\mathbf{X}\big(C(T/S)\big) = C\big(\mathbf{X}(T)/\,\mathbf{X}(S)\big).$$

**Example 2.7.** If $n = 1$, we have $C(T/S) = T$ and if $T_1/S_1$ and $T_2/S_2$ are étale extensions of degree 2, there is a canonical isomorphism

$$P\colon C(T_1 \times T_2/S_1 \times S_2) \xrightarrow{\sim} C(T_1/S_1) \otimes C(T_2/S_2).$$

[2]The notation $\Omega$ is used for $C$ in [12].

We call the $2^n$-dimensional algebra $C(T/S)$ the *Clifford algebra* of $T/S$. It admits a canonical involution $\underline{\sigma}$. If $\dim_F S$ is even $\underline{\sigma}$ is the identity on $\Delta(T)$. The canonical morphism $\delta$ of (2.3)

$$\delta\colon C\big(\mathbf{X}(T)/\,\mathbf{X}(S)\big) \to \Delta\big(\mathbf{X}(T)\big)$$

yields a canonical $F$-algebra homomorphism which we again denote by $\delta$,

$$\delta\colon \Delta(T) \to C(T/S),$$

so that $C(T/S)$ is an étale extension of degree $2^{n-1}$ of a quadratic étale $F$-algebra.

**Oriented étale algebras.** As for oriented $\Gamma$-sets we define *oriented étale algebras* as pairs $(S, \partial_S)$ where $S$ is an étale algebra and $\partial_S\colon \Delta(S) \xrightarrow{\sim} F \times F$ is an isomorphism of $F$-algebras. *Oriented extensions of étale algebras* are pairs $(S/S_0, \partial_S)$ where $S/S_0$ is an extension of étale algebras and $\partial_S\colon \Delta(S) \xrightarrow{\sim} F \times F$ is an isomorphism of $F$-algebras. We have corresponding groupoids $(\mathsf{\acute{E}t}_F^n)^+$, $(\mathsf{\acute{E}tex}_F^{d/n})^+$ and anti-equivalences

$$(\mathsf{Set}_\Gamma^n)^+ \equiv (\mathsf{\acute{E}t}_F^n)^+ \text{ and } (\mathsf{Cov}_\Gamma^{d/n})^+ \equiv (\mathsf{\acute{E}tex}_F^{d/n})^+.$$

Switching the orientation induces an involutive functor $\kappa$ on these groupoids.
The Clifford functor $C$ restricts to a pair of *spinor functors*

$$C_1, C_2\colon (\mathsf{\acute{E}tex}_F^{2/n})^+ \to \mathsf{\acute{E}tex}_F^{2/2^{n-2}} \tag{2.8}$$

if $n$ is even.

**Remark 2.9.** The terminology used above owes its origin to the fact that the Clifford functor is related to the theory of Clifford algebras in the framework of quadratic forms and central simple algebras with involution. We refer to [13] for details and more properties of the Clifford construction.

## 3. Cohomology

For any integer $n \geq 1$, we consider the $\Gamma$-set $\boldsymbol{n} = \{1, \dots, n\}$ with the trivial $\Gamma$-action and let $\mathfrak{S}_n$ denote the symmetric group on $\boldsymbol{n}$, i.e., the automorphism group of $\boldsymbol{n}$,

$$\mathfrak{S}_n = \operatorname{Aut}(\boldsymbol{n}).$$

Recall from [11, §28.A] that the cohomology set $H^1(\Gamma, \mathfrak{S}_n)$ (for the trivial action of $\Gamma$ on $\mathfrak{S}_n$) is the set of continuous group homomorphisms $\Gamma \to \mathfrak{S}_n$ up to conjugation. Letting $\operatorname{Iso}(\mathsf{Set}_\Gamma^n)$ denote the set of isomorphism classes in $\mathsf{Set}_\Gamma^n$, we have a canonical bijection of pointed sets

$$\operatorname{Iso}(\mathsf{Set}_\Gamma^n) \xrightarrow{\sim} H^1(\Gamma, \mathfrak{S}_n). \tag{3.1}$$

Cohomology sets can also be used to describe isomorphism classes of $\Gamma$-coverings: for any integers $n$, $d \geq 1$, the group of automorphisms of the $\Gamma$-covering with trivial $\Gamma$-action $\boldsymbol{d}/\boldsymbol{n}$ is the wreath product (of order $(d!)^n n!$)

$$\operatorname{Aut}(\boldsymbol{d}/\boldsymbol{n}) = \mathfrak{S}_d \wr \mathfrak{S}_n \quad (= \mathfrak{S}_d^n \rtimes \mathfrak{S}_n).$$

The same construction as above yields a canonical bijection

$$\operatorname{Iso}(\mathsf{Cov}_\Gamma^{d/n}) \xrightarrow{\sim} H^1(\Gamma, \mathfrak{S}_d \wr \mathfrak{S}_n), \tag{3.2}$$

where the $\Gamma$-action on $\mathfrak{S}_d \wr \mathfrak{S}_n$ is trivial; see [12, §4.2]. The automorphism group of the oriented $\Gamma$-covering $(\boldsymbol{d}/\boldsymbol{n}, \partial_{\boldsymbol{n}\times\boldsymbol{d}})$ is the group

$$(\mathfrak{S}_d \wr \mathfrak{S}_n)^+ = (\mathfrak{S}_d \wr \mathfrak{S}_n) \cap \mathfrak{A}_{dn}.$$

As a consequence we have

$$\operatorname{Iso}\big((\mathsf{Cov}_\Gamma^{d/n})^+\big) \xrightarrow{\sim} H^1\big(\Gamma, (\mathfrak{S}_d \wr \mathfrak{S}_n)^+\big), \tag{3.3}$$

where the $\Gamma$-action on $(\mathfrak{S}_d \wr \mathfrak{S}_n)^+$ is trivial. We now assume that $\Gamma$ is the absolute Galois group $\Gamma = \operatorname{Gal}(F_s/F)$ of a field $F$. The anti-equivalence $\mathsf{Set}_\Gamma^n \equiv \acute{\mathsf{E}}\mathsf{t}_F^n$ and the bijection (3.1) induce canonical bijections

$$\operatorname{Iso}(\acute{\mathsf{E}}\mathsf{t}_F^n) \cong \operatorname{Iso}(\mathsf{Set}_\Gamma^n) \cong H^1(\Gamma, \mathfrak{S}_n)$$

The bijection $\operatorname{Iso}(\acute{\mathsf{E}}\mathsf{t}_F^n) \cong H^1(\Gamma, \mathfrak{S}_n)$ may of course also be defined directly since

$$\operatorname{Aut}_{F\text{-alg}}(F^n) \cong \mathfrak{S}_n,$$

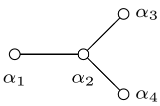
see [11, (29.9)]. Similarly, it follows from (3.2) and the anti-equivalence of groupoids $\acute{\mathsf{E}}\mathsf{tex}_F^{d/n} \equiv \mathsf{Cov}_\Gamma^{d/n}$ that we have canonical bijections of pointed sets:

$$\operatorname{Iso}(\acute{\mathsf{E}}\mathsf{tex}_\Gamma^{d/n}) \cong \operatorname{Iso}(\mathsf{Cov}_\Gamma^{d/n}) \cong H^1(\Gamma, \mathfrak{S}_d \wr \mathfrak{S}_n).$$

## 4. Triality

The Dynkin diagram $D_4$

(4.1)




has the permutation group $\mathfrak{S}_3$ as a group of automorphisms. The vertices of the diagram are labeled by the simple roots of the Lie algebra of type $D_4$. Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal basis of the Euclidean space $\mathbb{R}^4$. The simple roots are

$$\alpha_1 = e_1 - e_2,\ \alpha_2 = e_2 - e_3,\ \alpha_3 = e_3 - e_4 \ \text{ and } \alpha_4 = e_3 + e_4$$

(see [3]). The permutation $\alpha_1 \mapsto \alpha_4,\ \alpha_4 \mapsto \alpha_3,\ \alpha_3 \mapsto \alpha_1,\ \alpha_2 \mapsto \alpha_2$ is an automorphism of order 3 of the Dynkin diagram. Its extension to a linear automorphism of $\mathbb{R}^4$ is given by the orthogonal matrix

$$\omega = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & -1 & -1 & -1 \end{pmatrix}. \tag{4.2}$$

The matrix

$$\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{4.3}$$

extends the automorphism $\alpha_1 \mapsto \alpha_1,\ \alpha_4 \mapsto \alpha_3,\ \alpha_3 \mapsto \alpha_4,\ \alpha_2 \mapsto \alpha_2$ and the set $\{\omega, \nu\}$ generate a subgroup of $\mathrm{O}_4$ isomorphic to $\mathfrak{S}_3$, which restricts to the automorphism group of the Dynkin diagram. The group

$$W(D_4) = (\mathfrak{S}_2 \wr \mathfrak{S}_4)^+ = \mathfrak{S}_2^3 \rtimes \mathfrak{S}_4$$

is the Weyl group of the split adjoint algebraic group $\mathrm{PGO}_8^+$ which is of type $D_4$.

It is well known that the group of automorphisms of the Dynkin diagram $D_4$ induces outer automorphisms of the Weyl group. We recall a proof for completeness (and also to have an explicit description of these outer automorphisms).
The group $W(D_4) = \mathfrak{S}_2^3 \rtimes \mathfrak{S}_n$, as a subgroup of the orthogonal group $\mathrm{O}_4$, is generated by the reflections with respect to the roots of the Lie algebra of $\mathrm{PGO}_8^+$. Elements of $\mathfrak{S}_2 \wr \mathfrak{S}_4$ can be written as matrix products

$$w = D \circ P(\pi), \tag{4.4}$$

where $D$ is the diagonal matrix $\mathrm{Diag}(\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4)$, and $P(\pi)$ is the permutation matrix of $\pi \in \mathfrak{S}_n$. The group $\mathfrak{S}_2 \wr \mathfrak{S}_4$ fits into the exact sequence

$$1 \to \mathfrak{S}_2^4 \to \mathfrak{S}_2 \wr \mathfrak{S}_4 \xrightarrow{\beta} \mathfrak{S}_4 \to 1 \tag{4.5}$$

where $\beta$ maps $w = D \circ P(\pi)$ to $\pi$. Elements of $W(D_4)$ have a similar representation, with the supplementary condition $\prod_i \epsilon_i = 1$.

