On Compositions and Triality

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Dedicated with gratitude to Professor M. Kneser on his 65^{th} birthday

1. Introduction.

In this paper we develop a general theory of compositions for quadratic spaces of rank 8 with trivial Arf and Clifford invariants. Using this theory, and adapting a classical technique of C. Chevalley, we construct classes of examples of Cayley algebras over any affine scheme. As an application, for any field K of characteristic not 2 which admits a Cayley division algebra, we construct Cayley algebras over the polynomial ring $K[x, y]$ whose norms, restricted to trace zero elements, are indecomposable as quadratic spaces. These give rise to principal G_2–bundles on $\mathbf{A}^2_K$ with no reduction of the structure group to any proper connected reductive subgroup, thus settling one of the two cases left open by M.S. Raghunathan in [R], the other being that of principal F_4–bundles.

In brief, we proceed as follows: we define, for any quadratic space over a scheme X, a Clifford invariant with values in $H^2_{fl}(X, \mu_2)$ which generalizes the refined Clifford invariant introduced in [PS] for schemes with 2 invertible. Quadratic spaces with trivial Arf and Clifford invariants admit compositions via half–spin representations, which run parallel to the compositions described by C. Chevalley in [Ch_1] for quadratic spaces of maximal index over fields. If a rank 8 quadratic space and one of its half–spin representations represent 1, then, adapting Chevalley's techniques, we can construct a Cayley algebra whose norm is the given quadratic space. In this context, it is natural to consider rank 7 quadratic spaces q for which $1 \perp q$ occurs as a half–spin representation. A specific choice of such an admissible space $1 \perp q$ leads to the construction of a class of G_2–bundles on an affine scheme which admit a reduction of the structure group to $SU(3)$. By "twisting" these bundles through a glueing process developed in [P_2], we get nontrivial G_2–bundles over $\mathbf{A}^2_K$ with the property mentioned above.

The organisation of the paper is as follows: in Sections 2 and 3 we place in a general setting classical results on spin and half–spin representations of maximal isotropic forms. In this context the Clifford invariant plays an important role. Section 4 contains results on triality in the spirit of [BS_2]. Here we prove that the similarity class of a rank 8 quadratic space with trivial Arf and Clifford invariants is determined by its even Clifford algebra with involution. Sections 5 and 6 describe the construction of G_2–bundles with reduction of the structure group to $SU(3)$. Section 7 contains the construction of non-trivial G_2–bundles over $\mathbf{A}^2_K$.

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2. Involutions and similitudes.

Throughout this section, R denotes a commutative ring and unadorned tensor products are taken over R. For any R-algebra A we denote the group of units of A by $A^\times$. An R–linear involution τ of an Azumaya R-algebra A is said to be of *the first kind.* If $A = \mathrm{End}_R(V)$, V a faithfully projective R-module, there exist an invertible R-module I and an isomorphism

$$b : V \otimes I \xrightarrow{\sim} V^* = \mathrm{Hom}_R(V, R)$$

such that $\tau(\varphi) \otimes 1 = b^{-1}\varphi^* b$ and $b^* = \varepsilon b$ for some $\varepsilon \in \mu_2(R) = \{x \in R \,|\, x^2 = 1\}$, * denoting transposition. If $I = R$, $b : V \xrightarrow{\sim} V^*$ is an ε–symmetric bilinear form (in fact the *adjoint* of a form $b : V \times V \to R$, but we shall not distinguish between a form and its adjoint) and we call the pair (V, b) an *ε–symmetric bilinear space.* The corresponding involution of $\mathrm{End}_R(V)$ is denoted by τ_b and ε is the *type* of b.

A 1-symmetric bilinear space (I, d), with I invertible, is a *discriminant module.* The isometry classes of discriminant modules form a group, denoted $\mathrm{Disc}(R)$, under the tensor product. We denote the class of (I, d) by $[I, d]$. Let $\langle r\rangle_R$ be the discriminant module (R, d) with $d\,(1, 1) = r,\ r \in R^\times$. An isometry

$$t : (V \otimes I, b \otimes d) \xrightarrow{\sim} (V', b')$$

is a *similitude* with multiplier (I, d). Similitudes of quadratic spaces are defined correspondingly. If $(I, d) = \langle r\rangle_R,\ t$ is a similitude with multiplier r in the classical sense. The set of similitudes of (V, b) is a group. We denote it by $GO(V, b)$. For any similitude t, let

$$\mathrm{End}(t) : \mathrm{End}_R(V) \xrightarrow{\sim} \mathrm{End}_R(V')$$

be given by $\mathrm{End}(t)(\varphi) = t(\varphi \otimes 1)t^{-1}$, $\varphi \in \mathrm{End}_R(V)$.

(2.1) Lemma. *Any similitude $t : V \otimes I \xrightarrow{\sim} V'$ induces an isomorphism of algebras with involution*

$$\mathrm{End}(t) : (\mathrm{End}_R(V), \tau_b) \xrightarrow{\sim} (\mathrm{End}_R(V'), \tau_{b'})$$

and any such isomorphism is of the form $\mathrm{End}(t)$ *for some similitude t which is uniquely determined up to a unit of R.*

Proof: By Morita theory (see [KPS] or [K], p. 171). □

An involution τ_b of $\mathrm{End}_R(V)$ is of *orthogonal type* if b is the polar of a quadratic form q, i.e. $b(x, y) = q(x + y) - q(x) - q(y)$ for $x, y \in V$. In this case we denote the involution by τ_q. An *isomorphism* $\mathrm{End}_R(V) \xrightarrow{\sim} \mathrm{End}_R(V')$ *of algebras with involutions of orthogonal type*, is, by definition, of the form $\mathrm{End}(t)$ with $t : V \otimes I \xrightarrow{\sim} V'$ a similitude of quadratic forms, not just bilinear forms.

Let S be a quadratic etale R-algebra with conjugation σ_0. For any S-module W we denote by ${}^{\sigma_0}W$ the module W with the action of S twisted through σ_0, by $W^{(*)}$ the S-dual, by W^* the R-dual and by $W^\vee$, the module ${}^{\sigma_0}(W^{(*)})$. Accordingly, we set ${}^{\sigma_0}f$, $f^{(*)}$ and $f^\vee$ for an S-linear map f. If W is finitely generated projective over S, we identify $W^{\vee\vee}$ with W through the map $x \mapsto x^{\vee\vee}$, $x^{\vee\vee}(f) = \sigma_0(f(x))$. An involution τ of an Azumaya S-algebra A such that $\tau\,|\,_S = \sigma_0$ is of *the second kind.* If $A = \mathrm{End}_S(W)$, an involution τ of the second kind is of the form

$$\tau(\varphi) \otimes 1 = B^{-1}\varphi^\vee B$$

for some S-linear isomorphism $B : W \otimes I \xrightarrow{\sim} W^\vee$, where I is an invertible R-module and $B^\vee = B$. If $I = R$, B is a genuine hermitian form. We call a pair (W, B), with W finitely generated projective over S and $B : W \xrightarrow{\sim} W^\vee$ a nonsingular hermitian form, a *hermitian space* and denote the involution $\varphi \mapsto B^{-1}\varphi^\vee B$ of $\mathrm{End}_S(W)$ by τ_B.

A hermitian space of rank one over S is a *hermitian discriminant module.* Hermitian discriminant modules form a group with respect to tensor product over S. The identity element is the form $\langle 1\rangle_S = (S, d)$ with $d(x, y) = \sigma_0(x)y$. For any hermitian space (W, B) of rank n, $(\wedge^n W, \wedge^n B)$ is a hermitian discriminant module. We call it the *hermitian discriminant* of (W, B).

The trace map $\mathrm{tr}_{S/R} : S \to R$, defined by $\mathrm{tr}_{S/R}(s) = s + \sigma_0(s)$, induces an isomorphism $tr : W^{(*)} \xrightarrow{\sim} W^*$ of R-modules for any finitely generated projective S–module W. Identifying $W^\vee$ with $W^{(*)}$ as R–modules, trace yields an isomorphism $tr : W^\vee \xrightarrow{\sim} W^*$. To any S–hermitian form $B : W \to W^\vee$ corresponds an R–bilinear symmetric form $B_* = tr \circ B : W \xrightarrow{\sim} W^*$ over R. The form B_* is the polar form of the quadratic form $q_B(x) = B(x, x)$.

(2.2) Lemma. *Let W be a finitely generated projective S-module and let b be a symmetric R–bilinear nonsingular form over W. Then $b = B_*$ for some hermitian form B on W if and only if $b(sx, y) = b(x, \sigma_0(s)y)$ for $s \in S$, $x, y \in W$.*

Proof: Let $B : W \to W^\vee$ be defined as $B = tr^{-1} \circ b$, treating b as a linear map $W \to W^*$. Then B is S–linear if and only if $b(sx, y) = b(x, \sigma_0(s)y)$ for $s \in S$, $x, y \in W$ and, in this case, $b = B_*$. □

(2.3) Lemma. *Let W and b be as in 2.2. We have $b = B_*$ for some hermitian form B on W if and only if the involution τ_b induced by b restricts to σ_0 on the image of S in $\mathrm{End}_R(W)$. In this case τ_b restricts to the involution of the second kind τ_B on $\mathrm{End}_S(W)$.*

Proof: Let $B = tr^{-1} \circ b$. The condition $B : W \to W^{(*)}$ is σ_0–semilinear is equivalent to the condition $\tau_b(s) = \sigma_0(s)$ for $s \in S$. The rest of the assertions follows from 2.2. □

(2.4) Corollary. *Let (W, b) be as in 2.2. If τ_b restricts to σ_0 on the image of S, then τ_b is of orthogonal type.*

Proof: In fact we have $b = b_{q_B}$ with $q_B(x) = B(x, x)$. □

(2.5) Remark. A bilinear form b *admits* S if $b(sx, y) = b(x, \sigma_0(s)y)$ for $s \in S$, $x, y \in W$. The functor, which assigns to a S-hermitian space (W, B) the quadratic space (W, q_B) over R, is an isomorphism of the category of S-hermitian spaces with the category of quadratic spaces over R whose polars admit S (see [FM]).

Let (I, d) be a discriminant module and let (M, q) be a quadratic space over R. Let $C(q) = C_0(q) \oplus C_1(q)$ be the Clifford algebra of (M, q). We define a graded algebra structure on the R–module $C_0(q) \oplus C_1(q) \otimes I$ by

$$(c_0 + c_1 \otimes x)(c_0' + c_1' \otimes x') = c_0c_0' + c_1c_1' d(x, x') + c_0c_1' \otimes x' + c_1c_0' \otimes x.$$

(2.6) Lemma. *1) The canonical map $M \otimes I \to C_1(q) \otimes I$ induces a graded isomorphism of* *algebras*

$$C(q \otimes d) \xrightarrow{\sim} C_0(q) \oplus C_1(q) \otimes I.$$

2) Any similitude $t : M \otimes I \xrightarrow{\sim} M'$ *induces an isomorphism* $C_0(t) : C_0(q) \xrightarrow{\sim} C_0(q')$ *of algebras and a* $C_0(t)$*–semilinear isomorphism of bimodules* $C_1(t) : C_1(q) \otimes I \xrightarrow{\sim} C_1(q')$ *such that* $C_1(t)|_{M \otimes I} = t$.

Proof: 1) The existence of a homomorphism $C(q \otimes d) \to C_0(q) \oplus C_1(q) \otimes I$ follows from the universal property of the Clifford algebra. The map is an isomorphism since $C(q \otimes d)$ is an Azumaya algebra. 2) is a consequence of 1). □

Assume that M has even rank. Then the centre Z of $C_0(q)$ is a quadratic etale R–algebra. Let σ_0 be the unique R–linear nontrivial involution of Z. A similitude t of M is *proper* if $C_0(t)$ restricts to the identity of Z and is *improper* if it restricts to σ_0. If R is connected, any similitude is either proper or improper. We denote by $GO_+(q)$ the group of proper similitudes and by $GO_-(q)$ the set of improper similitudes of (M, q).

