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On the Dimension and Numerical Invariants of Algebras and Vector Products; a Tensor Categorical Approach

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Ai miei genitori.

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Desidero ringraziare innanzitutto il Professor Max-Albert Knus, che mi ha dato l'occasione di avvicinarmi al mondo delle forme quadratiche. I suoi preziosi consigli e validi suggerimenti sono stati fondamentali per la stesura della tesi. Inoltre, a lui va la mia gratitudine per avermi permesso di finire la tesi in Ticino e per aver pazientemente rispettato i miei altri impegni lavorativi. Sono inoltre grata ai Professori Markus Rost e Paul Balmer per gli utili commenti e per aver assunto il ruolo di co-esaminatori.

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Contents

Abstract

The aim of the thesis is to illustrate how tensor categorical methods used in the theory of knots and quantum groups can be applied to give a new approach to some classical algebraic results. Following ideas of Markus Rost, we determine the admissible dimensions of certain algebraic structures, like vector product algebras or composition algebras, by diagrammatic computations in a category of graphs attached to the given algebra. We essentially consider five types of algebraic structures: vector products and symmetric compositions, where we present results due to M. Rost, then J-algebras, 3-vector products and unital composition algebras. For example we show that 3-vector products can only occur in dimension $1, 2, 4$ and 8. Classically these results follow from the (known) structure of the algebras.

In the first part we use a construction of Turaev to associate in a natural way a category of graphs to an algebraic structure defined by a set of tensor identities. Closed graphs in the graph category represent numerical invariants of the given algebra, like the dimension.

In a second part of the thesis we apply the categorical formalism to specific situations and show how diagrammatic computations lead to conditions for the numerical invariants. Special feature of symmetric compositions is the appearance of a second basic numerical invariant, connected with a Casimir element. The presence of a bilinear form play an essential role in our constructions. Diagrammatic calculus applies particularly well for trace bilinear forms. The multiplication of the algebra can be represented by a trivalent vertex and graphical representations are simpler. We have trace forms in the cases of 2-vector products and symmetric compositions. For forms which are not trace forms, the representation of the multiplication by a graph is more delicate. In order to use multivalent vertices to represent products we need to refine Turaev's construction by working with a category of bicolored ribbon graphs. These refinements are used in the last chapters for 3-vector products and composition algebras with identity.

Riassunto

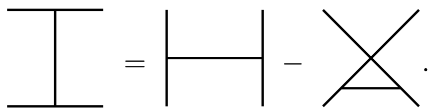
Lo scopo di questo lavoro è di illustrare l'applicazione di metodi tensoriali categorici, usati nella teoria dei nodi e dei gruppi quantici, a risultati algebrici classici. Motivati da idee di Markus Rost, determiniamo le possibili dimensioni di alcune strutture algebriche, come algebre di prodotti vettoriali e algebre di composizione, con calcoli diagrammatici nelle categorie di grafi ad esse associate. Consideriamo cinque tipi di strutture algebriche: prodotti vettoriali e composizioni simmetriche, dove vengono presentati risultati di M. Rost, e J-algebre, 3-prodotti vettoriali e algebre di composizione con identità. Per esempio, mostriamo che 3-prodotti vettoriali possono esistere solo in dimensione $1, 2, 4$ e 8. Classicamente questi risultati seguono dalla struttura (conosciuta) dell'algebra.

Nella prima parte utilizziamo una costruzione di Turaev per associare ad una struttura algebrica, definita da un insieme di identità tensoriali, una categoria di grafi. Nella categoria dei grafi, i grafi chiusi rappresentano invarianti numerici dell'algebra data, come per esempio la dimensione.

Nella seconda parte della tesi applichiamo il formalismo delle categorie e mostriamo come le operazioni diagrammatiche portano a determinare condizioni per gli invarianti numerici. Una peculiarità delle composizioni simmetriche è il manifestarsi di un secondo invariante numerico, correlato all'elemento Casimir. La presenza di una forma bilineare ha un ruolo essenziale nelle costruzioni. Il calcolo diagrammatico si applica efficacemente nel caso di forme bilineari associative: la moltiplicazione nell'algebra può venir rappresentata da un vertice trivalente e le rappresentazioni con i grafi risultano molto semplici. Forme bilineari associative compaiono in 2-prodotti vettoriali e in composizioni simmetriche. Nel caso di forme non associative la rappresentazione grafica della moltiplicazione è più delicata. Per poter rappresentare le moltiplicazioni con vertici multivalenti è necessario affinare la costruzione di Turaev, lavorando nella categoria dei grafi bicolorati. Questi perfezionamenti sono utilizzati negli ultimi capitoli, per 3-prodotti vettoriali e algebre di composizione con identità.