**Automorphisms of $W(D_4)$.** We view $W(D_4)$ as a subgroup of $\mathrm{O}_4$ as in (4.4). Conjugation with the matrices $\omega$ and $\nu$ on $\mathrm{O}_4$ induce by restriction outer automorphisms $\tilde{\omega}$ and $\tilde{\nu}$ of $W(D_4)$ and the set $\{\tilde{\omega}, \tilde{\nu}\}$ generates a group of automorphisms of $W(D_4)$ isomorphic to $\mathfrak{S}_3$ (see [5, p. 368]). The center of $W(D_4)$ is isomorphic to $\mathfrak{S}_2$ and is generated by

$$w_0 = \mathrm{Diag}(-1, -1, -1, -1) = -1. \tag{4.6}$$

Thus $W(D_4)/\langle w_0 \rangle$ acts on $W(D_4)$ as the group of inner automorphisms. Let $\psi$ be the automorphism of order 2 of $W(D_4)$ given by

$$\psi \colon D \cdot P(\pi) \mapsto D \cdot P(\pi) \cdot (w_0)^{\mathrm{sgn}(\pi)},$$

or equivalently by $x \mapsto x \det(x)$, $x \in W(D_4) \subset \mathrm{O}_4$. A proof of the following result can be found in [8] or in [7]:

**Proposition 4.7.**

$$\mathrm{Aut}\big(W(D_4)\big) \simeq \big((W(D_4)/\langle w_0 \rangle) \rtimes \mathfrak{S}_3\big) \times \langle \psi \rangle$$

**Corollary 4.8.** *The group $\mathfrak{S}_3$ acts as a group of automorphisms of the pointed set $H^1\big(F, W(D_4)\big) = \mathrm{Iso}\big((\mathsf{Cov}^{2/4})^+\big) = \mathrm{Iso}\big((\mathsf{\acute{E}tex}_\Gamma^{2/4})^+\big)$.*

*Proof.* Since $W(D_4) \simeq (\mathfrak{S}_2 \wr \mathfrak{S}_4)^+$ is the group of automorphisms of the oriented $\Gamma$-covering $(\mathbf{2}/\mathbf{4}, \partial_{\mathbf{4}\times\mathbf{2}})$, the set $\mathrm{Iso}\big((\mathsf{Cov}_\Gamma^{2/4})^+\big)$ is in bijection with the pointed cohomology set $H^1\big(F, W(D_4)\big)$. Inner automorphisms of $W(D_4)$ induce trivial action on $H^1\big(F, W(D_4)\big)$, hence the claim follows from Proposition 4.7. □

The action induced on $\mathrm{Iso}\big((\mathsf{Cov}_\Gamma^{2/4})^+\big)$ by the outer automorphism $\tilde{\nu}$ associates to the oriented covering 2/4-covering $(Z/Z_0, \partial_Z)$ the oriented covering $(Z/Z_0, \iota \circ \partial_Z)$ where $\iota \colon \mathbf{2} \to \mathbf{2}$ twists the orientation. We say that the action induced on $\mathrm{Iso}\big((\mathsf{Cov}^{2/4})^+\big)$ by an outer automorphism of order 3 of $W(D_4)$ is a *trialitarian action.* Since inner automorphisms act trivially on $W(D_4)$, it follows from Proposition 4.7 that there are only two possible trialitarian actions on $\mathrm{Iso}\big((\mathsf{Cov}_\Gamma^{2/4})^+\big)$: the one induced by $\tilde{\omega}$ and the one induced by $\tilde{\omega}^2 = \tilde{\omega}^{-1}$. We call triples of oriented 2/4-coverings which are permuted by the trialitarian action *trialitarian triples.*

**Triality and Γ-sets.** The functor $C$, which associates to a double covering its set of sections, leads for oriented 2/4-coverings of Γ-sets to two functors

$$C_1, C_2 \colon (\mathsf{Cov}_\Gamma^{2/4})^+ \to \mathsf{Cov}_\Gamma^{2/4}$$

(See Proposition 2.6). The functors $C_1$ and $C_2$ together with the functor $\kappa$, which interchanges the orientation, give an explicit description of the action of the group $\mathfrak{S}_3$ on the pointed set $\mathrm{Iso}\big((\mathsf{Cov}^{2/4})^+\big)$.

**Theorem 4.9.** *The functors $C_1$, $C_2 \colon (\mathsf{Cov}_\Gamma^{2/4})^+ \to \mathsf{Cov}_\Gamma^{2/4}$ take values in $(\mathsf{Cov}_\Gamma^{2/4})^+$ and satisfy natural equivalences $(C_1)^3 = 1$, $(C_1)^2 = C_2$, $C_1\kappa = \kappa\, C_2$. Moreover one of the functors $C_i$, $i = 1, 2$, induces the trialitarian action $\tilde{\omega}$ on $\mathrm{Iso}\big((\mathsf{Cov}_\Gamma^{2/4})^+\big)$, the other the action $\tilde{\omega}^2$.*

*Proof.* We use the 4-dimensional hypercube $\mathcal{K}$ in $\mathbb{R}^4$

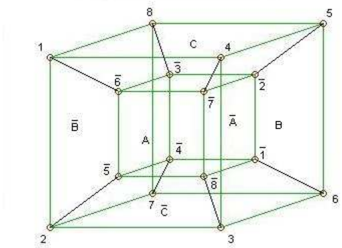



Figure 1

to describe triples $\{X/X_0,\, Y/Y_0,\, Z/Z_0\}$ of elements of $(\mathsf{Cov}_\Gamma^{2/4})^+$ which are permuted by $C^+$. We take for $Z$ the set of 3-dimensional cells in $\mathcal{K}$. We have

$$Z = \{A, \bar{A}, B, \bar{B}, C, \bar{C}, D, \bar{D}\}$$

where $A, \dots, \bar{C}$ are as in Figure 4, $D$ is the big cell and $\bar{D}$ the small cell inside. The involution permute a cell with its opposite cell and the set $Z_0$ is obtained by identifying pairs of opposite cells

$$Z_0 = \{\{A, \bar{A}\}, \{B, \bar{B}\}, \{C, \bar{C}\}, \{D, \bar{D}\}\}.$$

Next we decompose the set of vertices of $\mathcal{K}$ into two classes, two vertices being in the same class if the number of edges in any path connecting them is even. One class is

$$X = \{1, \bar{1}, 3, \bar{3}, 5, \bar{5}, 7, \bar{7}\}$$

and the other

$$Y = \{2, \bar{2}, 4, \bar{4}, 6, \bar{6}, 8, \bar{8}\}.$$

We get coverings $X/X_0$ and $Y/Y_0$ by identifying opposite vertices $v$ and $\bar{v}$.
A section of $Z/Z_0$ consists of a set of four cells which are not opposite two by two. Such four cells intersect in exactly one vertex and conversely each vertex lies in four cells. Thus we may identify $C(Z/Z_0)$ with the set of vertices of $\mathcal{K}$. With the notation in Figure 1 we have the following identification:

$$\begin{array}{llll}
1 = \{A, \bar{B}, C, D\} & \bar{1} = \{\bar{A}, B, \bar{C}, \bar{D}\} & 2 = \{A, \bar{B}, \bar{C}, D\} & \bar{2} = \{\bar{A}\ B, C, \bar{D}\} \\
3 = \{A, B, \bar{C}, D\} & \bar{3} = \{\bar{A}, \bar{B}, C, \bar{D}\} & 4 = \{A, B, C, D\} & \bar{4} = \{\bar{A}, \bar{B}, \bar{C}, \bar{D}\} \\
5 = \{\bar{A}, B, C, D\} & \bar{5} = \{A, \bar{B}, \bar{C}, \bar{D}\} & 6 = \{\bar{A}, B, \bar{C}, D\} & \bar{6} = \{A, \bar{B}, C, \bar{D}\} \\
7 = \{\bar{A}, \bar{B}, \bar{C}, D\} & \bar{7} = \{A, B, C, \bar{D}\} & 8 = \{\bar{A}, \bar{B}, C, D\} & \bar{8} = \{A, B, \bar{C}, \bar{D}\}
\end{array}$$

The Γ-action on $Z$ defines through this identification a Γ-action on the set of vertices of $\mathcal{K}$. This set defines a double covering by identifying opposite vertices. Using that $Z/Z_0$ is an oriented covering, one checks explicitly that the decomposition of $C(Z/Z_0)$ as the disjoint union $X/X_0 \sqcup Y/Y_0$ is Γ-compatible and that $X/X_0$ and $Y/Y_0$ are also oriented. Thus the functors $C_1$ and $C_2$ are given (up to a possible permutation) by the rule

$$C_1 \colon Z/Z_0 \to X/X_0 \qquad \text{and} \qquad C_2 \colon Z/Z_0 \to Y/Y_0.$$