3. The Clifford invariant and spin representations.

Most of the results of this section are valid over arbitrary algebraic schemes. However, to simplify the exposition, we restrict to affine schemes. Let (U, p) be a quadratic space over R of rank $2m$. The Clifford algebra $C(p)$ of (U, p) is an Azumaya algebra over R, the centre Z of the even Clifford algebra $C_0(p)$ is, as already observed, a quadratic etale R–algebra and $C_0(p)$ is an Azumaya algebra over Z. We call the involution τ of $C(p)$ which is the identity on U the *first involution* of $C(p)$ and the involution τ' such that $\tau'(x) = -x$ for $x \in U$ the *second involution* of $C(p)$. Let τ_0 be the restriction of τ (or τ') to $C_0(p)$. Then τ_0 restricts to the identity of Z if $\mathrm{rank}_R U \equiv 0\,(4)$ and to the unique nontrivial R–automorphism of Z if $\mathrm{rank}_R U \equiv 2\,(4)$. If not explicitly specified, we consider $C(p)$ as an algebra with the involution τ and $C_0(p)$ as an algebra with the involution τ_0. We recall that $\nu(c) = c\tau(c) \in R^\times$ for any $c \in C^\times$ with $cUc^{-1} \subset U$.

Let $O(p)$ be the group of isometries of (U, p) and let $SO(p) = O(p) \cap GO_+(p)$ be the *special orthogonal group*. Let $hC(p)^\times$ be the group of locally homogeneous units of $C(p)$, let

$$\mathrm{Pin}(p) = \{c \in hC(p)^\times \mid (-1)^{\deg(c)} cUc^{-1} \subset U \text{ and } c\tau(c) = 1\}$$

and let $\mathrm{Spin}(p) = \mathrm{Pin}(p) \cap C_0(p)$. We have exact sequences (see [B])

$$1 \to \mu_2(R) \to \mathrm{Pin}(p) \xrightarrow{\chi} O(p) \xrightarrow{SN} \mathrm{Disc}(R)$$

and

$$1 \to \mu_2(R) \to \mathrm{Spin}(p) \xrightarrow{\chi} SO(p) \xrightarrow{SN} \mathrm{Disc}(R)$$

where χ is the *vector representation*, i.e. $\chi_c(x) = (-1)^{\deg(c)} cxc^{-1}$, $x \in U$, and SN is the spinor norm.

In [PS] an invariant, called the *refined Clifford invariant*, with values in $H^2_{et}(X, \mu_2)$, $X = \mathrm{Spec}(R)$, was associated to a quadratic space over R, assuming that $2 \in R^\times$. Without the assumption 2 invertible, we define the *Clifford invariant*, with values in $H^2_{fl}(X, \mu_2)$, as follows: The above exact sequence yields an exact sequence of sheaves of groups

$$1 \to \mu_2 \to \mathrm{Pin}_{2m} \to O_{2m} \to 1$$

for the flat topology, where Pin and O are sheaves of flat sections of the group Pin, resp. the orthogonal group, associated to the hyperbolic quadratic form

$$q_H(x_1,\ldots,x_m,y_1,\ldots,y_m) = x_1y_1 + x_2y_2 + \ldots x_my_m.$$

Any rank $2m$ quadratic space (U,p) over X defines a class in $H^1_{fl}(X,O_{2m})$ and we define its image in $H^2_{fl}(X,\mu_2)$ under the connecting homomorphism $\partial : H^1_{fl}(X,O_{2m}) \to H^2_{fl}(X,\mu_2)$ (see [G], Remarque 4.2.10, p. 284) as the Clifford invariant of (U,p). One can verify that the Clifford invariant coincides, in the case 2 is invertible, with the refined Clifford invariant defined in [PS].

(3.1) Proposition. *Let (U,p) be a quadratic space over R of rank $2m$ with trivial Clifford invariant. There exists an isomorphism of algebras with involution*

$$\alpha : C(p) \xrightarrow{\sim} (\mathrm{End}_R(V),\tau_b)$$

for some ε–bilinear space (V,b). If $2m \equiv 0$ (8), the form b is the polar of a quadratic form q on V and the involution τ_b is of orthogonal type. Further, we have
1) $q(\alpha(x)(v)) = p(x)q(v)$ for $x \in U$ and $v \in V$.
2) $q(\alpha(c)(v)) = \nu(c)q(v)$ for $v \in V$ and $c \in C^\times$ with $cUc^{-1} \subset U$.

Proof: By [G], Proposition 4.2.8, p. 283, the Clifford invariant of (U,p) is trivial if and only if the class of (U,p) in $H^1_{fl}(X,O_{2m})$ is in the image of the canonical map $H^1_{fl}(X,\mathrm{Pin}_{2m}) \to H^1_{fl}(X,O_{2m})$. In this case we have an isomorphism $\alpha : C(p) \xrightarrow{\sim} (\mathrm{End}_R(V),\tau_b)$ for some ε–bilinear space (V,b). Let α_{ij} be a Čech 1–cocycle in Pin_{2m}, with respect to an affine covering $\{U_i\}$ of $X = \mathrm{Spec}(R)$ (for the flat topology), such that its image in O_{2m} defines the quadratic space (U,p). Let i,j be fixed and let $U_i \cap U_j = \mathrm{Spec}(S)$. The restriction of Clifford algebra $C(q_H)$ to $U_i \cap U_j$ is canonically isomorphic to $\mathrm{End}(\wedge(S^m))$ (see [Ch$_1$] or [B]) and α_{ij}, which is a unit of $C(q_H)$ restricted to $U_i \cap U_j$, corresponds to an element of $\mathrm{End}(\wedge(S^m))$ which preserves the bilinear form

$$b_0(x,y) = \begin{cases} 0 & \text{if} \quad k+\ell \neq m \\ \tau(x)y & \text{if} \quad k+\ell = m \end{cases}$$

for $x \in \wedge^k(S^m)$ and $y \in \wedge^\ell(S^m)$, τ denoting the involution of the exterior algebra $\wedge(S^m)$ which is the identity on S^m (see [PS]). This element defines a 1–cocycle with values in $O(\wedge(S^m),b_0)$ and yields a symmetric bilinear space (V,b). By the very construction we have an isomorphism $C(U,p) \simeq (\mathrm{End}_R(V),\tau_b)$. Further, if $2m \equiv 0$ (8) and $m = 2l$, let $q_0 : \wedge(R^m) \to \wedge^m(R^m) \simeq R$ be defined by

$$q_0(x) = \begin{cases} 0 & \text{if} \quad x \notin \wedge^\ell(R^{2\ell}) \\ (-1)^{\frac{\ell(\ell-1)}{2}}\exp(x)_{2\ell} & \text{if} \quad x \in \wedge^\ell(R^{2\ell}), \end{cases}$$

where exp is the exponential mapping as defined by Chevalley in [Ch$_2$]. On $U_i \cap U_j$, b_0 is the polar of q_0. Formulae 1) and 2) (over $U_i \cap U_j$) can be verified as in [Ch$_1$], Chapter III, Section 2.7. The element α_{ij} leaves in fact the restriction of q_0 to $U_i \cap U_j$ invariant, so that it defines a class (V,q) in $H^1_{fl}(X,O(q_0))$ as required. Formulae 1) and 2) hold since they hold locally. □

An isomorphism of algebras with involution

$$\alpha : C(p) \xrightarrow{\sim} (\mathrm{End}_R(V),\tau_q)$$

is a *spin representation* and (V,q) a *spin representation space*. We use the notation $\alpha(c)=\alpha_c$ for $c\in C(p)$. Given a spin representation α, we regard V as a Z–module through α, Z being the centre of $C_0(p)$. Since $C_0(p)$ is the centralizer of Z in $C(p)$ and since

$$C_1(p)=\{x\in C(p)\mid \sigma_0(z)x=xz,\,\forall z\in Z\},$$

α induces isomorphisms

$$\alpha_0: C_0(p)\xrightarrow{\sim}\mathrm{End}_Z(V)=V\otimes_Z V^{(*)} \text{ and } \alpha_1: C_1(p)\xrightarrow{\sim}\mathrm{Hom}_Z({}^{\sigma_0}V,V)=V\otimes_Z V^{\vee}.$$

For any $t\in SO(p)$, $C(t)$ is an automorphism of $C(p)$ and, by 2.1, $C(t)$ induces a similitude

$$\tilde{t}:(V,q)\otimes(I_t,d_t)\xrightarrow{\sim}(V,q).$$

In fact, the spinor norm $SN(t)$ of t is the class $[I_t,d_t]$ in $\mathrm{Disc}(R)$ (see [B]), so that $t\in SO(p)$ induces an isometry of (V,q) if and only if $SN(t)=1$ or, equivalently, if $t=\chi_c$ for some $c\in\mathrm{Spin}(p)$.

Let (U,p) be a quadratic space with trivial Clifford and Arf invariants (we recall that the *Arf invariant* is the isomorphism class of the centre Z of $C_0(p)$ if (U,p) has even rank; the Arf invariant is *trivial* if $Z\simeq R\times R$). Let $\alpha: C(p)\xrightarrow{\sim}\mathrm{End}_R(V)$ be a fixed spin representation and let $e\in Z$ be an idempotent generating $Z=R\times R$. For simplicity of presentation we restrict in the following to the case R connected. This implies that the pair of idempotents $(e,1-e)$ of Z is unique. We get a decomposition $V=E\oplus F$ with $E=\alpha_e V$ and $F=\alpha_{1-e}V$, the algebra $\mathrm{End}_R(V)$ has a corresponding block decomposition

$$\mathrm{End}_R(E\oplus F)=\begin{pmatrix}\mathrm{End}_R(E) & \mathrm{Hom}_R(F,E)\\ \mathrm{Hom}_R(E,F) & \mathrm{End}_R(F)\end{pmatrix}$$

and the gradation of $C(p)$ corresponds to the checker-board gradation of $\mathrm{End}_R(E\oplus F)$. Observe that $\mathrm{rank}_R E=\mathrm{rank}_R F$. If $\mathrm{rank}_R U\equiv 0$ (8), the involution τ_0 is the identity on $Z=R\times R$ and by 3.1 there exists nonsingular quadratic forms q_E and q_F on E, resp. F, such that the transport $\alpha\tau\alpha^{-1}$ of the involution τ of $C(p)$ is of the form τ_q with $q=q_E\perp q_F$. Let b_E and b_F be the polars of q_E and q_F respectively. We call (E,q_E), (F,q_F) a *pair of half-spin representation spaces*. We set

$$\alpha_c=\begin{pmatrix}\beta_c & \rho_c\\ \lambda_c & \gamma_c\end{pmatrix}\in\mathrm{End}_R(E\oplus F)\text{ for } c\in C(p)$$

and call $c\mapsto\beta_c$, $c\mapsto\gamma_c$ the *half-spin representations* of $C_0(p)$. For $u\in U$ the elements $\lambda_u\in\mathrm{Hom}_R(E,F)$, $\rho_u\in\mathrm{Hom}_R(F,E)$ satisfy $\lambda_u\rho_u=p(u)\cdot 1_F$ and $\rho_u\lambda_u=p(u)\cdot 1_E$. Let $\lambda(u,x)=\lambda_u(x)$ and $\rho(u,y)=\rho_u(y)$ for $u\in U$, $x\in E$ and $y\in F$. The maps $\lambda: U\times E\to F$ and $\rho: U\times F\to E$ are bilinear and 3.1 implies that

$$q_F(\lambda(u,x))=p(u)q_E(x)\ \text{ and }\ q_E(\rho(u,y))=p(u)q_F(y).$$

A triple of nonsingular quadratic spaces (U,p), (E,q_E), (F,q_F), with a bilinear map λ as above, is a *composition of quadratic forms*. Thus any quadratic space of rank $8m$ with trivial Arf and Clifford invariants gives rise to a composition $\lambda: U\times E\to F$. The converse also holds:

(3.2) Proposition. *Let $\lambda : U \times E \to F$ be a composition of quadratic spaces (U,p), (E,q_E) and (F,q_F) such that $\mathrm{rank}_R U = 8m$ and $\mathrm{rank}_R E = \mathrm{rank}_R F = 2^{4m-1}$. Then (U,p) has trivial Arf and Clifford invariants and (E,q_E), (F,q_F) is a pair of half-spin representation spaces of (U,p).*

Proof: We view λ as a map $U \to \mathrm{Hom}_R(E,F)$ and put $\lambda_u(x) = \lambda(u,x)$. Let $\rho_u = b_E^{-1}\lambda_u^* b_F$. Then $u \mapsto \left(\begin{smallmatrix} 0 & \rho_u \\ \lambda_u & 0 \end{smallmatrix}\right) \in \mathrm{End}_R(E \oplus F)$ extends to an isomorphism $C(p) \xrightarrow{\sim} \mathrm{End}_R(E \oplus F)$ of graded algebras and the involution τ_q with $q = \left(\begin{smallmatrix} q_E & 0 \\ 0 & q_F \end{smallmatrix}\right)$ corresponds to τ. □

(3.3) Remark. In view of the Radon-Hurwitz formula, the half-spin representation spaces E and F are spaces of the smallest possible rank which admit composition with U.

If $\lambda : U \times E \to F$ and $\lambda' : U' \times E' \to F'$ are compositions, an *isometry* $\lambda \xrightarrow{\sim} \lambda'$ of compositions is a triple (t, t_2, t_1) of isometries $t : U \xrightarrow{\sim} U'$, $t_2 : E \xrightarrow{\sim} E'$ and $t_1 : F \xrightarrow{\sim} F'$ such that $t_1 \circ \lambda = \lambda' \circ (t, t_2)$.

(3.4) Proposition. *Let $c \in C_0(p)^\times$. The following conditions are equivalent:*
1) $c \in \mathrm{Spin}(p)$.
2) $cUc^{-1} \subset U$, β_c is an isometry of (E,q_E) and γ_c is an isometry of (F,q_F).
3) $(\chi_c, \beta_c, \gamma_c)$ is an isometry of the composition λ.

Proof: The equivalence of 1) and 2) follows from 3.1. If $cuc^{-1} \in U$, we have $\gamma_c \circ \lambda = \lambda \circ (\chi_c, \beta_c)$ since $\lambda_{cuc^{-1}} = \gamma_c \lambda_u \beta_c^{-1}$. Thus 3) is also equivalent to 2). □

(3.5) Proposition. *Let (U,p), (U',p') be quadratic spaces with trivial Clifford and Arf invariants and let $\lambda : U \times E \to F$, $\lambda' : U' \times E' \to F'$, be compositions given by half-spin representations. Let $t : (U,p) \xrightarrow{\sim} (U',p') \otimes (I,d)$ be a similitude. There exist a discriminant module (J,k) and either similitudes $t_2 : E \otimes J \xrightarrow{\sim} E'$, $t_1 : F \otimes I \otimes J \xrightarrow{\sim} F'$ or similitudes $t_2 : E \otimes J \xrightarrow{\sim} F'$, $t_1 : F \otimes I \otimes J \xrightarrow{\sim} E'$ such that (t, t_1, t_2) is an isometry of $\lambda \otimes 1 : U \times E \otimes I \to F \otimes I \otimes J$ with λ' or an isometry of $\rho \otimes 1 : U \times F \otimes I \to E \otimes I \otimes J$ with λ'. Furthermore t determines the pair (t_1, t_2) up to a unit of R.*

Proof: Let $\alpha : C(p) \xrightarrow{\sim} \mathrm{End}_R(E \oplus F)$, $\alpha' : C(p') \xrightarrow{\sim} \mathrm{End}_R(E' \oplus F')$ be the spin representations induced by λ, λ' respectively, as in 3.2. Then $\alpha' \circ C_0(t) \circ \alpha^{-1} : \mathrm{End}_R(E \oplus F) \xrightarrow{\sim} \mathrm{End}_R(E' \oplus F')$ is an isomorphism of algebras with involution (of orthogonal type). If e, e', are idempotents of $C(p)$ and $C(p')$ corresponding to the half–spin representations of (U,p), (U',p'), respectively, we have either $C(t)(e) = e'$ or $= 1 - e'$. This corresponds to the two described cases in the claim, which then follows from 2.1. □

(3.6) Corollary. *Let $\lambda : U \times E \to F$, $\rho : U \times F \to E$ be compositions given by a pair of half-spin representation spaces (E,F) of the quadratic space (U,p).*
1) If $t : U \otimes I \xrightarrow{\sim} U$ is a proper similitude of (U,p), with multiplier (I,d), there exist a discriminant module (J,k) and similitudes $t_2 : E \otimes J \xrightarrow{\sim} E$, $t_1 : F \otimes I \otimes J \xrightarrow{\sim} F$ such that (t, t_2, t_1) is an isometry of $\lambda \otimes 1 \otimes 1 : U \otimes I \times E \otimes J \to F \otimes I \otimes J$ with λ.
2) If $t : U \otimes I \xrightarrow{\sim} U$ is an improper similitude of (U,p), with multiplier (I,d), there exist a discriminant module (J,k) and similitudes $t_2 : E \otimes J \xrightarrow{\sim} F$, $t_1 : F \otimes I \otimes J \xrightarrow{\sim} E$ such that

(t, t_2, t_1) *is an isometry of* $\lambda \otimes 1 \otimes 1 : U \otimes I \times E \otimes J \to F \otimes I \otimes J$ *with* ρ.

We next assume that the quadratic space (U, p) represents a unit, i.e. there exists $u_1 \in U$ such that $p(u_1) \in R^\times$. Replacing p by $p(u_1)^{-1}p(x)$, we may as well assume that the form p represents 1. Then $\lambda_{u_1} : (E, b_E) \xrightarrow{\sim} (F, b_F)$ is an isometry with inverse ρ_{u_1}. Replacing λ by $\rho_{u_1} \circ \lambda$, we get a composition $\lambda : U \times E \to E$ such that u_1 acts as identity on E and a spin representation $\alpha : C(p) \xrightarrow{\sim} \mathrm{End}_R(E \oplus E)$ such that $\alpha_{u_1} = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

(3.7) Remark. A similitude $(E, q_E) \xrightarrow{\sim} (F, q_F)$ may exist even if (U, p) does not represent a unit. Let $R = \mathbf{R}[x, y]$ be the polynomial ring in two variables over the field of real numbers, let (R^n, p_n) be an indecomposable quadratic space over R of rank n such that its reduction modulo (x, y) is the diagonal form $\langle 1, \ldots, 1\rangle$. Such spaces exist for $n \geq 3$, by [P_2]. Then $p_3 \perp p_5$ does not represent a unit and has trivial Arf and Clifford invariants. The isometry $t = -1 \perp 1$ switches the two factors of the centre $R \times R$ of $C_0(p)$ since it is improper. Thus $C(t)$ induces a similitude $t_2 : E \xrightarrow{\sim} F$ for any pair of half-spin representation (E, F).

4. Triality.

Let (U, p) be a quadratic space of rank 8 with trivial Arf and Clifford invariants. Let $\alpha : C(p) \xrightarrow{\sim} \mathrm{End}_R(E \oplus F)$ be a half-spin representation. The two quadratic spaces (E, q_E) and (F, q_F) also have rank 8. We construct six compositions relating U, E and F. We put $\lambda_1 = \lambda$, $\rho_1 = \rho$, where λ and ρ are as in Section 3, and define λ_2, ρ_2, λ_3, ρ_3 as follows. The map ρ_2 is given by $\rho_2(x, u) = \lambda_1(u, x)$. Let $T : U \times E \times F \to R$ be the trilinear form

$$(u, x, y) \mapsto b_F(\lambda_1(u, x), y) = b_E(x, \rho_1(u, y)).$$

For fixed $(x, y) \in E \times F$, we define $f_{(x,y)} \in U^*$ by $f_{(x,y)}(u) = T(u, x, y)$. Since p is nonsingular, there exists $\lambda_2(x, y) \in U$ such that $f_{(x,y)}(u) = p(\lambda_2(x, y), u)$ for all $u \in U$. By definition of λ_2 and ρ_2, we have

$$b_p(\lambda_2(x, y), u) = b_F(y, \rho_2(x, u)).$$

Finally, we set $\lambda_3(y, u) = \rho_1(u, y)$ and define $\rho_3 : F \times E \to U$ through the trilinear form T, i.e.

$$b_p(\rho_3(y, x), u) = b_E(x, \lambda_3(y, u)).$$

To check that all these maps are compositions of the corresponding quadratic forms, we can localize and follow Chevalley's computations ([Ch_1], p. 120).

For any composition $\mu : X \times Y \to W$ we denote by μ_x the linear map $Y \to W$ given by $\mu_x(y) = \mu(x, y)$. For the proof of the following result, we shall use the identities

$$\begin{array}{rcccl} \lambda_{2,x}\rho_{2,x} & = & q_E(x) \cdot 1 & = & \rho_{2,x}\lambda_{2,x} \\ \lambda_{3,y}\rho_{3,y} & = & q_F(y) \cdot 1 & = & \rho_{3,y}\lambda_{3,y} \end{array}$$

for $x \in E$ and $y \in F$.

(4.1) Proposition. *The pair* (λ_2, ρ_2) *induces an isomorphism*

$$\alpha_2 : C(q_E) \xrightarrow{\sim} (\mathrm{End}_R(U \oplus F), \tau_{q_2}), \text{ where } q_2 = \left(\begin{smallmatrix} p & 0 \\ 0 & q_F \end{smallmatrix}\right),$$

and (λ_3,ρ_3) *induces an isomorphism*

$$\alpha_3 : C(q_F) \xrightarrow{\sim} (\mathrm{End}_R(U\oplus E),\tau_{q_3}),\ \ \textit{where}\ q_3 = \left(\begin{smallmatrix} p & 0 \\ 0 & q_E \end{smallmatrix}\right).$$

Proof: The map α_2 is induced by $x \mapsto \left(\begin{smallmatrix} 0 & \rho_{2,x} \\ \lambda_{2,x} & 0 \end{smallmatrix}\right)$ and α_3 is induced by $y \mapsto \left(\begin{smallmatrix} 0 & \rho_{3,y} \\ \lambda_{3,y} & 0 \end{smallmatrix}\right)$. □

(4.2) Corollary. *Let R be a connected ring. Two quadratic spaces of rank 8 over R with trivial Arf and Clifford invariants are similar if and only if their even Clifford algebras are isomorphic as algebras with involution.*

Proof: Let (U,p) and (U',p') be the two spaces, let

$$\begin{array}{rcl} \alpha_0 &:& C_0(p) \xrightarrow{\sim} \mathrm{End}_R(E)\times \mathrm{End}_R(F) \\ \alpha_0' &:& C_0(p') \xrightarrow{\sim} \mathrm{End}_R(E')\times \mathrm{End}_R(F') \end{array}$$

be induced by half-spin representations and let $\psi : C_0(p) \xrightarrow{\sim} C_0(p')$ be an isomorphism of algebras with involution. Since R is connected, we have $\alpha_0'\psi\alpha_0^{-1}(1,\ 0) = (1,0)$ or $= (0,1) \in R\times R$. By relabelling E' and F', we may assume that $\alpha_0'\psi\alpha_0^{-1}$ maps $\mathrm{End}_R(E)$ to $\mathrm{End}_R(E')$ and $\mathrm{End}_R(F)$ to $\mathrm{End}_R(F')$. Thus $\alpha_0'\psi\alpha_0^{-1}$ is an isomorphism of algebras with involutions $\mathrm{End}_R(E)\times\mathrm{End}_R(F) \xrightarrow{\sim} \mathrm{End}_R(E')\times \mathrm{End}_R(F')$ over $R\times R$ and, by 2.1, ψ induces similitudes

$$\begin{array}{rcl} \varphi_2 &:& (E,q_E)\otimes(I_2,d_2) \xrightarrow{\sim} (E',q_{E'}) \\ \varphi_3 &:& (F,q_F)\otimes(I_3,d_3) \xrightarrow{\sim} (F',q_{F'}) \end{array}$$

of quadratic forms, for some discriminants modules (I_2,d_2), (I_3,d_3). In turn, by 2.6, φ_2 and φ_3 induce isomorphisms of algebras with involution

$$C_0(\varphi_2) : C_0(q_E) \xrightarrow{\sim} C_0(q_{E'}),\quad C_0(\varphi_3) : C_0(q_F) \xrightarrow{\sim} C_0(q_{F'}),$$

so that by 4.1,

$$(\mathrm{End}_R(U)\times \mathrm{End}_R(F),\tau_{p\times q_F}) \simeq (\mathrm{End}_R(U')\times\mathrm{End}_R(F'),\tau_{p'\times q_{F'}}).$$