A section of $X/X_0$ is a set of four vertices in $X$ which are not two by two opposite. Such four vertices either lie on a 3-dimensional cell or are adjacent to exactly one vertex in the complementary set $Y$. A similar claim holds for a section of $Y/Y_0$. This leads to identifying:

$$\begin{array}{ll} A = \{1,3,\bar 5,\bar 7\} = \{2,4,\bar 6,\bar 8\} & \bar A = \{\bar 1,\bar 3,5,7\} = \{\bar 2,\bar 4,6,8\} \\ B = \{\bar 1,3,5,\bar 7\} = \{\bar 2,4,6,\bar 8\} & \bar B = \{1,\bar 3,\bar 5,7\} = \{2,\bar 4,\bar 6,8\} \\ C = \{1,\bar 3,5,\bar 7\} = \{\bar 2,4,\bar 6,8\} & \bar C = \{\bar 1,3,\bar 5,7\} = \{2,\bar 4,6,\bar 8\} \\ D = \{1,3,5,7\} = \{2,4,6,8\} & \bar D = \{\bar 1,\bar 3,\bar 5,\bar 7\} = \{\bar 2,\bar 4,\bar 6,\bar 8\} \end{array} \tag{4.10}$$

and

$$\begin{array}{ll} 1 = \{A,\bar B,C,D\} = \{2,4,\bar 6,8\} & \bar 1 = \{\bar A,B,\bar C,\bar D\} = \{\bar 2,\bar 4,6,\bar 8\} \\ 3 = \{A,B,\bar C,D\} = \{2,4,6,\bar 8\} & \bar 3 = \{\bar A,\bar B,C,\bar D\} = \{\bar 2,\bar 4,\bar 6,8\} \\ 5 = \{\bar A,B,C,D\} = \{\bar 2,4,6,8\} & \bar 5 = \{A,\bar B,\bar C,\bar D\} = \{2,\bar 4,\bar 6,\bar 8\} \\ 7 = \{\bar A,\bar B,\bar C,D\} = \{2,\bar 4,6,8\} & \bar 7 = \{A,B,C,\bar D\} = \{\bar 2,4,\bar 6,\bar 8\} \\ 2 = \{A,\bar B,\bar C,D\} = \{1,3,\bar 5,7\} & \bar 2 = \{\bar A,B,C,\bar D\} = \{\bar 1,\bar 3,5,\bar 7\} \\ 4 = \{A,B,C,D\} = \{1,3,5,\bar 7\} & \bar 4 = \{\bar A,\bar B,\bar C,\bar D\} = \{\bar 1,\bar 3,\bar 5,7\} \\ 6 = \{\bar A,B,\bar C,D\} = \{\bar 1,3,5,7\} & \bar 6 = \{A,\bar B,C,\bar D\} = \{1,\bar 3,\bar 5,\bar 7\} \\ 8 = \{\bar A,\bar B,C,D\} = \{1,\bar 3,5,7\} & \bar 8 = \{A,B,\bar C,\bar D\} = \{\bar 1,3,\bar 5,\bar 7\}, \end{array} \tag{4.11}$$

hence the existence of decompositions $C(X/X_0) = Y/Y_0 \sqcup Z/Z_0$ and $C(Y/Y_0) = Z/Z_0 \sqcup X/X_0$ which, in fact, are decompositions as Γ-sets. The functor $\kappa$ switches the two parts of the decomposition of the Clifford covering, hence $C_1\kappa = \kappa C_2$. To show the last claim of Theorem 4.9 we use use coordinates. Let $(e_1, e_2, e_3, e_4)$ be the standard basis of the Euclidean space $\mathbb{R}^4$. The set

$$\{\frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2}\}$$

is the set of vertices of an hypercube $\mathcal{K}$ in $\mathbb{R}^4$. The set of oriented directions

$$\{\pm e_1, \pm e_2, \pm e_3, \pm e_4\}$$

is in bijection with the set $Z$ of 3-dimensional cells in $\mathcal{K}$ and we identify both sets. Then $Z_0$ is the set obtained by identifying opposite directions. The set $C(Z/Z_0)$ of sections can be naturally identified with

$$\{\frac{\pm e_1 \pm e_2 \pm e_3 \pm e_4}{2}\},$$

which is the set of vertices of the hypercube $\mathcal{K}$. The decomposition $C(Z/Z_0) = X \sqcup Y$ corresponds to

$$X = \left\{\pm\frac{e_1+e_2+e_3+e_4}{2}, \pm\frac{e_1+e_2-e_3-e_4}{2}, \pm\frac{e_1-e_2+e_3-e_4}{2}, \pm\frac{-e_1+e_2+e_3-e_4}{2}\right\}$$

and

$$Y = \left\{\pm\frac{e_1+e_2+e_3-e_4}{2}, \pm\frac{e_1+e_2-e_3+e_4}{2}, \pm\frac{e_1-e_2+e_3+e_4}{2}, \pm\frac{e_1-e_2-e_3-e_4}{2}\right\}.$$

We get coverings by setting $X_0 = X/\{\pm 1\}$ and $Y_0 = Y/\{\pm 1\}$. Now it is easy to check that the orthogonal matrix $\omega$ in (4.2) maps cyclically $Z/Z_0$ to $X/X_0$, $X/X_0$ to $Y/Y_0$ and $Y/Y_0$ to $Z/Z_0$, hence the last claim. □

**Remark 4.12.** The explicit choice of $\omega$ as the matrix in (4.2) determines the orientation of $Z/Z_0$ since it determines the order in the pair $(C_1, C_2)$.

**Remark 4.13.** The rôle of the hypercube $\mathcal{K}$ in the proof of Theorem 4.9 is explained by the fact that the group $W = \mathfrak{S}_2^4 \rtimes \mathfrak{S}_4 = \mathfrak{S}_2 \wr \mathfrak{S}_4$ is the group of automorphisms of $\mathcal{K}$. The subgroup $W(D_4) = (\mathfrak{S}_2 \wr \mathfrak{S}_4)^+ = \mathfrak{S}_2^3 \rtimes \mathfrak{S}_4$ consists of the automorphisms of $\mathcal{K}$ respecting the decomposition of the set of vertices as $X \sqcup Y$, i.e., automorphisms of the half-hypercube.

**Hurwitz' Quaternions.** Hurwitz' quaternions can be applied to give an alternate description of trialitarian automorphisms of $W(D_4) = (\mathfrak{S}_2 \wr \mathfrak{S}_4)^+$. Let $\{1, i, j, k\}$ be the standard basis of the skew field of real quaternions $\mathbb{H}$ and let

$$Z = \{\pm 1, \pm i, \pm j, \pm k\}, \quad Z_0 = Z/\{\pm 1\},$$

so that $Z/Z_0$ is a double covering with $|Z| = 8$. The set $C(Z/Z_0)$ can be identified with the subset

$$C(Z/Z_0) = \{\frac{\pm 1 \pm i \pm j \pm k}{2}\}$$

of $\mathbb{H}$. The involution $\underline{\sigma}$ maps a quaternion to its negative. The union $Z \cup C(Z/Z_0) \subset \mathbb{H}$ is the group $\mathbb{H}^1$ of Hurwitz quaternions of norm 1. The element

$$\rho = -\frac{1}{2}(1 + i + j + k)$$

is of order 3 in $\mathbb{H}^1$ and conjugation by $\rho$ permutes $i$, $j$ and $k$ cyclically. The set $Z$ is in fact the underlying set of the quaternionic group $\mathfrak{Q}_8$ and

$$\mathbb{H}^1 = \mathfrak{Q}_8 \rtimes \mathfrak{C}_3$$

where the cyclic group of three elements $\mathfrak{C}_3$ operates on $\mathfrak{Q}_8$ via conjugation with $\rho$. We set

$$X = \rho Z = \left\{ \pm\frac{1+i+j+k}{2}, \pm\frac{1-i-j+k}{2}, \pm\frac{1+i-j-k}{2}, \pm\frac{1-i+j-k}{2} \right\}$$

and

$$Y = \rho^2 Z = \left\{ \pm\frac{1-i-j-k}{2}, \pm\frac{1+i+j-k}{2}, \pm\frac{1-i+j+k}{2}, \pm\frac{1+i-j+k}{2} \right\}$$

and get coverings $X/X_0$ and $Y/Y_0$ by going modulo $\{\pm 1\}$. It follows that

$$C(Z/Z_0) = X/X_0 \sqcup Y/Y_0$$

and multiplication with $\rho$ permutes cyclically the three double covering $Z/Z_0$, $X/X_0$ and $Y/Y_0$. Thus we get a trialitarian triple with respect to $\rho$. Fixing $\{1, i, j, k\}$ as a basis of $\mathbb{R}^4$, multiplication by $\rho$ is given by the matrix

$$\rho = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 & 1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ -1 & 1 & -1 & -1 \end{pmatrix} \tag{4.14}$$

and conjugation with $\rho$ induces a trialitarian action $\tilde{\rho}$ on $W(D_4)$. Observe that

$$\rho\omega^{-1} = -1 \cdot \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in W(D_4)$$

so that $\tilde{\rho} = \tilde{\omega} \circ \mathrm{Int}(\alpha)$, $\alpha \in W(D_4)$, in compliance with Proposition 4.7.

**Triality and étale algebras.** Triality acts on the set $\mathrm{Iso}(\acute{\mathsf{E}}\mathsf{tex}_\Gamma^{2/4})$ of isomorphism classes of oriented extensions of étale algebras $S/S_0$ with $\dim_F S_0 = 4$ and $\dim_F S = 8$. Such isomorphisms classes are given by homomorphisms $\Gamma \to W(D_4)$ up to conjugation. We classify in Table 1 extensions $S/S_0$ according to the image of $\Gamma$ in $W(D_4)$. The following notation is used in the table.
We view $W(D_4)$ as part of the split exact sequence:

$$1 \to \mathfrak{S}_2^3 \to W(D_4) \xrightarrow{\beta} \mathfrak{S}_4 \to 1. \tag{4.15}$$