We either have

$$(\mathrm{End}_R(U),\tau_p) \simeq (\mathrm{End}_R(U'),\tau_{p'})$$

or

$$(\mathrm{End}_R(U),\tau_p) \simeq (\mathrm{End}_R(F'),\tau_{q_{F'}}) \text{ and } (\mathrm{End}_R(F),\tau_{q_F}) \simeq (\mathrm{End}_R(U'),\tau_{p'}).$$

Since F and F' are similar, we get in any case an isomorphism

$$(\mathrm{End}_R(U),\tau_p) \simeq (\mathrm{End}_R(U'),\tau_{p'})$$

and, as claimed, (U,p) and (U',p') are similar. The other direction follows by 2.6. □

If U and E represent units, we may as well assume that they represent 1 (by scaling p and q_E). Let $u_1 \in U$ be such that $p(u_1) = 1$ and let $x_1 \in E$ be such that $q_E(x_1) = 1$. Then $y_1 = \lambda_1(u_1,x_1) \in F$ is such that $q_F(y_1) = 1$. We define a composition $\circ : U\times U \to U$ by

$$\begin{array}{rcl} u\circ v &=& \lambda_2(\rho_1(u,y_1),\lambda_1(v,x_1)) \\ &=& \lambda_2(\lambda_3(y_1,u),\rho_2(x_1,v)) \end{array}$$

for $u, v \in U$. By construction $(\lambda_{3,y_1}, \rho_{2,x_1}, 1_U)$ is an isometry of the composition $\circ$ with the composition $\lambda_2 : E \times F \to U$ and we have $p(u \circ v) = p(u)p(v)$ for $u, v \in U$. We get

$$\begin{aligned} u_1 \circ v &= \lambda_2(\rho_1(u_1, y_1), \lambda_1(v, x_1)) \\ &= \lambda_2(\rho_{1,u_1}(\lambda_1(u_1, x_1)), \rho_2(x_1, v)) \\ &= \lambda_{2,x_1}\rho_{2,x_1} v = v \end{aligned}$$

and similarly $v \circ u_1 = v$ for all $v \in U$. Thus $\circ$ admits u_1 as a unit element. A space (U, p) of rank 8 with a composition $U \times U \to U$ which admits a unit element is a *Cayley algebra*. The construction of a Cayley algebra given above, out of a half-spin representation, is in [Ch$_1$] for (U, p) a quadratic space of maximal index over a field. We call it the *Chevalley construction.*

(4.3) Question. We obtain a composition $\circ : U \times U \to U$ assuming that the quadratic spaces (U, p) and (E, q_E) represent units. Conversely, given a composition $\circ : U \times U \to U$, does (U, p) represent a unit? This is the case if U is of rank 4. We do not know the answer if $\mathrm{rank}_R U = 8$.

Let $\mathfrak{C}$ be a Cayley algebra with composition $\circ$, norm $\mathfrak{n}$ and unit element u_1. For any $x \in \mathfrak{C}$ we set $\overline{x} = b_{\mathfrak{n}}(x, u_1)u_1 - x$. We have $\overline{\overline{x}} = x$ and one can check as in [Ch$_1$], p. 124, 125, [BS$_1$], or in [K], Chapter V, §7, that $\overline{x \circ y} = \overline{y} \circ \overline{x}$, $\overline{x}x = x\overline{x} = \mathfrak{n}(x)u_1$, $x \circ (\overline{x} \circ y) = (x \circ \overline{x}) \circ y = \mathfrak{n}(x)y$ and that $\mathfrak{C}$ is an alternative algebra. We shall also use the formula $b_{\mathfrak{n}}(x \circ y, z) = b_{\mathfrak{n}}(y, \overline{x} \circ z)$, which holds for any Cayley algebra (see [BS$_1$]). The map $x \mapsto \overline{x}$ is the *conjugation* of $\mathfrak{C}$.

(4.4) Proposition. *For any composition algebra $\mathfrak{C}$, the map $x \mapsto \left(\begin{smallmatrix} 0 & \mu_{\overline{x}} \\ \mu_x & 0 \end{smallmatrix}\right)$, with $\mu_x(y) = x \circ y$, induces isomorphisms of algebras with involutions*

$$C(\mathfrak{C}, \mathfrak{n}) \xrightarrow{\sim} (\mathrm{End}_R(\mathfrak{C} \oplus \mathfrak{C}), \tau_{\widetilde{\mathfrak{n}}}) \ \textit{and} \ \ (C_0(\mathfrak{C}, \mathfrak{n}), \tau) \xrightarrow{\sim} (\mathrm{End}_R(\mathfrak{C}), \tau_{\mathfrak{n}}) \times (\mathrm{End}_R(\mathfrak{C}), \tau_{\mathfrak{n}}),$$

where $\widetilde{\mathfrak{n}} = \left(\begin{smallmatrix} \mathfrak{n} & 0 \\ 0 & \mathfrak{n} \end{smallmatrix}\right)$.

Proof: The existence of a homomorphism follows from the universal property of the Clifford algebra. It is an isomorphism since $C(\mathfrak{C}, \mathfrak{n})$ is an Azumaya algebra. The claims about the involutions follow from the formula $b_{\mathfrak{n}} \circ \mu_x = \mu_{\overline{x}}^* \circ b_{\mathfrak{n}}$ (where $b_{\mathfrak{n}}$ stands for the adjoint), which is equivalent to $b_{\mathfrak{n}}(x \circ y, z) = b_{\mathfrak{n}}(y, \bar{x} \circ z)$. As already observed, this last formula holds for any Cayley algebra. □

(4.5) Proposition. *Let $t : \mathfrak{C} \otimes I \xrightarrow{\sim} \mathfrak{C}$ be a similitude with multiplier (I, d). There exist a discriminant module (J, k) and similitudes*

$$t_2 : \mathfrak{C} \otimes J \xrightarrow{\sim} \mathfrak{C}, \ t_1 : \mathfrak{C} \otimes I \otimes J \xrightarrow{\sim} \mathfrak{C}$$

such that:
1) $t_1(x \circ y \otimes \xi \otimes \eta) = t(x \otimes \xi) \circ t_2(y \otimes \eta)$ if t is an proper similitude and
2) $t_1(x \circ y \otimes \xi \otimes \eta) = t(y \otimes \xi) \circ t_2(x \otimes \eta)$ if t is an improper similitude.
Conversely, if 1) holds, t is proper and, if 2) holds, t is improper. Furthermore t determines the pair (t_1, t_2) up to a common unit of R.

Proof: This is just a reformulation of 3.5.

Let t be a similitude $t : \mathfrak{C} \otimes I \xrightarrow{\sim} \mathfrak{C}$ with multiplier (I, d). Following [BS$_2$], p. 161, we define $\hat{t} : \mathfrak{C} \otimes I^* \xrightarrow{\sim} \mathfrak{C}$ by

$$\hat{t}(x \otimes \xi) = \overline{t(\overline{x} \otimes d^{-1}(\xi))}.$$

We have

$$\begin{aligned} \mathfrak{n}(\hat{t}(x \otimes \xi)) &= \mathfrak{n}(t(\overline{x} \otimes d^{-1}(\xi))) \\ &= \mathfrak{n}(x)d(d^{-1}(\xi))(d^{-1}(\xi)) \\ &= \mathfrak{n}(x)d^{-1}(\xi, \xi) \end{aligned}$$

so that $\hat{t}$ is a similitude with multiplier (I^*, d^{-1}). Since $\bar{x} = \chi_{u_1}(x)$ and $C(\chi_{u_1})|_Z = \sigma_0$, $\hat{t}$ is proper if t is proper and is improper if t is improper.

(4.6) Proposition. *With the notations of 4.5 we have*
1) If $t \in GO_+(\mathfrak{n})$, then t_1, $t_2 \in GO_+(\mathfrak{n})$ and

$$\begin{aligned} t(x \circ y \otimes \xi)k(\eta, \eta) &= t_1(x \otimes \xi \otimes \eta) \circ \hat{t}_2(y \otimes k(\eta)) \\ t_2(x \circ y \otimes \eta)d(\xi, \xi) &= \hat{t}(x \otimes \xi) \circ t_1(y \otimes \eta \otimes \xi) \\ \hat{t}(x \circ y \otimes d(\xi))k(\eta, \eta) &= t_2(x \otimes \eta) \circ \hat{t}_1(y \otimes d(\xi) \otimes k(\eta)) \end{aligned}$$

2) If $t \in GO_-(\mathfrak{n})$, then t_1, $t_2 \in GO_-(\mathfrak{n})$ and

$$\begin{aligned} t(x \circ y \otimes \xi)k(\eta, \eta) &= t_1(y \otimes \xi \otimes \eta) \circ \hat{t}_2(x \otimes k(\eta)) \\ t_2(x \circ y \otimes \eta)d(\xi, \xi) &= \hat{t}(y \otimes \xi) \circ t_1(x \otimes \eta \otimes \xi) \\ \hat{t}(x \circ y \otimes d(\xi))k(\eta, \eta) &= t_2(y \otimes \eta) \circ \hat{t}_1(x \otimes d(\xi) \otimes k(\eta)). \end{aligned}$$

If $t \in SO(\mathfrak{n})$ is such that $SN(t) = 1$, then t_1, t_2 can be taken in $\mathrm{Ker}SN \subset SO(\mathfrak{n})$.

Proof: The verification of the formulas is a straightforward generalization of corresponding computations of [BS$_2$] and we only check the first one. In the formula

$$t_1(x \circ y \otimes \xi \otimes \eta) = t(x \otimes \xi) \circ t_2(y \otimes \eta),$$

we replace x by $x \circ y$ and y by $\overline{y}$. We get

$$\begin{aligned} t(x \circ y \otimes \xi) \circ t_2(\overline{y} \otimes \eta) &= t(x \circ y \otimes \xi) \circ \overline{\hat{t}_2(y \otimes k(\eta)))} \\ &= t_1(x \otimes \xi \otimes \eta)\mathfrak{n}(y). \end{aligned}$$

Multiplying by $\hat{t}_2(y \otimes k(\eta))$ gives

$$t(x \circ y \otimes \xi)k(\eta, \eta)\mathfrak{n}(y) = t_1(x \otimes \xi \otimes \eta) \circ \hat{t}_2(y \otimes k(\eta))\mathfrak{n}(y)$$

Viewing y as "generic", we may divide both sides with $\mathfrak{n}(y)$. This is the first formula. The claim about the "parity" then follows from 3.6. If $t \in \mathrm{Ker}SN$, then t_1, t_2 can be taken in $SO(q)$ (see the discussion after the proof of 3.1) and in fact $t_1, t_2 \in \mathrm{Ker}SN$, since $SN(\hat{t}\ ) = 1$. □

(4.7) Remark. Let R be a connected ring with $\mathrm{Pic}(R) = 0$. Let $\circ$ and $*$ be two compositions giving rise to the same norm on $\mathfrak{C}$ and with the same identity element u_1. By 4.6 there exist similitudes $t_1, t_2 : M \xrightarrow{\sim} M$ such that

$$x * t_2(y) = t_1(x \circ y) \quad \text{or} \quad t_2(y) * x = t_1(x \circ y).$$

We may assume that $x * t_2(y) = t_1(x \circ y)$. Setting $x = u_1$ we get $t_1 = t_2$ and setting $y = u_1$ we see that $t_1(x) = x \circ u$ with $u = t_2(u_1)$, so that

$$x * y = (x \circ (y \circ u^{-1})) \circ u.$$

Conversely, this formula can be used to construct different compositions on $\mathfrak{C}$ with the same identity element u_1. Thus we may have on the same quadratic space (U, p) different Cayley compositions $\circ$ and $*$ with the same identity element u_1. This is in contrast with quadratic or quaternion algebras, the other types of composition algebras. However, even if different, the two multiplications could be isomorphic.