where $\beta$ is as in (4.5). The action of the Galois group $\Gamma$ factors through a subgroup $G$ of $W(D_4)$. For each subgroup $G$ of $W(D_4)$ we denote by $G_1$ the restriction $G \cap \mathfrak{S}_2^3$ and by $G_0$ the projection $\beta(G)$. The center of $W(D_4)$, generated by $w_0 = \mathrm{Diag}(-1,-1,-1,-1) = -1$ is denoted by $C$ and we set $w_1 = \mathrm{Diag}(1,-1,1,-1)$, $w_2 = \mathrm{Diag}(1,-1,-1,1)$ and $w_3 = \mathrm{Diag}(-1,-1,1,1)$ for special elements of the subgroup $\mathfrak{S}_2^3 \subset W(D_4)$. We denote by $\mathfrak{S}_n$ the permutation group of $n$ elements, $\mathfrak{A}_n$ is the alternating subgroup, $\mathfrak{C}_n$ is cyclic of order $n$, $\mathfrak{D}_n$ is the dihedral group of order $2n$, $\mathfrak{V}_4$ is the kleinian Vierergruppe, and $\mathfrak{Q}_8$ is the quaternionic group with eight elements. In Column $S$ we summarize the various possibilities for étale algebras of dimension 8 associated to the class of a cocycle $\alpha\colon \Gamma \to G$ and in Column $S_0$ étale algebras of dimension 4 associated to the class of the induced cocycle $\beta \circ \alpha\colon \Gamma \to G_0$. The entry $K$ in one of the column $S$ or $S_0$ denotes a quadratic field extension, $L_0$ is a cubic extension (if $L_0$ is cyclic or not can be deduced from the corresponding group in the next column) and $L$ denotes a Galois extension of degree 6 with group $\mathfrak{S}_3$. We use symbols $E$, resp. $E_0$ for separable field extensions having $G$, resp. $G_0$ as Galois groups. The symbol $R(E)$ stands for the cubic resolvent of $E$ and $\lambda^2 E$ for the second lambda power of $E$ (see [12], where it is denoted $\Lambda_2(E)$ or [9], where it is denoted $E(2)$[3]). We write $\overline{E}_0$, $\overline{L}_0$, for the Galois closure of $E_0$, resp. $L_0$. The symbol $\ell$ gives the number of groups in the conjugacy class of $G$, MS refers to the maximal subgroups of $G$ and in column $T$ we give the two conjugacy classes which are the images of the class of $G$ under the trialitarian automorphisms $\bar{\omega}$ and $\bar{\omega}^2$.
The table was generated with help of the algebra computational system Magma [2]. We worked with $W(D_4)$ represented as a matrix group, as described above. Entries $N, |G|, \ell, MS$ in the table were given by Magma. The computation of the entry $T$, the explicit representation of the group as exact sequence and the decomposition of the étale algebras as products of fields were checked case by case. Some concrete computations of trialitarian triples were made in [1] and [18] using the explicit description of the trialitarian action given in the proof of Theorem 4.9.

[3]The corresponding $\Gamma$-set $\lambda^2 X$ is obtained by removing the diagonal from $X \times X$ and dividing by the involution $(x, y) \mapsto (y, x)$.

| $N$ | $S_0$ | $S$ | $G_1 \to G \to G_0$ | $\lvert G\rvert$ | $\ell$ | $MS$ | $T$ |
|---|---|---|---|---|---|---|---|
| 1 | $F^4$ | $F^8$ | $1 \to 1 \to 1$ | 1 | 1 | 1 | $1, 1$ |
| 2 | $F^4$ | $K^4$ | $C \to \mathfrak{S}_2 \to 1$ | 2 | 6 | 1 | $2, 2$ |
| 3 | $K^2$ | $K^4$ | $1 \to \mathfrak{S}_2 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 2 | 6 | 1 | $3, 5$ |
| 4 | $K^2$ | $K^4$ | $1 \to \mathfrak{S}_2 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 2 | 6 | 1 | $4, 3$ |
| 5 | $F^4$ | $F^2 \times K^2$ | $\langle w_1\rangle \to \mathfrak{S}_2 \to 1$ | 2 | 6 | 1 | $5, 4$ |
| 6 | $F^2 \times K$ | $F^2 \times K^2$ | $1 \to \mathfrak{S}_2 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 2 | 12 | 1 | $6, 6$ |
| 7 | $F^2 \times K$ | $F^2 \times K^2$ | $1 \to \mathfrak{S}_2 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 2 | 12 | 1 | $7, 7$ |
| 8 | $F \times E_0$ | $F^2 \times S_0^2$ | $1 \to \mathfrak{C}_3 \to \mathfrak{C}_3$ | 3 | 16 | 1 | $8, 8$ |
| 9 | $F^4$ | $K_1^2 \times K_2^2$ | $\langle C, w_1\rangle \to \mathfrak{S}_2^2 \to 1$ | 4 | 3 | 2 5 | $11, 10$ |
| 10 | $K^2$ | $K \otimes K_1^2$ | $C \to \mathfrak{S}_2^2 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 4 | 3 | 2 3 | $9, 11$ |
| 11 | $K^2$ | $K \otimes K_1^2$ | $C \to \mathfrak{S}_2^2 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 4 | 3 | 2 4 | $10, 9$ |
| 12 | $K_1 \otimes K_2$ | $K_1 \otimes K_2^2$ | $1 \to \mathfrak{S}_2^2 \to \mathfrak{V}_4$ | 4 | 4 | 4 | $14, 13$ |
| 13 | $F^4$ | $K_1^2 \times K_2^2$ | $\langle w_1, w_2\rangle \to \mathfrak{S}_2^2 \to 1$ | 4 | 4 | 5 | $12, 14$ |
| 14 | $K_1 \otimes K_2$ | $K_1 \otimes K_2^2$ | $1 \to \mathfrak{S}_2^2 \to \mathfrak{V}_4$ | 4 | 4 | 3 | $13, 12$ |
| 15 | $F^2 \times K$ | $F^4 \times K_1 \otimes K$ | $\langle w_1\rangle \to \mathfrak{S}_2^2 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 4 | 6 | 5 6 | $18, 17$ |
| 16 | $K_1 \times K_2$ | $K_1 \otimes K_2^2$ | $1 \to \mathfrak{S}_2^2 \to \mathfrak{S}_2^2$ | 4 | 6 | 4 7 | $21, 19$ |
| 17 | $K_1 \times K_2$ | $K_1^2 \times K_2^2$ | $1 \to \mathfrak{S}_2^2 \to \mathfrak{S}_2^2$ | 4 | 6 | 3 6 | $15, 18$ |
| 18 | $K_1 \times K_2$ | $K_1^2 \times K_2^2$ | $1 \to \mathfrak{S}_2^2 \to \mathfrak{S}_2^2$ | 4 | 6 | 4 6 | $17, 15$ |
| 19 | $F^2 \times K$ | $K_1^2 \times K_1 \times K$ | $\langle w_1\rangle \to \mathfrak{S}_2^2 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 4 | 6 | 5 7 | $16, 21$ |
| 20 | $K \times K$ | $E \times E$ | $C \to \mathfrak{C}_4 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 4 | 2 | 2 | $20, 20$ |
| 21 | $K_1 \times K_2$ | $K_1 \otimes K_2^2$ | $1 \to \mathfrak{S}_2^2 \to \mathfrak{S}_2^2$ | 4 | 6 | 3 7 | $19, 16$ |
| 22 | $K_1 \times K_2$ | $K_1^2 \times K_1 \otimes K_2$ | $1 \to \mathfrak{S}_2^2 \to \mathfrak{S}_2^2$ | 4 | 12 | 4 6 7 | $23, 27$ |
| 23 | $K_1 \times K_2$ | $K_1^2 \times K_1 \otimes K_2$ | $1 \to \mathfrak{S}_2^2 \to \mathfrak{S}_2^2$ | 4 | 12 | 3 6 7 | $27, 22$ |
| 24 | $K \times K$ | $K^2 \times K \otimes K_1$ | $\langle w_1\rangle \to \mathfrak{S}_2^2 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 4 | 12 | 3 4 5 | $24, 24$ |
| 25 | $F^2 \times K$ | $F^4 \times E$ | $\langle w_1\rangle \to \mathfrak{C}_4 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 4 | 12 | 5 | $26, 28$ |
| 26 | $E_0$ | $E_0 \times E_0$ | $1 \to \mathfrak{C}_4 \to \mathfrak{C}_4$ | 4 | 12 | 4 | $28, 25$ |
| 27 | $F^2 \times K$ | $K_1^2 \times K_1 \otimes K$ | $\langle -w_1\rangle \to \mathfrak{S}_2^2 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 4 | 12 | 5 6 7 | $22, 23$ |
| 28 | $E_0$ | $E_0 \times E_0$ | $1 \to \mathfrak{C}_4 \to \mathfrak{C}_4$ | 4 | 12 | 3 | $25, 26$ |
| 29 | $F^2 \times K$ | $K^2 \times K \otimes K_1$ | $C \to \mathfrak{S}_2^2 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 4 | 12 | 2 6 7 | $29, 29$ |
| 30 | $F \times E_0$ | $F^2 \times E_0 \otimes \Delta(E_0)$ | $1 \to \mathfrak{S}_3 \to \mathfrak{S}_3$ | 6 | 16 | 7 8 | $30, 30$ |
| 31 | $F \times E_0$ | $F^2 \times E$ | $C \to \mathfrak{C}_6 \to \mathfrak{C}_3$ | 6 | 16 | 2 8 | $31, 31$ |
| 32 | $F \times E_0$ | $F^2 \times E_0^2$ | $1 \to \mathfrak{S}_3 \to \mathfrak{S}_3$ | 6 | 16 | 6 8 | $32, 32$ |
| 33 | $E_0$ | $E$ | $C \to \mathfrak{S}_2^3 \to \mathfrak{V}_4$ | 8 | 1 | 11 12 | $34, 35$ |
| 34 | $E_0$ | $E$ | $C \to \mathfrak{S}_2^3 \to \mathfrak{V}_4$ | 8 | 1 | 10 14 | $35, 33$ |
| 35 | $F^4$ | $K_1 \times K_2 \times K_3 \times K_{123}$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2^3 \to 1$ | 8 | 1 | 9 13 | $33, 34$ |
| 36 | $E_0$ | $E$ | $C \to \mathfrak{Q}_8 \to \mathfrak{V}_4$ | 8 | 2 | 20 | $36, 36$ |
| 37 | $K \times K$ | $E_1 \times E_2$ | $\langle C, w_1\rangle \to \mathfrak{S}_2 \times \mathfrak{C}_4 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 8 | 3 | 3 20 | $38, 40$ |
| 38 | $E_0$ | $E_0 \otimes K$ | $C \to \mathfrak{S}_2 \times \mathfrak{C}_4 \to \mathfrak{C}_4$ | 8 | 3 | 11 20 | $40, 37$ |
| 39 | $K \times K$ | $K \otimes K_1 \times K \otimes K_2$ | $\langle C, w_1\rangle \to \mathfrak{S}_2^3 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 8 | 3 | 9 10 11 24 | $39, 39$ |
| 40 | $E_0$ | $E_0 \otimes K$ | $C \to \mathfrak{S}_2 \times \mathfrak{C}_4 \to \mathfrak{C}_4$ | 8 | 3 | 10 20 | $37, 38$ |
| 41 | $K \times K$ | $E \times E$ | $\langle C, w_2\rangle \to \mathfrak{D}_4 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 8 | 6 | 9 11 20 | $49, 45$ |
| 42 | $F^2 \times K$ | $K_1 \times E$ | $\langle C, w_1\rangle \to \mathfrak{S}_2 \times \mathfrak{C}_4 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 8 | 6 | 9 25 | $44, 47$ |
| 43 | $K_1 \times K_2$ | $K \otimes K_1 \times K \otimes K_2$ | $C \to \mathfrak{S}_2^3 \to \mathfrak{S}_2^2$ | 8 | 6 | 10 17 21 23 29 | $48, 46$ |
| 44 | $E_0$ | $E$ | $C \to \mathfrak{S}_2 \times \mathfrak{C}_4 \to \mathfrak{C}_4$ | 8 | 6 | 11 26 | $47, 42$ |
| 45 | $K \times K$ | $E \times E$ | $\langle C, w_2\rangle \to \mathfrak{D}_4 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 8 | 6 | 9 10 20 | $41, 49$ |
| 46 | $K_1 \times K_2$ | $K \otimes K_1 \times K \otimes K_2$ | $C \to \mathfrak{S}_2^3 \to \mathfrak{S}_2^2$ | 8 | 6 | 11 16 18 22 29 | $43, 48$ |
| 47 | $E_0$ | $E_0 \otimes K$ | $C \to \mathfrak{S}_2 \times \mathfrak{C}_4 \to \mathfrak{C}_4$ | 8 | 6 | 10 28 | $42, 44$ |
| 48 | $F^2 \times K$ | $K_1^2 \times K \otimes K_2$ | $\langle C, w_1\rangle \to \mathfrak{S}_2^3 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 8 | 6 | 9 15 19 27 29 | $46, 43$ |
| 49 | $E_0$ | $E$ | $C \to \mathfrak{D}_4 \to \mathfrak{V}_4$ | 8 | 6 | 10 11 20 | $45, 41$ |