(4.8) Proposition. *We have*

$$\mathrm{Spin}(\mathfrak{C}, \mathfrak{n}) \simeq \{(t_0, t_1, t_2) \,|\, t_i \in SO(\mathfrak{C}, \mathfrak{n}) \text{ with } t_1(x \circ y) = t_0(x) \circ t_2(y)\}$$

and the canonical map $\mathrm{Spin}(\mathfrak{C}, \mathfrak{n}) \to SO(\mathfrak{C}, \mathfrak{n})$ *corresponds to* $(t_0, t_1, t_2) \mapsto t_0$.

Proof: In view of 3.4 and 4.6, for $c \in \mathrm{Spin}(\mathfrak{C}, \mathfrak{n})$, the assignment $c \mapsto (\chi_c, \beta_c, \gamma_c)$ gives the required bijection. □

(4.9) Remark. The results 4.5, 4.6 and 4.8 for forms over fields of characteristic not 2 are in [BS$_2$] or [S]. The proofs given there use the theorem of Cartan-Dieudonné.

(4.10) Lemma. *Let* $\mathfrak{C}, \mathfrak{C}'$ *be Cayley algebras with identities* u_1, u_1' *and let* (t, t_2, t_1) *be an isometry* $(\mathfrak{C}, \mathfrak{n}) \xrightarrow{\sim} (\mathfrak{C}', \mathfrak{n}')$. *The following conditions are equivalent:*
1) $t = t_1 = t_2$
2) $t(u_1) = t_1(u_1) = t_2(u_1) = u_1'$.

Proof: 2) is a consequence of 1) since $t(x) = t(x \circ u_1) = t(x) \circ t(u_1)$, for all $x \in \mathfrak{C}$, implies $t(u_1) = u_1'$ and 1) follows from 2) since $t_1(y) = t_1(u_1 \circ y) = u_1 \circ t_2(y) = t_2(y)$ and similarly $t_1(y) = t(y)$. □

An isometry of $(\mathfrak{C}, \mathfrak{n})$ satisfying the equivalent properties of 4.10 is an *automorphim* of the composition algebra. The group of automorphisms of the composition algebra $\mathfrak{C}$ is denoted by $G_2(\mathfrak{C})$.

Let Γ be $\mathrm{Ker}\, SN \subset SO(\mathfrak{C}, \mathfrak{n}))\,(\simeq \mathrm{Spin}(\mathfrak{C}, \mathfrak{n}))$ modulo its centre) and let $[t] \in \Gamma$ be the class of $t \in \mathrm{Ker} SN$. We define

$$\varphi_1([t]) = [t_1],\ \varphi_2([t]) = [t_2] \text{ and } \epsilon([t]) = [\hat{t}].$$

(4.11) Proposition. *Let R be connected. The maps φ_1, φ_2 and ϵ are automorphisms of Γ. They generate an action of the symmetric group $\mathcal{S}_3$ on Γ and $G_2(\mathfrak{C}) = (\Gamma)^{\mathcal{S}_3}$.*

Proof: The first claim is as in [BS$_2$], p. 161. Let $[t] \in (\Gamma)^{\mathcal{S}_3}$. We get r_1, $r_2 \in \mu_2(R)$ such that $r_1 t(x \circ y) = t(x) \circ r_2 t(y)$. If $r_1 = r_2$ the map t is multiplicative and if $r_1 = -r_2$ the map $-t$ is multiplicative. □

5. Spaces of rank 6.

Let (M, q) be a quadratic space of rank 6 over R with Arf invariant Z and trivial Clifford invariant. Let $\alpha : C(q) \xrightarrow{\sim} \mathrm{End}_R(V)$ be a spin representation space for (M, q). Then V is a Z-module through α and is projective of rank 4, Z being a separable R–algebra. Since $C_0(q) = C(q)^Z = \{x \in C(q) \,|\, xz = zx, \forall z \in Z\}$ α restricts to $\alpha_0 : C_0(q) \xrightarrow{\sim} \mathrm{End}_Z(V)$ and, since the rank of M is congruent to 2 modulo 4, τ_0 restricts to the nontrivial automorphism σ_0 on Z. By 2.3 there exists a nonsingular Z–hermitian form $B : V \xrightarrow{\sim} V^\vee$ on V such that α is an isomorphism $(C_0(q_0), \tau_0) \xrightarrow{\sim} (\mathrm{End}_Z(V), \tau_B)$ of algebras with involution. Furthermore we have

$$(C(q), \tau') \xrightarrow{\sim} \mathrm{End}_R(V), \tau_{q_B}),$$

where τ' is the <u>second</u> involution of $C(q)$, i.e. such that $\tau'(x) = -x$ for $x \in V$ (see [K], p. 241), and $q_B(x) = B(x, x)$. Thus the spin representation space (V, q_B) is induced from the hermitian space (V, B). It follows from general results of [KPS], §8, that a hermitian space (V, B) of rank 4 induces in this way a spin representation space for a quadratic space of rank 6 if and only if its hermitian discriminant is trivial. In this section we give a direct proof of this fact, without using the machinery of [KPS]. We begin with some preliminaries. Let V be a rank 4 projective module over R and let

$$\mathrm{pf} : \wedge^2 V \to \wedge^4 V$$

be its pfaffian. If V is free with basis $\{e_1, e_2, e_3, e_4\}$, we recall that

$$\mathrm{pf}(\sum_{i<j} a_{ij}(e_i \wedge e_j) = \mathrm{pf}(\alpha)(e_1 \wedge e_2 \wedge e_3 \wedge e_4),$$

where $\alpha \in M_4(R)$ is the alternating matrix with $(i \times j)$-entry a_{ij} for $i < j$, and $\mathrm{pf}(\alpha)$ is the classical pfaffian of the matrix α. If $\wedge^4 V$ is free and $\lambda : \wedge^4 V \xrightarrow{\sim} R$ is an isomorphism, the composite $\mathrm{pf}_\lambda = \lambda \circ \mathrm{pf}$ is a quadratic form on the space $\wedge^2 V$ of rank 6. We describe its Clifford algebra. We identify $\wedge^2 V$ with

$$\mathrm{Alt}(V \otimes V) = \{\xi \in V \otimes V \,|\, \xi = \eta - \omega_V(\eta), \eta \in V \otimes V\},$$

ω_V the switch of $V \otimes V$, through the map $x \wedge y \mapsto x \otimes y - y \otimes x$ and view pf_λ as a quadratic form on $\mathrm{Alt}(V \otimes V)$. If V is free with basis $\{e_1, e_2, e_3, e_4\}$ and $\lambda(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$, we have

$$\mathrm{pf}_\lambda(\sum_{i<j} a_{ij}(e_i \otimes e_j - e_j \otimes e_i)) = \mathrm{pf}(\alpha).$$

Let $\alpha^\circ = (a^\circ_{ij})$ be the alternating matrix such that $\alpha^\circ\alpha = \alpha^\circ\alpha = \mathrm{pf}(\alpha)$. Let $\{e_1^*, e_2^*, e_3^*, e_4^*\}$ be the dual basis of V^*. The map

$$\pi : \mathrm{Alt}(V \otimes V) \to \mathrm{Alt}(V^* \otimes V^*) \otimes \wedge^4 V$$

given by $\sum_{i<j} a_{ij}(e_i \otimes e_j - e_j \otimes e_i) \mapsto \sum_{i<j} a^\circ_{ij}(e_i^* \otimes e_j^* - e_j^* \otimes e_i^*) \otimes e_1 \wedge e_2 \wedge e_3 \wedge e_4$ is independent of the choice of the basis, hence π is defined for any rank 4 projective R–module V and we have

$$\begin{aligned} \pi(\xi)\xi &= 1 \otimes \mathrm{pf}(\xi) \in \mathrm{End}_R(V^*) \otimes \wedge^4 V \\ \xi\pi(\xi) &= 1 \otimes \mathrm{pf}(\xi) \in \mathrm{End}_R(V) \otimes \wedge^4 V, \end{aligned}$$

where we identify $W' \otimes W^*$ with $\mathrm{Hom}_R(W, W')$ for any finitely generated projective R-modules W and W'. The products $\pi(\xi)\xi$ and $\xi\pi(\xi)$ then are given by the corresponding compositions of maps. We write (using the same identification)

$$\mathrm{End}_R(V \oplus V^*) = \begin{pmatrix} V \otimes V^* & V^* \otimes V^* \\ V \otimes V & V^* \otimes V \end{pmatrix},$$

where the product on the right hand side is induced by (2×2)–matrix multiplication.

(5.1) Proposition. *Let* $\lambda : \wedge^4 V \xrightarrow{\sim} R$ *be an isomorphism and let*

$$\pi_\lambda = (1 \otimes \lambda) \circ \pi : \mathrm{Alt}(V \otimes V) \to \mathrm{Alt}(V^* \otimes V^*).$$

1) The map $\mathrm{Alt}(V \otimes V) \to \mathrm{End}_R(V \oplus V^*)$ *given by*

$$\xi \mapsto \left(\begin{smallmatrix} 0 & \pi_\lambda(\xi) \\ \xi & 0 \end{smallmatrix}\right),\ \xi \in \mathrm{Alt}(V \otimes V),$$

induces an isomorphism of algebras with involution

$$\alpha : (C(\mathrm{pf}_\lambda), \tau') \xrightarrow{\sim} (\mathrm{End}_R(V \oplus V^*), \tau_h),$$

where h is the hyperbolic quadratic form on $V \oplus V^*$, i.e. $H((x, f)) = f(x)$ for $x \in V$ and $f \in V^*$.
2) The centre Z of $C_0(\mathrm{pf}_\lambda))$ is isomorphic to $R \times R$ and the restriction of α to $C_0(\mathrm{pf}_\lambda)$ is an isomorphism

$$\alpha_0 : C_0(\mathrm{pf}_\lambda) \xrightarrow{\sim} (\mathrm{End}_R(V) \times \mathrm{End}_R(V^*), \tau_H),$$

where $\tau_H(\phi, \psi) = (\psi^*, \phi^*)$.
3) The isomorphism $(\lambda, \lambda^{*-1}) : \wedge^4_{R\times R}(V \times V^*) = \wedge^4 V \times \wedge^4 V^* \xrightarrow{\sim} R \times R$ *is an isometry*

$$(\wedge^4_{R\times R}(V \times V^*), \wedge^4 H) \xrightarrow{\sim} \langle 1 \rangle_{R\times R}$$

of $(R \times R)$–hermitian discriminant modules.

Proof: 1) follows from the universal property of the Clifford algebra and 2) is a consequence of 1). We check 3): the hermitian form

$$\wedge^4 H : \wedge^4 V \times \wedge^4 V^* \to (\wedge^4 V \times \wedge^4 V^*)^{(*)} = (\wedge^4 V)^* \times (\wedge^4 V^*)^*$$

is given by $(\xi, x) \mapsto (\xi, x)$ after identifying $(\wedge^4 V)^*$ with $\wedge^4 V^*$ through the map which is locally given by $(e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*)(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$ for a local basis $\{e_1, e_2, e_3, e_4\}$ of V. Then $\lambda(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$ implies $\lambda^*(1) = e_1^* \wedge e_2^* \wedge e_3^* \wedge e_4^*$ and (λ, λ^{*-1}) is as required. □

(5.2) Proposition. *Let (M, q) be a quadratic space of rank 6 with trivial Clifford invariant, let Z be the centre of $C_0(q)$ and let (V, B) be a Z–hermitian space inducing the spin representation* $\alpha : C(q) \xrightarrow{\sim} \mathrm{End}_R(V)$. *There exists an isometry* $\lambda : (\wedge^4_Z V, \wedge^4 B) \xrightarrow{\sim} \langle 1 \rangle_Z$ *such that*

$$(Z \otimes M, Z \otimes q) \xrightarrow{\sim} (\mathrm{Alt}(V \otimes_Z V), \mathrm{pf}_\lambda).$$

In particular (V, B) has trivial hermitian discriminant.

Proof: The representation α induces an isomorphism $C(Z \otimes M) \xrightarrow{\sim} \mathrm{End}_Z(Z \otimes V)$. Let σ_0 be the nontrivial R–automorphism of Z and let $\nu : Z \otimes V \xrightarrow{\sim} V \oplus V^{(*)}$ be given by $\gamma(z \otimes v) = (zv, B(\sigma_0(z)v))$. The map $\beta = \mathrm{End}(\nu) \circ (1_Z \otimes \alpha)$ is an isomorphism

$$\beta : (C(Z \otimes M, Z \otimes q), \tau') \xrightarrow{\sim} (\mathrm{End}_Z(V \oplus V^{(*)}), \tau_h).$$

Let $x \in Z \otimes M$ and let

$$\beta(x) = \begin{pmatrix} 0 & \beta_2(x) \\ \beta_1(x) & 0 \end{pmatrix} \in \begin{pmatrix} 0 & V^{(*)} \otimes_Z V^{(*)} \\ V \otimes_Z V & 0 \end{pmatrix} \subset \mathrm{End}_Z(V \oplus V^{(*)}).$$

Since $\tau_h \beta(x) = \beta(\tau(x)) = -\beta(x)$, $\beta_1(x)$ is contained in the set of antisymmetric tensors of $V \otimes_Z V$. Thus we get $\beta_1(Z \otimes M) = \mathrm{Alt}(V \otimes_Z V)$ if 2 is invertible. In general, we get $\beta_1(Z \otimes M) = \mathrm{Alt}(V \otimes_Z V))$ by 5.1 and faithfully flat descent. Similarly we get $\beta_2(Z \otimes M) = \mathrm{Alt}(V^{(*)} \otimes_Z V^{(*)})$. The map $\gamma = \beta_2 \beta_1^{-1} : \mathrm{Alt}(V \otimes_Z V) \to \mathrm{Alt}(V^{(*)} \otimes_Z V^{(*)})$ has the property that $\gamma(\xi)\xi \in Z$ for $\xi \in \mathrm{Alt}(V \otimes_Z V)$, in fact $\gamma(\xi)\xi = (Z \otimes q)(x)$ for $\xi = \beta_1(x)$, $x \in Z \otimes V$. By [KPS], Lemma 1.3, there exists an isomorphism $\lambda : \wedge_Z^4 V \xrightarrow{\sim} Z$ such that $\gamma = \pi_\lambda$ and β_1 is an isometry $(Z \otimes M, z \otimes q) \xrightarrow{\sim} (\mathrm{Alt}(V \otimes_Z V), \mathrm{pf}_\lambda)$. The fact that λ is an isometry $(\wedge_Z^4 V, \wedge^4 B) \xrightarrow{\sim} \langle 1 \rangle_Z$ follows from 5.1. □

By 5.2, the condition that the hermitian discriminant is trivial is necessary for a hermitian spin representation space of rank 4. We next check that it is sufficient.

(5.3) Proposition. *Let S/R be a quadratic etale R-algebra with conjugation σ_0 and let (E, B) be a hermitian space of rank 4 of S such that $(\wedge_S^4 E, \wedge^4 B) \simeq \langle 1 \rangle_S$. There exists a quadratic space (M, q) of rank 6 over R and an isomorphism $\alpha : (C(q), \tau') \xrightarrow{\sim} (\mathrm{End}_R(E), \tau_{q_B})$, with $q_B(x) = B(x, x)$, such that $\alpha_0(C_0(q), \tau_0) = (\mathrm{End}_S(E), \tau_B)$ and $\alpha_0(Z) = S$.*

Proof: Let λ be an isometry $(\wedge_S^4 E, \wedge^4 B) \xrightarrow{\sim} \langle 1 \rangle_S$. In view of 5.2, it is natural to define (M, q) as a descent (from S to R) of the quadratic space $(\mathrm{Alt}(E \otimes_S E), \mathrm{pf}_\lambda)$. The descent is the composite

$$\sigma : \mathrm{Alt}(E \otimes_S E) \xrightarrow{B \otimes B} \mathrm{Alt}(E^\vee \otimes_S E^\vee) \xrightarrow{i \otimes i} \mathrm{Alt}(E^{(*)} \otimes_S E^{(*)}) \xrightarrow{\pi_\lambda^{-1}} \mathrm{Alt}(E \otimes_S E),$$

where $i : E^\vee \xrightarrow{\sim} E^{(*)}$ is the tautological map $x \mapsto x$. Observe that i is σ_0–semilinear. To check that $\sigma^2 = 1$, we may assume that E is free with basis $\{e_1, e_2, e_3, e_4\}$ over S and that $\lambda(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$. Through this choice we identify $\mathrm{Alt}(E \otimes_S E)$ with $\mathrm{Alt}_4(S)$, the set of alternating (4×4)–matrices with entries in S, and, through the choice of the dual basis, we identify $\mathrm{Alt}(E^{(*)} \otimes_S E^{(*)})$ with $\mathrm{Alt}_4(S)$. For any matrix $X = (x_{ij}) \in M_n(S)$, let $\overline{X} = (\sigma_0(x_{ij}))$. If U is the matrix of B with respect to the given basis, $B \otimes B$ corresponds to $X \mapsto UXU^t$. The fact that B is hermitian implies that $\overline{U}^t = U$ and the fact that λ is an isometry $(\wedge_S^4 E, \wedge^4 B) \xrightarrow{\sim} \langle 1 \rangle_S$ with $\lambda(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 1$ implies that $\det(U) = 1$. The tautological map i is given by $X \mapsto \overline{X}$ and pf_λ is given by $X \mapsto X^\circ$, where $X^\circ \in \mathrm{Alt}_4(S)$ is such that $XX^\circ = XX^\circ = \mathrm{pf}(X)$. Observe that $(X^\circ)^\circ = X$. Thus we have $\sigma(X) = (\overline{UXU^t})^\circ$. The formula $\mathrm{pf}(UXU^t) = \det(U)\mathrm{pf}(X)$ implies $(UXU^t)^\circ = \det(U)(U^t)^{-1} X^\circ U^{-1}$ and we get $\sigma(X) = U^{-1} \overline{X}^\circ U^{t-1}$. It follows that

$$\sigma^2(X) = \left(\overline{U(U^{-1}\overline{X}^\circ U^{t-1})U^t}\right)^\circ = \overline{\overline{X}^\circ}^\circ = X.$$

By definition of descent, we set

$$M = \{\xi \in \mathrm{Alt}(E \otimes_S E) \,|\, \sigma(\xi) = \xi\} \text{ and } q = \mathrm{pf}_\lambda|_M.$$

Let

$$\varphi = \mathrm{End}\left(\begin{smallmatrix} 1 & 0 \\ 0 & B^{-1} \end{smallmatrix}\right) : \mathrm{End}_S(E \oplus E^{(*)}) \xrightarrow{\sim} \mathrm{End}_S(E \oplus {}^{\sigma_0}E) = S \otimes \mathrm{End}_R(E).$$

We claim that the inclusion $M \to \mathrm{Alt}(E \otimes_S E) \to \mathrm{End}_S(E \oplus E^{(*)}) \xrightarrow{\varphi} \mathrm{End}_S(E \oplus {}^{\sigma_0}E) = S \otimes \mathrm{End}_R(E)$ induces an isomorphism $C(q) \xrightarrow{\sim} \mathrm{End}_R(E)$. We show that $(\sigma_0 \otimes 1)\varphi = \varphi\, C(\sigma)$. This will imply that φ maps $C(q)$, which is the descent for the datum $C(\sigma)$, onto $\mathrm{End}_R(E)$, which is the descent for the datum $\sigma_0 \otimes 1$. By 5.2, $\mathrm{Alt}(E \otimes_S E)$ is identified with the set

$$\left(\begin{smallmatrix} 0 & \pi_\lambda(\xi) \\ \xi & 0 \end{smallmatrix}\right) \in \mathrm{End}_S(E \oplus E^{(*)}),\ \xi \in \mathrm{Alt}(E \otimes_S E).$$

It follows from $\sigma^2 = 1$ that $\pi_\lambda^{-1} \circ (iB \otimes iB) = (iB \otimes iB)^{-1} \circ \pi_\lambda$ on $\mathrm{Alt}(E \otimes_S E)$, thus

$$\begin{aligned} C(\sigma)\begin{pmatrix} 0 & \pi_\lambda(\xi) \\ \xi & 0 \end{pmatrix} &= \begin{pmatrix} 0 & (iB \otimes iB)(\xi) \\ (iB \otimes iB)^{-1}\pi_\lambda(\xi) & 0 \end{pmatrix} \\ &= \mathrm{End}\begin{pmatrix} 0 & (iB)^{-1} \\ iB & 0 \end{pmatrix}\begin{pmatrix} 0 & \pi_\lambda(\xi) \\ \xi & 0 \end{pmatrix} \\ &= \varphi^{-1} \circ (\sigma_0 \otimes 1) \circ \varphi \begin{pmatrix} 0 & \pi_\lambda(\xi) \\ \xi & 0 \end{pmatrix} \end{aligned}$$

The claim then follows from the fact that $\mathrm{Alt}(E \otimes_S E)$ generates the Clifford algebra $C(\mathrm{pf}_\lambda) = \mathrm{End}_S(E \oplus E^{(*)})$. Similar arguments show that $\alpha_0(C_0(q)) = \mathrm{End}_S(E)$ and $\alpha_0(Z) = S$. The involution τ on $\mathrm{End}_S(E \oplus E^{(*)})$ is τ_h, $h = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Thus its transport to $\mathrm{End}_S(E \oplus {}^{\sigma_0}E)$ is $\tau_{B'}$ with $B' = \left(\begin{smallmatrix} 0 & {}^{\sigma_0}B \\ B & 0 \end{smallmatrix}\right)$. Since $1 \otimes B_* = (B, {}^{\sigma_0}B)$, $\tau_{B'}$ descends to τ_{q_B} with $q_B(x) = B(x, x)$. Similarly τ restricts to τ_B on $C_0(q) \xrightarrow{\sim} \mathrm{End}_S(E)$. □

For a hermitian space (E, B) with trivial hermitian discriminant and of rank n over S, we define $SU(E, B)$ to be the subgroup of isometries $t \in \mathrm{U}(E, B)$ such that $\wedge^n t \circ \lambda = \lambda$, where $\lambda : (\wedge^n_S E, \wedge^n B) \xrightarrow{\sim} \langle 1 \rangle_S$ is a fixed isometry. We denote by t_* the isometry of the quadratic form q_B induced by t. If (E, B) is as in 5.3, we have

(5.4) Proposition. *1) For any $t \in SU(E, B)$, there exists $t_0 \in SO(q)$ such that $C(t_0) = \mathrm{End}(t_*)$, $C_0(t_0) = \mathrm{End}(t)$.*
2) $\mathrm{Spin}(q) = SU(E, B)$.