| $N$ | $S_0$ | $S$ | $G_1 \to G \to G_0$ | $\lvert G\rvert$ | $\ell$ | $MS$ | $T$ |
|---|---|---|---|---|---|---|---|
| 50 | $F^2 \times K$ | $K^2 \times E$ | $\langle w_1, w_2\rangle \to \mathfrak{D}_4 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 8 | 6 | 13 19 25 | 55, 51 |
| 51 | $E_0$ | $E$ | $1 \to \mathfrak{D}_4 \to \mathfrak{D}_4$ | 8 | 12 | 14 21 28 | 50, 55 |
| 52 | $E_0$ | $E_0^2$ | $1 \to \mathfrak{D}_4 \to \mathfrak{D}_4$ | 8 | 12 | 14 17 28 | 54, 57 |
| 53 | $K_1 \times K_2$ | $K \otimes K_1 \times K \otimes K_2$ | $\langle w_1\rangle \to \mathfrak{S}_2^3 \to \mathfrak{S}_2^2$ | 8 | 12 | 16 19 21 22 23 24 27 | 53, 53 |
| 54 | $F^2 \times K$ | $F^2 \times K \times E$ | $\langle w_1, w_2\rangle \to \mathfrak{D}_4 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 8 | 12 | 13 15 25 | 57, 52 |
| 55 | $E_0$ | $\bar{E}_0$ | $1 \to \mathfrak{D}_4 \to \mathfrak{D}_4$ | 8 | 12 | 12 16 26 | 51, 50 |
| 56 | $K_1 \times K_2$ | $K_1^2 \times E$ | $\langle w_1\rangle \to \mathfrak{S}_2^3 \to \mathfrak{S}_2^2$ | 8 | 12 | 15 17 18 22 23 24 27 | 56, 56 |
| 57 | $E_0$ | $E_0^2$ | $1 \to \mathfrak{D}_4 \to \mathfrak{D}_4$ | 8 | 12 | 12 18 26 | 52, 54 |
| 58 | $E_0$ | $E_0^2$ | $1 \to \mathfrak{A}_4 \to \mathfrak{A}_4$ | 12 | 4 | 8 14 | 59, 60 |
| 59 | $F \times E_0$ | $F^2 \times E_0^2$ | $\langle w_1, w_2\rangle \to \mathfrak{S}_2^2 \rtimes \mathfrak{C}_3 \to \mathfrak{C}_3$ | 12 | 4 | 8 13 | 60, 58 |
| 60 | $E_0$ | $E_0^2$ | $1 \to \mathfrak{A}_4 \to \mathfrak{A}_4$ | 12 | 4 | 8 12 | 58, 59 |
| 61 | $F \times E_0$ | $K \times K \otimes E_0$ | $C \to \mathfrak{S}_2 \times \mathfrak{S}_3 \to \mathfrak{S}_3$ | 12 | 4 | 29, 30, 31, 32 | 61, 61 |
| 62 | $K_1 \times K_2$ | $E^2$ | $\langle C, w_1\rangle \to [2^2]4 \to \mathfrak{S}_2^2$ | 16 | 3 | 39 42 | 66, 63 |
| 63 | $E_0$ | $E$ | $\langle C, w_1\rangle \to [2^2]4 \to \mathfrak{C}_4$ | 16 | 3 | 39 47 | 62, 66 |
| 64 | $K \times K$ | $E_1 \times E_2$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2 \times \mathfrak{D}_4 \to \mathfrak{S}_2 \subset \mathfrak{V}_4$ | 16 | 3 | 35 37 39 41 45 | 68, 67 |
| 65 | $K_1 \times K_2$ | $K_1 \otimes K_3 \times K_2 \otimes K_4$ | $\langle C, w_1\rangle \to \mathfrak{S}_2^4 \to \mathfrak{S}_2^2$ | 16 | 3 | 39 43 46 48 53 56 | 65, 65 |
| 66 | $E_0$ | $E$ | $\langle C, w_1\rangle \to [2^2]4 \to \mathfrak{C}_4$ | 16 | 3 | 39 44 | 63, 62 |
| 67 | $E_0$ | $E$ | $\langle C, w_1\rangle \to \mathfrak{S}_2 \times \mathfrak{D}_4 \to \mathfrak{V}_4$ | 16 | 3 | 34 39 40 45 49 | 64, 68 |
| 68 | $E_0$ | $E$ | $\langle C, -w_2\rangle \to \mathfrak{S}_2 \times \mathfrak{D}_4 \to \mathfrak{V}_4$ | 16 | 3 | 33 38 39 41 49 | 67, 64 |
| 69 | $K_1 \times K_2$ | $E^2$ | $\langle C, w_1\rangle \to [2^2]4 \to \mathfrak{S}_2^2$ | 16 | 6 | 37 42 48 | 73, 72 |
| 70 | $E_0$ | $E$ | $\langle C, w_1\rangle \to \mathfrak{Q}_8 : 2 \to \mathfrak{V}_4$ | 16 | 6 | 36 37 38 40 41 45 49 | 70, 70 |
| 71 | $F^2 \times K$ | $K_1^2 \times E$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2 \times \mathfrak{D}_4 \to \mathfrak{S}_2 \not\subset \mathfrak{V}_4$ | 16 | 6 | 35 42 48 50 54 | 75, 74 |
| 72 | $E_0$ | $E$ | $\langle C, w_1\rangle \to [2^2]4 \to \mathfrak{V}_4$ | 16 | 6 | 40 43 47 | 69, 73 |
| 73 | $E_0$ | $E$ | $C \to [2^2]4 \to \mathfrak{D}_4$ | 16 | 6 | 38 44 46 | 72, 69 |
| 74 | $E_0$ | $E_0 \otimes K$ | $C \to \mathfrak{S}_2 \times \mathfrak{D}_4 \to \mathfrak{D}_4$ | 16 | 6 | 34 43 47 51 52 | 71, 75 |
| 75 | $E_0$ | $E_0 \otimes K$ | $C \to \mathfrak{S}_2 \times \mathfrak{D}_4 \to \mathfrak{D}_4$ | 16 | 6 | 33 44 46 55 57 | 74, 71 |
| 76 | $E_0$ | $E_0^2$ | $1 \to \mathfrak{S}_4 \to \mathfrak{S}_4$ | 24 | 4 | 32 52 58 | 79, 81 |
| 77 | $F \times R(E)$ | $\Delta(E) \times \lambda^2 E$ | $\langle w_1, w_3\rangle \to \mathfrak{S}_2^2 \rtimes \mathfrak{S}_3 \to \mathfrak{S}_3$ | 24 | 4 | 30 50 59 | 84, 82 |
| 78 | $F \times E_0$ | $\Delta(E) \times \lambda^2 E$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2^3 \rtimes \mathfrak{C}_3 \to \mathfrak{C}_3$ | 24 | 4 | 31 35 59 | 83, 80 |
| 79 | $F \times R(E)$ | $F^2 \times \lambda^2 E$ | $\langle w_1, w_3\rangle \to \mathfrak{S}_2^2 \rtimes \mathfrak{S}_3 \to \mathfrak{S}_3$ | 24 | 4 | 32 54 59 | 76, 81 |
| 80 | $E_0$ | $E_0 \otimes K$ | $C \to \mathfrak{S}_2 \times \mathfrak{A}_4 \to \mathfrak{A}_4$ | 24 | 4 | 31 34 58 | 78, 83 |
| 81 | $E_0$ | $E_0^2$ | $1 \to \mathfrak{S}_4 \to \mathfrak{S}_4$ | 24 | 4 | 32 57 60 | 76, 79 |
| 82 | $E_0$ | $E_0 \otimes \Delta(E_0)$ | $1 \to \mathfrak{S}_4 \to \mathfrak{S}_4$ | 24 | 4 | 30 51 58 | 77, 84 |
| 83 | $E_0$ | $E$ | $C \to \mathfrak{S}_2 \times \mathfrak{A}_4 \to \mathfrak{A}_4$ | 24 | 4 | 31 33 60 | 78, 80 |
| 84 | $E_0$ | $E_0 \otimes (E_0)$ | $1 \to \mathfrak{S}_4 \to \mathfrak{S}_4$ | 24 | 4 | 30 55 66 | 82, 77 |
| 85 | $E_0$ | $E$ | $\mathfrak{S}_2 \to \tilde{\mathfrak{A}}_4 \to \mathfrak{A}_4$ | 24 | 4 | 31 36 | 85, 85 |
| 86 | $E_0$ | $E$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2^3 \rtimes \mathfrak{V}_4 \to \mathfrak{V}_4$ | 32 | 1 | 64 67 68 70 | 86, 86 |
| 87 | $E_0$ | $E$ | $\langle C, w_1\rangle \to \mathfrak{S}_2^2 \rtimes \mathfrak{D}_4 \to \mathfrak{D}_4$ | 32 | 3 | 63 65 67 72 74 | 92, 90 |
| 88 | $E_0$ | $E$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2^3 \rtimes \mathfrak{C}_4 \to \mathfrak{C}_4$ | 32 | 3 | 63 64 66 | 91, 89 |
| 89 | $E_0$ | $E$ | $\langle C, w_1\rangle \to \mathfrak{S}_2^3 \rtimes \mathfrak{C}_4 \to \mathfrak{D}_4$ | 32 | 3 | 62 66 67 | 88, 91 |
| 90 | $E_0$ | $E$ | $\langle C, w_1\rangle \to \mathfrak{S}_2^2 \rtimes \mathfrak{D}_4 \to \mathfrak{D}_4$ | 32 | 3 | 65 66 68 73 75 | 92, 87 |
| 91 | $E_0$ | $S$ | $\langle C, w_1\rangle \to \mathfrak{S}_2^3 \rtimes \mathfrak{C}_4 \to \mathfrak{D}_4$ | 32 | 3 | 62 63 68 | 89, 88 |
| 92 | $K_1 \times K_2$ | $E_1 \times E_2$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2^2 \rtimes \mathfrak{D}_4 \to \mathfrak{S}_2^2$ | 32 | 3 | 62 64 65 69 71 | 87, 90 |
| 93 | $F \times R(E)$ | $\Delta(E) \times K * \lambda^2 E$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2 \times \mathfrak{S}_4 \to \mathfrak{S}_3$ | 48 | 4 | 61 71 77 78 79 | 94, 95 |
| 94 | $E_0$ | $E_0 \otimes K$ | $C \to \mathfrak{S}_2 \times \mathfrak{S}_4 \to \mathfrak{S}_4$ | 48 | 4 | 61 75 81 83 84 | 95, 93 |
| 95 | $E_0$ | $E_0 \otimes K$ | $C \to \mathfrak{S}_2 \times \mathfrak{S}_4 \to \mathfrak{S}_4$ | 48 | 4 | 61 74 76 80 82 | 93, 94 |
| 96 | $E_0$ | $E$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2^3 \rtimes \mathfrak{D}_4 \to \mathfrak{D}_4$ | 64 | 3 | 86 87 88 89 90 91 92 | 96, 96 |
| 97 | $E_0$ | $E$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2^3 \rtimes \mathfrak{A}_4 \to \mathfrak{A}_4$ | 96 | 1 | 78 80 83 85 86 | 97, 97 |
| 98 | $E_0$ | $E$ | $\mathfrak{S}_2^3 \to \mathfrak{S}_2^3 \rtimes \mathfrak{S}_4 \to \mathfrak{S}_4$ | 192 | 1 | 93 94 95 96 97 | 98, 98 |