Proof: By construction $t \otimes t$ is an isometry of $(\mathrm{Alt}(E \otimes_S E), \mathrm{pf}_\lambda)$ and $t \otimes t$ commutes with the descent σ. Thus t induces an isometry t_0 of (M, q) and $C(t_0) = \mathrm{End}(t_*)$, $C_0(t_0) = \mathrm{End}(t)$ holds. Since t is S–linear, $t_0 \in SO(q)$. Since $C_0(q) = \mathrm{End}_Z(E)$ with τ_0 induced by B, for any $t \in \mathrm{End}_Z(E)$, the condition $t\tau_0(t) = 1$ is equivalent to $t \in U(E, B)$. This, together with 1) implies that $\mathrm{Spin}(q) = SU(E, B)$. □

6. Cayley algebras arising from rank 3 hermitian spaces.

Let S be a quadratic etale R–algebra with norm $n = n_{S/R}$ and let (E, B) be a hermitian space of rank 4 over S with trivial discriminant. Let (M, q) be the quadratic space of rank 6 and α :

$(C(q),\tau') \xrightarrow{\sim} (\mathrm{End}_R(E), \tau_{q_B})$ the spin representation given by 5.3. Let $(U,p) = (S,n) \perp (M,-q)$ and let

$$\tilde{\alpha} : S \oplus M \to \mathrm{End}_R(E \oplus E), \quad \tilde{\alpha}(s,x) = \begin{pmatrix} 0 & \alpha_{\sigma_0(s)} + \alpha_x \\ \alpha_s - \alpha_x & 0 \end{pmatrix}$$

where, for $s \in S$, $\alpha_s : E \to E$ is the multiplication by s.

(6.1) Lemma. *The map $\tilde{\alpha}$ extends to an isomorphism of algebras with involution*

$$\tilde{\alpha} : (C(p),\tau) \xrightarrow{\sim} (\mathrm{End}_R(E \oplus E), \tau_{\tilde{q}}), \text{ where } \tilde{q} = \left(\begin{smallmatrix} q_B & 0 \\ 0 & q_B \end{smallmatrix}\right).$$

In particular $\tilde{\alpha}$ induces compositions $\lambda, \rho : (U,p) \times (E,q_B) \to (E,q_B)$ of quadratic forms.

Proof: 1) The existence of $\tilde{\alpha}$ follows from the universal property of the Clifford algebra, the fact that it is an isomorphism follows from the fact that $C(p)$ is an Azumaya algebra. We have

$$\begin{aligned} \tau_{\tilde{q}}\tilde{\alpha}(s,x) &= b_{\tilde{q}}^{-1} \begin{pmatrix} 0 & \alpha_{\sigma_0(s)} - \alpha_x \\ \alpha_s + \alpha_x & 0 \end{pmatrix}^* b_{\tilde{q}} \\ &= \begin{pmatrix} 0 & b_{q_B}^{-1}(\alpha_s + \alpha_x)^* b_{q_B} \\ b_{q_B}^{-1}(\alpha_{\sigma_0(s)} - \alpha_x)^* b_{q_B} & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & \alpha_{\sigma_0(s)} - \alpha_x \\ \alpha_s + \alpha_x & 0 \end{pmatrix} = \tilde{\alpha}\tau(s,x) \end{aligned}$$

since $b_{q_B}^{-1}\alpha_x^* b_{q_B} = -\alpha_x$ and $b_{q_B}^{-1}\alpha_s^* b_{q_B} = B^{-1}\alpha_s^* B = \alpha_{\sigma_0(s)}$. □

Let (E', B') be a hermitian space of rank 3 over S with trivial hermitian discriminant and let

$$(E,B) = \langle 1 \rangle_S \perp (E', B').$$

Putting as above $q_B(x) = B(x,x)$, it follows that

$$(E, q_B) = (S, n_{S/R}) \perp (E', q'_B)$$

and the composition $U \times E \to E$ restricts on $S \times S \to S$ to the given algebra structure of S. Let $u_1 = (1,0) \in U = S \perp M$ and let $x_1 = (1,0) \in E = S \perp E'$. Let $\mathrm{Cay}(S,E')$ be the Cayley algebra with underlying quadratic space (U,p) and composition $\circ$ given by the Chevalley construction applied to $\lambda : U \times E \to E$ for the choice of u_1 and x_1.

(6.2) Proposition. *1) The composition $\circ$ of $\mathrm{Cay}(S,E')$ restricts on S to the multiplication map and defines the structure of an S–module on M.*
2) There exists a hermitian structure $\widetilde{B}$ on M as an S–module such that the map $\phi : U \to E$ given by $\phi(u) = \lambda(u,x_1)$ restricts to an isometry $(M,\widetilde{B}) \xrightarrow{\sim} (E',B')$.

Proof: 1) The first claim follows from the fact that the composition $U \times E \to E$ restricts to the multiplication on S. The composition $\circ : S \times U \to U$ satisfies the associativity condition $(\lambda\lambda') \circ u = \lambda \circ (\lambda' \circ u)$: since S is quadratic over R, it is enough to verify this for $\lambda = \lambda' = z$ a generator of S over R. Then $z^2 \circ u = z \circ (z \circ u)$ since Cayley algebras are alternative. Thus U is an S-module and the fact that M is an S–module follows from $M = S^\perp \subset U$, since

$b_p(sm, s_1) = b_p(m, \overline{s}s_1)$ holds (U being a Cayley algebra with norm p). The fact that ϕ is S–linear then is obvious. The form $\widetilde{B}$ is the pull-back $tr^{-1} \circ b$ of the adjoint of the polar of $-q$ (see Section 2, in particular Remark 2.5). $\square$

(6.3) Corollary. *Let $\mathfrak{C}$ be Cayley algebra with norm $\mathfrak{n}$ and let S be an etale quadratic subalgebra of $\mathfrak{C}$. There exists a nonsingular hermitian form B on $\mathfrak{C}$ such that $q_B = \mathfrak{n}$ and $\mathfrak{C} = \mathrm{Cay}(S, S^{\perp})$.*

(6.4) Remark. Let $X = \mathrm{Spec}(R)$. The association $E \mapsto q$ in 5.3 corresponds to the map $H^1_{et}(X, SU_4) \to H^1_{et}(X, SO_6)$ induced by the homomorphism $SU_4 \to \mathrm{Spin}_4 \to SO_6$ of group schemes and $(S, E') \mapsto \mathrm{Cay}(S, E')$ corresponds to $H^1_{et}(X, SU_3) \to H^1_{et}(X, G_2)$ induced by the inclusion $SU_3 \to G_2$ (see [J], Theorem 3, p. 16, or Collected Works, Vol. 2, p. 356).

7. Composition over affine spaces.

In this section K is a field of characteristic not 2. Let $R = K[X_1, \ldots, X_n]$ be the polynomial ring in n variables over K. Let $\mathfrak{C}$ be a Cayley algebra over R with underlying module U. By a theorem of Quillen-Suslin, we may write $U = \overline{U} \otimes K[X_1, \ldots X_n]$, where $\overline{U}$ is the K-space $U/(X_1, \ldots, X_n)U$. For any R-linear map t we denote its reduction modulo $(X_1, \ldots, X_n)$ by $\bar{t}$. We say that $\mathfrak{C}$ is *extended from K* if there exists an isomorphism of Cayley algebras $\mathfrak{C} \xrightarrow{\sim} \overline{\mathfrak{C}} \otimes K[X_1, \ldots, X_n]$.

(7.1) Lemma. *Let R be a domain and let $R[X]$ be the polynomial ring over R. Let $\mathfrak{C}$ be a Cayley algebra over $R[X]$ and let $\overline{\mathfrak{C}}$ be its reduction modulo X. Suppose the norm $\mathfrak{n}_{\overline{\mathfrak{C}}}$ is anisotropic. If $t : \mathfrak{C} \to \overline{\mathfrak{C}} \otimes_R R[X]$ is an isometry such that $\bar{t} = 1_{\overline{\mathfrak{C}}}$, then t is an isomorphism of Cayley algebras.*

Proof: Let u_1 be the identity element of $\mathfrak{C}$. Then $\bar{u}_1 \in \overline{\mathfrak{C}}$ is the identity element of $\overline{\mathfrak{C}}$. Let $t(u_1) = \bar{u}_1 \otimes 1 + v_1 \otimes X + v_2 \otimes X^2 + \ldots + v_k \otimes X^k$. We claim that $v_i = 0$ for $i \geq 1$. Suppose $v_k \neq 0$. Since t is an isometry, $\mathfrak{n}_{\overline{\mathfrak{C}} \otimes R[X]}(\bar{u}_1 \otimes 1 + v_1 \otimes X + \ldots v_k \otimes X^k) = 1$. The left hand side is a polynomial in X with leading term $\mathfrak{n}_{\overline{\mathfrak{C}}}(v_k)X^{2k}$, so that $\mathfrak{n}_{\overline{\mathfrak{C}}}(v_k) = 0$. Since $\mathfrak{n}_{\mathfrak{C}}$ is anisotropic, we get $v_k = 0$, a contradiction. Thus $t(u_1) = \bar{u}_1 \otimes 1$. By 3.6, there exist similitudes $t_1, t_2 : \mathfrak{C} \to \overline{\mathfrak{C}} \otimes R[X]$ such that $t_1(x \circ y) = t(x) \,\tilde{\circ}\, t_2(y)$, $\tilde{\circ}$ denoting the multiplication $\overline{\circ} \otimes 1$ of $\overline{\mathfrak{C}} \otimes R[X]$. Since $\bar{u}_1 \otimes 1$ is the identity for $\tilde{\circ}$, $t_1(y) = t(u_1) \,\tilde{\circ}\, t_2(y) = t_2(y)$, so that $t_1 = t_2$. Since $\bar{t} = 1$ and $\bar{t}$ determines $\bar{t}_1$ and $\bar{t}_2$ up to scalars (see 4.5), $\bar{t}_1$ is a scalar. Scaling t_1, we may assume that $\bar{t}_1 = 1$. Since $t_1 : \mathfrak{C} \to \overline{\mathfrak{C}} \otimes R[X]$ is an isometry with $\bar{t}_1 = 1$, as above, we get $t_1(u_1) = \overline{u}_1 \otimes 1$. Then $t = t_1 = t_2$ and, by 4.10, t is an isomorphism of Cayley algebras. $\square$

(7.2) Corollary. *Let K be a field of characteristic not 2. Let $\mathfrak{C}$ be a Cayley algebra over $K[X_1, \ldots, X_n]$. If the norm $\mathfrak{n}_{\mathfrak{C}}$ is anisotropic and extended from K, then $\mathfrak{C}$ is isomorphic to $\overline{\mathfrak{C}} \otimes_K K[X_1, \ldots, X_n]$.*

(7.3) Remark. The same arguments as in 7.1 can be used to show that, for any Cayley algebra $\mathfrak{C}$ over a domain R with $\mathfrak{n}_{\mathfrak{C}}$ anisotropic, the natural map $G_2(\mathfrak{C}) \to G_2(\mathfrak{C} \otimes R[X_1, \ldots, X_n])$ is an isomorphism.

(7.4) Proposition. *Let $\mathfrak{C}$ be a Cayley algebra over $K[X_1, \ldots, X_n]$. If its norm form $\mathfrak{n}$ is isotropic, the algebra $\mathfrak{C}$ is extended from K.*

Proof: As above, let $R = K[X_1, \ldots, X_n]$. An isotropic quadratic space over R is extended from K (see [O]). Since a Cayley algebra with zero divisors over a field is split, the form $\overline{\mathfrak{n}}$ is hyperbolic, so that $\mathfrak{n}$ is hyperbolic. Let $t : (\mathfrak{C}, \mathfrak{n}) \xrightarrow{\sim} H(P) = P \oplus P^*$, with $P = R^4$, be an isometry. We get, for u_1 the identity element, $t(u_1) = (p_1, q_1)$, $p_1 \in P$, $q_1 \in P^*$ and the pair (p_1, q_1) is hyperbolic. The element $t^{-1}(p_1)$ generates a split separable quadratic R–algebra $S = R \times R \subset M$. In particular S is extended from K. By 6.2, $(M, q) = (S, n)^\perp$ is a S-module of rank 3 and carries a nonsingular S–hermitian form B such that $q(x) = B(x, x)$. Since $\overline{q}$ is hyperbolic, q is isotropic and by [O] is extended as a quadratic space. It follows that q represents any unit, in particular -1 and B can be decomposed as $\langle -1 \rangle_S \perp B_1$. Since B_1 has hermitian discriminant -1, it is hyperbolic ([K], p. 304), hence extended, and B is extended. Since S and B are extended, 6.3 implies that $\mathfrak{C}$ is extended. □

(7.5) Corollary. *Any composition algebra over $K[X]$ is extended from K.*

Proof: By a theorem of Harder, anisotropic spaces over $K[X]$ are extended from K. □

(7.6) Remark. 7.1, 7.4 are special cases of [RR] and 7.2 is a special case of [R] (for the group G_2). Another proof of 7.5 is in [Pe].