Table 1

**Trialitarian triples and fixed points.** Triples of isomorphisms classes of étale algebras which are permuted under triality are called *trialitarian triples.* They correspond to the triples listed in Columns $N$ and $T$ of Table 1. Triples of cohomology classes invariant under triality do not necessarily correspond to triples of étale algebras fixed under triality since conjugacy of homomorphisms is stronger than conjugacy of the image groups. In Table 1 the conjugacy classes 1, 2, 6, 7, 8, 20, 29, 30, 31, 32, 36, 61 and 85 give rise to triples fixed under triality, the classes 39, 65, 70, 86, 96 and 98 not. Maximal cases are given by the dihedral group $\mathfrak{D}_6$ of 12 elements ($N = 61$) and the double covering $\tilde{\mathfrak{A}}_4$ of $\mathfrak{A}_4$ ($N = 85$). We next show that these cases correspond to subgroups of $W(D_4)$ fixed under triality. The two trialitarian automorphisms $\tilde{\omega}$ (see (4.2)) and $\tilde{\rho}$ (see (4.14)) of $W(D_4)$ are not conjugate in $\mathrm{Aut}\big(W(D_4)\big)$, not even in $\mathrm{O}_4$, since they have different traces. However:

**Proposition 4.16.** *Any trialitarian automorphism of $W(D_4)$ is conjugate in the group $\mathrm{Aut}\big(W(D_4)\big)$ to either $\tilde{\omega}$ or $\tilde{\rho}$.*

*Proof.* The conjugation class of $\tilde{\rho}$ contains 16 elements and the conjugation class of $\tilde{\omega}$ contains 32 elements. In view of Proposition 4.7 any trialitarian automorphism of $W(D_4)$ is conjugate to an element in $\mathrm{O}_4$ of the form $u = \omega \cdot w$ or $u = \omega^2 \cdot w$, $w \in W(D_4)$, such that $u^3 = 1$. There are 48 elements $u$ of this form, hence the claim. □

Conjugate trialitarian automorphisms have isomorphic groups of fixed points, thus it follows from Proposition 4.16 that there are two types of groups of fixed points, those contained in the subgroup $\mathrm{Fix}(\tilde{\omega})$ of $W(D_4)$ of elements fixed by $\tilde{\omega}$ and those contained in $\mathrm{Fix}(\tilde{\rho})$.

**Proposition 4.17.** *1) The 2-dimensional subspace of $\mathbb{R}^4$ generated by the set of elements $\{e_1 - e_3, e_2 - e_3\}$ is fixed under $\omega$.*
*2) The set $\{e_1 - e_3, e_2 - e_3\}$ generates a root system of type $G_2$ and the group $\mathrm{Fix}(\tilde{\omega})$ is the corresponding Weyl group, which is the dihedral group $\mathfrak{D}_6$ of order 12.*
*3) The group $\mathrm{Fix}(\tilde{\omega})$ belongs to class $N = 61$ in Table 1.*

*Proof.* By explicit computation. □

**Proposition 4.18.** *The group $\mathrm{Fix}(\tilde{\rho})$ is isomorphic to the group of order 24 of Hurwitz quaternions $\mathbb{H}^1 = \mathfrak{Q}_8 \rtimes \mathfrak{C}_3$. This group is isomorphic to the double covering $\tilde{\mathfrak{A}}_4$ of $\mathfrak{A}_4$ and belongs to the conjugacy class $N = 85$ in Table 1.*

*Proof.* The matrix $\rho$ is obtained by choosing $\{1, i, j, k\}$ as basis of $\mathbb{R}^4$ and letting $-\frac{1}{2}(1 + i + j + k)$ operate by left multiplication in $\mathbb{H}$. The group $\mathbb{H}^1$ has a representation in $W(D_4)$ by right multiplication which obviously commute with the action of $\rho$. Hence $\mathrm{Fix}(\tilde{\rho})$ contains a copy of $\mathbb{H}^1$. The claim then follows from the fact that $\mathrm{Fix}(\tilde{\rho})$ has 24 elements. □

Observe that fixed étale algebras in class $N = 61$ are of the form $K \times (E_0 \otimes K)$, where $K$ is quadratic and $E_0$ is cubic. Hence they are not fields over $F$, in contrast to algebras in class $N = 85$.

The fact that there are two types of triples invariant under triality can easily be seen by Galois cohomology. Since the Galois group $\Gamma = \mathrm{Gal}(F_s/F)$ acts trivially on $W(D_4)$, the cohomology set $H^1\big(F, W(D_4)\big)$ can be identified with the set of equivalence classes of continuous group homomorphisms $\mathrm{Hom}_{\mathrm{cont}}\big(\Gamma, W(D_4)\big)/\sim$, where the equivalence relation is defined as

$$\psi \sim \varphi \Leftrightarrow \exists \alpha \in W(D_4) \mid \psi(\gamma) = \alpha\varphi(\gamma)\alpha^{-1},\ \forall \gamma \in \Gamma.$$

Any outer automorphism $\xi$ of order 3 of $W(D_4)$ acts on $H^1\big(F, W(D_4)\big)$ as

$$\xi([\varphi]) = [\xi \circ \varphi].$$

Since two outer automorphisms of order 3 differ (up to inverses) by an inner automorphism, the subset $H^1\big(F, W(D_4)\big)^{C_3}$ of cohomology classes fixed under the action of an outer automorphism of order 3 is independent of the choice of the automorphism. For any fixed outer automorphism $\xi$ of order 3, let $\big(W(D_4)\big)^{\xi}$ be the subgroup of elements of $W(D_4)$ fixed under $\xi$. There is a canonical map

$$H^1(F, W(D_4)^{\xi}) \to H^1\big(F, W(D_4)\big)^{C_3}. \tag{4.19}$$

**Proposition 4.20.** *Any class in $H^1\big(F, W(D_4)\big)^{C_3}$ lies in the image of the map (4.19) for $\xi = \tilde{\omega}$ or $\xi = \tilde{\rho}$ as in (4.16).*

*Proof.* Conjugate automorphisms define the same image in $H^1\big(F, W(D_4)\big)^{C_3}$, hence the claim follows from Proposition 4.16. □

**Trialitarian resolvents.** Trialitarian triples of étale algebras can be viewed as one étale algebra with two attached resolvents. For example, let $E$ be a quartic separable field with Galois group $\mathfrak{S}_4$. The field $E \otimes \Delta(E)$ is octic with the same Galois group $\mathfrak{S}_4$ and the extension $E \otimes \Delta(E)/E$ corresponds to Class $N = 82$ in Table 1. Class $N = 77$ in the same trialitarian triple corresponds to the extension

$$\big(\Delta(E) \times \lambda^2 E\big)/\big(F \times R(E)\big),$$

where $R(E)$ is the cubic resolvent of $E_0$ and $\lambda^2 E$ is the second lambda power of $E$, as defined in [12] .
In this section we compute the two resolvent polynomials for an étale algebra given by a polynomial. We assume that the base field has characteristic different from 2.

**Proposition 4.21.** *Let $S/S_0$ be an extension of étale $F$-algebras such that* $\dim_F S_0 = 4$ *and* $\dim_F S = 8$.
*1) There exists an element $x \in S$ such $x$ generates $S$ and $x^2$ generates $S_0$.*
*2) There exists a polynomial*

$$f_4(x) = x^4 + ax^3 + bx^2 + cx + d.$$

*with coefficients in $F$ such that $S_0 \xrightarrow{\sim} F[x]/\big(f_4(x)\big)$ and $S \xrightarrow{\sim} F[x]/\big(f_8(x)\big)$, where $f_8(x) = f_4(x^2)$.*
*3) The algebra $S$ has trivial discriminant if and only if $d$ is a square in $F$.*

*Proof.* We may assume that $S$ is a field. To prove 1) we are looking for elements $x$ of $S$ such that $\mathrm{Tr}_{S/S_0}(x) = 0$ and such that the discriminant of the characteristic polynomial of $x^2$ is not zero. These elements form an Zariski open subset of the space of trace zero elements. One checks that this open subspace is not empty by going to an algebraic closure of $F$.
2) follows from 1) and 3) follows from a discriminant formula (see [4, p. 51]) for the discriminant $d(f_8)$ of $f_8$:

$$d(f_8) = d \cdot \big(2^2 d(f_4)\big)^2$$

(recall that $\Delta(S) \xrightarrow{\sim} F[x]/\big(x^2 - d(f_8)\big)$). □

Let $S/S_0$ with trivial discriminant be given as in Proposition 4.21, by a polynomial

$$f_4(x) = x^4 + ax^3 + bx^2 + cx + e^2 \tag{4.22}$$

and let $\{y_1, y_2, y_3, y_4\}$ be the set of zeroes of $f_4$ in a separable closure $F_s$ of $F$. The set $\{\pm x_i = \pm\sqrt{y_i},\ i = 1, \dots 4\}$ is the set of zeroes of $f_8$. With this notation, we have:

**Proposition 4.23.** *Let $(S'/S'_0, S''/S''_0)$ be extensions of étale algebras associated by triality to $S/S_0$. Let $\xi$ be the column vector $[x_1, x_2, x_3, x_4]^T$ and let $\omega$ be the matrix* (4.2)*.*
*1) The components of the two vectors $\pm\xi' = \pm\omega\xi$ are the zeroes of a polynomial $f'_4$ defining $S'/S'_0$ as in* (4.21) *and the components of $\pm\xi'' = \pm\omega^2\xi$ are the zeroes a polynomial $f''_4$ defining $S''/S''_0$.*
*2) The polynomials $f'_4$ and $f''_4$ have the form*

$$\begin{aligned} f'_4(x) &= x^4 + ax^3 + (\tfrac{3}{8}a^2 - \tfrac{1}{2}b + 3e)x^2 + \\ &\quad (\tfrac{1}{16}a^3 - \tfrac{1}{4}ab + c + \tfrac{1}{2}ae)x + \left(\tfrac{1}{16}a^2 - \tfrac{1}{4}b - \tfrac{1}{2}e\right)^2 \\ f''_4(x) &= x^4 + ax^3 + (\tfrac{3}{8}a^2 - \tfrac{1}{2}b - 3e)x^2 + \\ &\quad (\tfrac{1}{16}a^3 - \tfrac{1}{4}ab + c - \tfrac{1}{2}ae)x + \left(\tfrac{1}{16}a^2 - \tfrac{1}{4}b + \tfrac{1}{2}e\right)^2 \end{aligned}$$

*where the coefficients $a, b, c$ and $e$ are as in* (4.22)*.*

*Proof.* The set of components $\{\pm x_i,\ i = 1, \dots, 4\}$ of the two vectors $\xi$ and $-\xi$ is the set of zeroes of the polynomial $f_8$ defining $S$. If $\alpha : \Gamma \to W(D_4) \subset \mathrm{O}_4$ is the cocycle corresponding to $S/S_0$, the group $\alpha(\Gamma)$ permutes the elements $\pm x_i$ through left matrix multiplication on $\xi$. The cocycle corresponding to $S'/S'_0$ is given by

$$\alpha' : \gamma \mapsto \omega\alpha(\gamma)\omega^{-1}, \gamma \in \Gamma$$

Thus $\alpha'(\Gamma)$ permutes the components of $\pm\xi' = \pm\omega\xi = \pm[x'_1, x'_2, x'_3, x'_4]^T$ and $\{\pm x'_i, i = 1, \dots, 4\}$ is the set of zeroes of $f'_8$. It follows that

$$f'_8(x) = f'_4(x^2) = \prod_i (x - x'_i)(x + x'_i) = \prod_i (x^2 - {x'_i}^2)$$

The $x'_i$ are the components of $\xi' = \omega\xi$. Thus the coefficients of $f'_8(x)$ can be expressed as functions of the $x_i$. Using that the symmetric functions in the $x_i$ can be expressed as functions of the coefficients of $f_8$ one gets (for example with Magma [2]) the expression given in Proposition 4.23 for $f'_i$. Similar computations with $\omega^2$ instead of $\omega$ lead to the formula for $f''_4$. □

**Remark 4.24.** Observe that we move from $f'_8$ to $f''_8$ by replacing $e$ by $-e$, as it should be.

**Triality and Witt invariants of étale algebras.** The results of this section were communicated to us by Serre, [15]. They are based on results of [14] and [9]. Similar results can be obtained for cohomological invariants of étale algebras instead of Witt invariants. Let $k$ be a fixed base field of characteristic not 2 and $F/k$ be a field extension. Let $WGr(F)$ be the Witt-Grothendieck ring and $W(F)$ the Witt ring of $F$, viewed as functors of $F$. We recall that elements of $WGr(F)$ are formal differences $q - q'$ of isomorphism classes of nonsingular quadratic forms over $F$ and that the sum and product are those induced by the orthogonal sum and the tensor product of quadratic forms. The Witt ring $W(F)$ is the quotient of

$WGr(F)$ by the ideal consisting of integral multiples of the 2-dimensional diagonal form $\langle 1, -1\rangle$.

Some of the following considerations hold for oriented quadratic extensions $S/S_0$ of étale algebras of arbitrary dimension. To simplify notation we assume from now on that $\dim_F S = 8$.

Let $\mathsf{\acute{E}tex}^{2/4}$ be the functor which associates to $F$ the set $\mathsf{\acute{E}tex}_F^{2/4}$ of isomorphism classes of oriented quadratic extensions $S/S_0$ of étale algebras of dimension $n$.

A *Witt invariant* on $\mathsf{\acute{E}tex}^{2/4}$, more precisely on $W(D_4)$, is a map

$$H^1\big(F, W(D_4)\big) \to W(F)$$

for each $F$, subject to compatibility and specialization conditions (see [9]). The set of Witt invariants $\mathrm{Inv}\,\big(W(D_4), W\big) = \mathrm{Inv}(\mathsf{\acute{E}tex}^{2/4}, W)$ is a module over $W(k)$. The aim of this section is to describe this set and how triality acts on it.

A main tool is the following splitting principle, which is a special case of a variant of the splitting principle for étale algebras (see [9, Theorem 24.]), and which was proved by Serre in his mini-course at the Ascona Conference, February 2007, [14].

**Theorem 4.25.** *If $a \in \mathrm{Inv}(\mathsf{\acute{E}tex}^{2/4}, W)$ satisfies $a(S/S_0) = 0$ for every product of two biquadratic algebras*

$$S = F\big(\sqrt{x}, \sqrt{y}\big) \times F\big(\sqrt{z}, \sqrt{t}\big), \quad S_0 = F\big(\sqrt{xy}\big) \times F\big(\sqrt{zt}\big).$$

*over every extension $F$ of $k$, then $a = 0$.*

A construction of Witt invariants is through trace forms. Let $S/S_0 \in \mathsf{\acute{E}tex}^{2/4}$ and let $\sigma$ be the involution of $S$. We may associate two nonsingular quadratic trace forms to $S/S_0$:

$$\begin{array}{lllll} Q(x) & = & Q_S(x) & = & \mathrm{Tr}_{S/F}(x^2) \\ Q'(x) & = & Q'_S(x) & = & \mathrm{Tr}_{S_0/F}\big(x\sigma(x)\big), \ x \in S \end{array}$$

The decomposition

$$S = \mathrm{Sym}(S, \sigma) \oplus \mathrm{Skew}(S, \sigma)$$

leads to orthogonal decompositions

$$Q = Q^+ \perp Q^-, \quad Q' = Q^+ \perp -Q^-$$

and $Q^+ = \langle 2\rangle \cdot Q_{S_0}$, hence the forms $Q^+$ and $Q^-$ define two Witt invariants attached to $S/S_0$. The étale algebras associated to $S/S_0$ by triality lead to corresponding invariants. To simplify notations let $S/S_0 = S_1/S_{0,1}$ and let $S_i/S_{0,i}$, $i = 2, 3$, be the associated étale algebras. We denote the corresponding Witt invariants by $Q_i^+ = Q_{S_i}^+$ and $Q_i^- = Q_{S_i}^-$, $i = 1$, 2 and 3.