Corollary 7.5 does not hold for polynomial rings in more than one variable:

(7.7) Theorem. *Let K be a field of characteristic not 2 which admits a non-split Cayley algebra $\mathfrak{C}_0$. There exists an infinite sequence of non-isomorphic Cayley algebras $(\mathfrak{C}_i, \circ_i)$ over $K[X, Y]$, whose reductions modulo (X, Y) are isomorphic to $\mathfrak{C}_0$, and such that the restriction of the norm to $\mathfrak{C}_i' = \{x \in \mathfrak{C}_i \mid x + \overline{x} = 0\}$ is indecomposable as a rank 7 quadratic space.*

(7.8) Theorem. *For all i $\mathfrak{C}_i$ is a principal G_2-bundle over $\mathbf{A}^2_K$ whose structure group cannot be reduced to any proper reductive connected subgroup.*

We first prove 7.8 and postpone the proof of 7.7. Theorem 7.8 is a consequence of 7.7 and of the following Lemmas 7.9, 7.10 and 7.11 communicated to us by Raghunathan. Let G be a simple algebraic group of type G_2 over a field K and let $\rho : G \to GL(V)$ be its 7–dimensional representation. Let H be a connected reductive subgroup of G which is not abelian.

(7.9) Lemma. *If the representation $\rho\,|_H$ is irreducible, it is absolutely irreducible.*

Proof: If $\rho|_H$ is reducible over the algebraic closure $\overline{K}$, then it has at least 2 distinct irreducible components of different dimensions, $\dim_K V$ being a prime and H not being abelian. The corresponding isotypical components descend to give a decomposition of $\rho|_H$ over K. □

(7.10) Lemma. *Let K be an algebraically closed field. let G and $\rho : G \to GL(V)$ be as above. Let $u : SL_2 \to G$ be any homomorphism. Then $\rho \circ u$ cannot be irreducible.*

Proof: As observed in [Ch$_1$], Chapter IV, Section 4.2, $\rho(G)$ leaves a nonzero cubic form invariant. Thus it suffices to show that the natural 7–dimensional representation of SL_2 does not leave any nonzero cubic form invariant. Denoting this representation by V again, we need to show that the 3rd symmetric power $S^3(V)$ has no nonzero SL_2–invariant submodule. In fact we will show that $S^2(V) \otimes V$ has no nonzero SL_2–invariant submodule. If $S^2(V) \otimes V = \mathrm{Hom}_K(V^*, S^2(V))$ contains an invariant element, then $S^2(V)$ contains $V^* \simeq V$ as an SL_2–submodule. It is easy to see from the Clebsch–Gordan formula that $S^2(V) \simeq C \oplus D \oplus E \oplus F$, where C is the trivial representation, D, resp. E, resp. F is the irreducible representation of dimension 5, resp. 9, resp. 13. Thus $S^2(V)$ does not contains V (which has dimension 7) as an irreducible SL_2–submodule. $\square$

(7.11) Lemma. *Let H be a proper reductive connected subgroup of G. Then H acts reducibly on V.*

Proof: If H is abelian, it acts reducibly on V. Suppose that H is not abelian. By Lemma 7.9, it is enough to check that H acts reducibly over $\overline{K}$. Hence we assume that $K = \overline{K}$. If $H \simeq SL_2$, this follows from Lemma 7.10. Next suppose that H is locally isomorphic to $SL_2 \times G_m$. If $\rho \circ u$ is irreducible as a representation of H, then $\rho \circ u \mid_{SL_2}$ is necessarily isotypical, since G_m commutes with SL_2. Since 7 is a prime, V has to be irreducible as a SL_2–module as well, a contradiction. This means that we need only to consider the case where H is semisimple of rank 2. But then looking at root systems shows that H has to be of type B_2 or A_2. From Weyl's dimension formula we get that the irreducible representations of dimension ≤ 7 are of dimension 4 and 5 in the case of B_2 and of dimension 3 and 6 in the case of A_2. Thus there are no irreducible representations of dimension 7 for B_2 or A_2. $\square$

We cut the proof of 7.7 in steps. Some preliminaries and some notations are needed. For any module N over a commutative ring R, any $s \in R$ and any R–linear homomorphim f, we denote by N_s, resp. f_s the localization with respect to the multiplicative set $\{1, s, s^2, \ldots\}$.

(7.12) Lemma. *Let L be a quadratic field extension of K and let $\langle \lambda_1, \lambda_2, \lambda_3 \rangle$ be an anisotropic hermitian space over L. There exists an infinite sequence $\{f_i\}_{i\geq 1}$ of polynomials in $K[X]$ with $(f_i, f_j) = 1$ for $i \neq j$ and indecomposable hermitian spaces (N_i, B_i) over $L[X, Y]$, whose reductions modulo (X, Y) are isometric to $\langle \lambda_1, \lambda_2, \lambda_3 \rangle$, and such that*
1) the quadratic spaces $q_i = q_{B_i}$ are indecomposable over $K[X, Y]$.
2) $(N_i, B_i)_{f_i}$ is extended from $L[X]_{f_i}[Y]$ for all i.

Proof: The construction uses the techniques developed in [P_2] for quadratic spaces. We first construct B_1. There exist indecomposable anisotropic hermitian spaces B'_i of rank 2 over $L[X, Y]$, polynomials $f'_i \in K[X]$ such that $(f'_i, f'_j) = 1$ for $i \neq j$ and isometries

$$L[X]_{f'_i}[Y] \otimes_{L[X,Y]} B'_i \xrightarrow{\sim} L[X]_{f'_i}[Y] \otimes_K \langle \lambda_1, \lambda_2 \rangle.$$

(see [K], p. 449). We get an indecomposable hermitian space B_1 of rank 3 by glueing the space $(B'_1)_{f_2} \perp \langle \lambda_3 \rangle$, defined over $L[X]_{f'_2}[Y]$, with $(B'_2)_{f_1} \perp \langle \lambda_3 \rangle$, defined over $L[X]_{f'_1}[Y]$, over $L[X]_{f'_1 f'_2}[Y]$ as in [P_2]. We claim that $q_1 = q_{B_1}$ is indecomposable as a quadratic space over $K[X, Y]$. Suppose that $q_1 = q' \perp q''$ with q', q'' quadratic spaces over $K[X, Y]$. Since B_1 is indecomposable it does not represent units and q_1 does not represent units either. By [P_2] or [K], Lemma 10.1.3, p. 450, q' and q'' do not represent units. Hence, in view of the fact that rank 2 spaces over $K[X, Y]$ are extended ([P_1]), each should be of rank 3. Since $(B_1)_{f'_2} = (B'_1)_{f'_2} \perp \langle \lambda_3 \rangle$ over $K[X]_{f'_2}[Y]$, we have $(q_{B_1})_{f'_2} = (q_{B'_1})_{f'_2} \perp \langle \lambda_3 \rangle \otimes \langle 1, -u \rangle \simeq (q' \perp q'')_{f'_2}$. Thus by [$P_2$] or [K], Lemma 10.1.3, p. 450, one of $(q')_{f'_2}$ or $(q'')_{f'_2}$ must represent a unit, hence is diagonalizable for the same reasons as above. This would imply, by the following Lemma 7.13, that $(B'_1)_{f'_2}$ represents a unit and, being of rank 2, is extended from $L[X]_{f'_2}$. Since $(B'_1)_{f'_1}$ is extended from $L[X]_{f'_1}$, B'_1 is locally extended from $L[X]$ and it follows from [BCW] that B'_1 is extended from $L[X]$, contradicting the choice of B'_1. Finally we set $f_1 = f'_1 f'_2$. To get B_i we repeat the construction for B_1, taking a pair B'_{2i-1}, B'_{2i} and the corresponding polynomials f'_{2i-1}, f'_{2i} and setting $f_i = f'_{2i-1} f'_{2i}$. □

(7.13) Lemma. *Let q, q' be indecomposable quadratic spaces over $R[Y]$, R a domain, and let q_1 be a quadratic space over $R[Y]$ such that $q \perp q_1 \simeq q' \perp \langle v_1, \ldots, v_r \rangle$ for units $v_1, \ldots, v_r \in R^\times$. If $q \perp q_1$ is anisotropic, then $q_1 \simeq \langle v_1, \ldots, v_r \rangle$.*

Proof: The claim is a straightforward generalization of [P$_2$] or [K], Lemma 10.1.3, p. 450. □

Proof of 7.7: Let $\mathfrak{C}_0$ be a non-split Cayley algebra over K. We write the norm $\mathfrak{n}_0$ of $\mathfrak{C}_0$ as a three-fold Pfister form $\langle 1, -\lambda \rangle \otimes \langle 1, -\mu \rangle \otimes \langle 1, -\nu \rangle$. Let $L = K(\sqrt{\lambda})$ and let f_i, B_i be as in 7.12 for the anisotropic hermitian space $\langle -\mu, -\nu, \mu\nu \rangle$ over L. Let $S = L[X,Y]$, $R = K[X,Y]$ and let $U_i = \mathrm{Cay}(S, N_i)$ be the Cayley algebra associated to the rank 3 hermitian space (N_i, B_i) (Section 6). Let p_i be the norm of U_i. We have

$$(U_i, p_i) = (S, n_{S/R}) \perp (N_i, q_i)$$

with $q_i = q_{B_i}$ and we get isometries

$$\pi_i : (q_i)_{f_i} \xrightarrow{\sim} \langle -\mu, \lambda\mu, \nu, -\nu\lambda, -\nu\mu, \nu\lambda\mu \rangle \otimes K[X]_{f_i}[Y]$$

over $K[X]_{f_i}[Y]$ such that $\overline{\pi}_i = 1$, bar denoting the reduction modulo Y. We now construct $\mathfrak{C}_1$ by glueing $U_1 = \mathrm{Cay}(S, N_1)$ and $U_2 = \mathrm{Cay}(S, N_2)$ by an isomorphism

$$\theta : (U_1)_{f_1 f_2} \xrightarrow{\sim} (U_2)_{f_1 f_2}$$

over $K[X]_{f_1 f_2}[Y]$ defined as follows. Let ψ be an automorphism of the algebra $\mathfrak{C}_0$ such that $\psi(\langle -\lambda \rangle) \subset \langle 1, -\lambda \rangle^\perp$. Such automorphisms always exist: take one in the quaternion subalgebra $\langle 1, -\lambda \rangle \otimes \langle 1, -\mu \rangle$ of $\mathfrak{C}_0$ and extend it to $\mathfrak{C}_0$ by the Cayley–Dickson process. We set

$$\theta = (1 \perp 1 \perp \pi_2)^{-1} \circ \psi \circ (1 \perp 1 \perp \pi_1).$$

Since $\overline{\pi}_i = 1$, it follows from 7.1 that the maps $1 \perp 1 \perp \pi_i$ are isomorphisms of Cayley algebras. Thus θ is a Cayley algebra isomorphism and $\mathfrak{C}_1$, obtained by glueing $U_1 = \mathrm{Cay}(S, N_1)$ and $U_2 = \mathrm{Cay}(S, N_2)$ through θ, is a Cayley algebra. It follows as in [P$_2$] that $\mathfrak{C}'_1 = \langle 1 \rangle_R{}^\perp$ is indecomposable. We get finally $\mathfrak{C}_i$ by glueing similarly U_{2i-1} and U_{2i} for each i. □

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