Another construction of Witt invariants is through orthogonal representations. Let $\mathrm{O}_n$ be the orthogonal group of the $n$-dimensional form $\langle 1, \ldots, 1\rangle$. Quadratic forms over $F$ of dimension $n$ are classified by the cohomology set $H^1(F, \mathrm{O}_n)$. Thus any group homomorphism $W(D_4) \to \mathrm{O}_n$ gives rise to a Witt invariant. In particular we get a Witt invariant $q$ associated with the orthogonal representation $W(D_4) \to \mathrm{O}_4$ described in (4.4). Moreover the group $W(D_4)$ has three normal subgroups $H_i$ of type $(2, 2, 2)$ (i.e., isomorphic to $\mathfrak{S}_2^3$), corresponding to the classes $N = 33$, 34, 35 of Table 1. Since the factor groups are isomorphic to $\mathfrak{S}_4$, the canonical representation $\mathfrak{S}_4 \to \mathrm{O}_4$ through permutation matrices leads to three Witt invariants $q_1$, $q_2$, $q_3$.

**Proposition 4.26.** *1) The Witt invariant $q$ is invariant under triality and coincide with $Q_i^-$, $i = 1, 2, 3$.*
*2) We have $q_i = Q_i^+$, $i = 1, 2, 3$, and the three invariants $q_1$, $q_2$, $q_3$ are permuted by triality.*

*Proof.* The fact that $q$ is invariant under triality follows from the fact that triality acts on $W(D_4)$ by an inner automorphism of $\mathrm{O}_4$. Moreover the trialitarian action on $W(D_4)$ permutes the normal subgroups $H_i$, hence the invariants $q_i$. For the other claims we may assume by the splitting principle that $S_1$ is a product of two biquadratic algebras

$$S_1 = F\big(\sqrt{x}, \sqrt{y}\big) \times F\big(\sqrt{z}, \sqrt{t}\big), \quad S_{0,1} = F\big(\sqrt{xy}\big) \times F\big(\sqrt{zt}\big). \tag{4.27}$$

An explicit computation, using for example the description of triality given in the proof of Theorem 4.9 (see [1] or [18]) shows that

$$S_2 = F\big(\sqrt{x}, \sqrt{z}\big) \times F\big(\sqrt{y}, \sqrt{t}\big), \quad S_{0,2} = F\big(\sqrt{xz}\big) \times F\big(\sqrt{yt}\big)$$

and

$$S_3 = F\big(\sqrt{x}, \sqrt{t}\big) \times F\big(\sqrt{y}, \sqrt{z}\big), \quad S_{0,3} = F\big(\sqrt{xt}\big) \times F\big(\sqrt{yz}\big).$$

We get

$$Q_i^- = \langle x, y, z, t\rangle$$

for $i = 1, 2, 3$, and

$$Q_1^+ = \langle 1, 1, xy, zt\rangle,\ Q_2^+ = \langle 1, 1, xz, yt\rangle,\ Q_3^+ = \langle 1, 1, xt, yz\rangle.$$

The equalities $q = Q_i^-$ and $q_i = Q_i^+$ follow from the fact that the corresponding cocycles are conjugate in $\mathrm{O}_4$. □

Further basic invariants are the constant invariant $\langle 1\rangle$ and the discriminant

$$\langle d\rangle = \mathrm{Disc}(q) = \mathrm{Disc}(q_i),\ i = 1, 2, 3,$$

which corresponds to the 1-dimensional representation $\det\colon W(D_4) \to \mathrm{O}_1 = \pm 1$. Since $Q_i^+(1/2) = 1$, the quadratic forms $q_i = Q_i^+$ represent 1, one can replace them by 3-dimensional invariants $\ell_i = (1/2)^\perp \subset q_i$, $i = 1, 2, 3$.
In the following result $\lambda^2 q$ denotes the second exterior power of the quadratic form $q$ (see [9]). If $q = \langle \alpha_1, \ldots, \alpha_n\rangle$ is diagonal, then $\lambda^2 q$ is the $n(n-1)/2$-dimensional form $\lambda^2 q = \langle \alpha_1\alpha_2, \ldots, \alpha_{n-1}\alpha_n\rangle$.

**Theorem 4.28.** *1) The $W(k)$-module $\mathrm{Inv}\big(W(D_4), W\big) = \mathrm{Inv}\big(\mathsf{Étex}^{2/4}, W\big)$ is free over $W(k)$ with basis*

$$\big(\langle 1\rangle, \langle d\rangle, q, \langle d\rangle \cdot q, \ell_1, \ell_2, \ell_3\big). \tag{4.29}$$

*2) The elements $\langle 1\rangle$, $\langle d\rangle$, $q$, $\langle d\rangle{\cdot}q$ are fixed under triality and the elements $\ell_1, \ell_2$ and $\ell_3$ are permuted.*
*3) The following nonlinear relations hold among elements of (4.29):*

$$\begin{aligned} \langle d\rangle &= \mathrm{Disc}(q) = \mathrm{Disc}(q_i), \\ \lambda^2 q &= \ell_1 + \ell_2 + \ell_3 - \langle 1, 1, 1\rangle \\ \langle 1, d\rangle \cdot q &= q \cdot (\ell_i - 1),\ i = 1, 2, 3. \end{aligned}$$

*Proof.* 1) The proof follows the pattern of the proof of [9, Theorem 29.2]. Let $G = \mathfrak{S}_2^4$. An arbitrary element of $H^1(F,G)$ is given by a 4-tuple $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \in \big(F^\times/F^{\times 2}\big)^4$. For $I$ a subset of $[1,4] = \{1,2,3,4\}$, we write $\alpha_I$ for the product of the $\alpha_i$ for $i \in I$. By [9, Theorem 27.15] $\mathrm{Inv}(G,W)$ is a free $W(k)$-module with basis $(\alpha_I)_{I\subset[1,4]}$. It follows that (4.29) is also a basis of $\mathrm{Inv}(G,W)$ over $W(k)$. Let $a$ be an element of $\mathrm{Inv}\big(\mathsf{\acute{E}tex}^{2/4}, W\big)$ and let $S_\alpha$ be the algebra (4.27) for $\alpha_1 = x, \alpha_2 = y, \alpha_3 = z, \alpha_4 = t$. The map $\alpha \mapsto a(S_\alpha)$ is a Witt invariant of $G$, hence by [9, Theorem 27.15] can be uniquely written as a linear combination

$$\sum c_I \cdot \langle \alpha_I \rangle \text{ with } c_I \in W(k),$$

hence as a linear combination over $W(k)$ of the elements in (4.29). The restriction map $\mathrm{Inv}\big(W(D_4), W\big) \to \mathrm{Inv}(G,W)$ is injective by the splitting principle. Moreover by [9, Prop. 13.2], its image is contained in the $W(k)$-submodule of $\mathrm{Inv}(G,W)$ fixed by the normalizer $N$ of $G$ in $W(D_4)$. This normalizer is the subgroup $\mathfrak{S}_2^3 \rtimes \mathfrak{D}_4$ corresponding to class $N = 96$ in Table 1. The elements in (4.29) are invariant under $N$. This shows claim 1). Claim 2) follows from Proposition 4.26 and 3) is easy to check for a product of biquadratic extensions. □

## 5. Trialitarian étale algebras

A trialitarian triple $(S/S_0, S'/S'_0, S''/S''_0)$ of étale algebras can be viewed as an extension $T/T_0$ of algebras over the base $F^3$, satisfying a certain "trialitarian" property. Let $L$ be an arbitrary cubic étale algebra over $F$ and let $T/T_0$ be an extension of étale algebras over $L$ such that $T$ is free over $L$ of rank 8 and $T_0$ is free over $L$ of rank 4. The Clifford algebra $C(T/T_0)$ can be viewed as an object over $L$. For $T/T_0 = (S/S_0, S'/S'_0, S''/S''_0)$, a trialitarian triple, we have

$$C(T/T_0) = (T/T_0)^\rho \times (T/T_0)^{\rho^2}, \tag{5.1}$$

where $\rho$ is the cyclic permutation $(S/S_0, S'/S'_0, S''/S''_0) \mapsto (S'/S'_0, S''/S''_0, S/S_0)$. We generalize property (5.1) for arbitrary cubic étale extensions following a construction for central simple algebras given in [11]. Let $\Delta(L)$ be the discriminant of $L$. The $F$-algebra $L \otimes \Delta(L)$ is Galois with group $\mathfrak{S}_3$. Let $\rho$ be an automorphism of $L \otimes \Delta(L)$ of order 3. The algebra $T \otimes \Delta(L)$, for $T$ as above, is a $L \otimes \Delta(L)$-algebra and we denote by ${}^\rho\big(T \otimes \Delta(L)\big)$ the $L$-algebra $T \otimes \Delta(L)$, where the action of $L$ is twisted through $\rho$. We say that $E/E_0$ is a *trialitarian étale $F$-algebra* if there exists an isomorphism

$$\alpha_T \colon C(T/T_0) \xrightarrow{\sim} {}^\rho\big(T \otimes \Delta(L)\big) \tag{5.2}$$

such that

$${}^{\rho^2}\alpha_{{}^\rho(T\otimes\Delta(L))} = \alpha_{T\otimes\Delta(L)}.$$

Thus a trialitarian $F$-algebra is characterized by the data $(L, E/E_0, \alpha_E)$ and an isomorphism $(L, E/E_0, \alpha_E) \xrightarrow{\sim} (L', E'/E'_0, \alpha'_E)$ is a pair of isomorphisms $\varphi \colon L \xrightarrow{\sim} L'$, $\psi \colon E/E_0 \xrightarrow{\sim} E'/E'_0$ compatible with (5.2). One shows as in [11] for trialitarian central simple algebras that the set

$$H^1\big(\Gamma, (\mathfrak{S}_2^3 \rtimes \mathfrak{S}_4) \rtimes \mathfrak{S}_3\big)$$

classifies isomorphism classes of trialitarian étale $F$-algebras.

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