Introduction

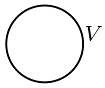
Diagrams occur in various fields of pure and applied mathematics. Examples are Feynman diagrams, flow charts, diagrams of knots or links. Penrose introduced a graphical notation for calculating with tensors (see [PR84]). Recent developments in knot theory and the theory of quantum groups have led to consider categories whose morphisms are represented by diagrams or graphs. The technique of diagrammatic morphisms is now an important ingredient in comprehending and visualizing certain types of categories with structure (see for example [BÁ02], [Bla03], [BN95] or [Tur94]). A historical account of the subject can be found in [Yet03]. A typical feature is to represent structural identities by graph relations. Examples are the Homfly relations in skein theory (see [Bla03]) or the IHX identity (see [BN95]):



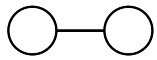
Observe that the IHX identity is a geometric translation of the Jacobi identity of a Lie algebra.

The aim of the thesis is to illustrate how the technique of diagrammatic morphisms can be applied to give a new approach to some classical algebraic results. Inspired by ideas of Markus Rost, we determine the admissible dimensions of certain algebraic structures, like vector product algebras or composition algebras, by diagrammatic computations in a category of graphs attached to the given algebra. Classically these results follow from the known structure of the algebras.
In Chapter I we start by recalling the most important definitions and results on braided tensor categories and give the example of the braid category. This category gives a first example of diagrammatic calculus. Then we give the definition of a ribbon category $\mathcal{C}$, i.e., a braided tensor category with a twist and a compatible duality, and we introduce the notion of category of ribbon graphs $\mathrm{Rib}_{\mathcal{C}}$ associated to C, as in Turaev [Tur94, Chapter I]. Every morphism of $\mathrm{Rib}_{\mathcal{C}}$ is depicted diagrammatically by a ribbon graph composed of bands, coupons and annuli; the graph is $\mathcal{C}$-colored in the sense that each band and each annulus are equipped with an object of $\mathcal{C}$ and each coupon is equipped with a morphism of $\mathcal{C}$. An important result on ribbon categories

is the existence (and uniqueness) of a specific functor $\mathcal{R}\colon \mathrm{Rib}_{\mathcal{C}} \to \mathcal{C}$. The passage to the category $\mathrm{Rib}_{\mathcal{C}}$ is the tool which allows to apply diagrammatic calculus. The concepts of trace and dimension are discussed; for example the dimension of an object V of $\mathcal{C}$ is depicted by a circle diagram:

$$d := \dim V =$$  .

Chapter II describes the categorical approach for algebras of tensor type. Such an algebra is a data $A = (V, b, m, \mathfrak{C})$, where V is a vector space over a field F of characteristic not 2, b is a symmetric nonsingular bilinear form on V, $m\colon V \otimes V \to V$ is is a bilinear multiplication, and b and m satisfy a given set $\mathfrak{C}$ of multilinear relations. We associate to the data $A = (V, b, m, \mathfrak{C})$ a category of ribbon graphs Rib_Γ, enriched with a linear structure, and in which the relations $\mathfrak{C}$ become graph identities. Closed graphs of Rib_Γ form a ring E_Γ (the ring of numerical invariants) and the aim of the thesis is to compute E_Γ for different algebras of tensor type. The dimension is a typical numerical invariant.
The presence of a nonsingular symmetric bilinear form is essential. A case of special interest is when the form is associative with respect to the multiplication, i.e., when $b\big(m(x,y),z\big) = b\big(x,m(y,z)\big)$ holds for all $x, y, z \in V$. Such a form is also sometimes called a trace form. In this situation the coupon representing the multiplication can be replaced by an oriented trivalent vertex and computations become much simpler.
Chapter III and Chapter IV essentially give a diagrammatic categorical presentation of results of M. Rost [Ros95] (see also [Boo98]). In Chapter III it is shown by diagrammatic computations that for the dimension d of a vector product algebra holds the equation $d(d-1)(d-3)(d-7) = 0$. Chapter IV is dedicated to the study of symmetric composition algebras. These are composition algebras where the bilinear form is associative. It appears that, beside d, the numerical invariant depicted by

$$e =$$ 

is another generator of the ring of numerical invariants. The values of these invariants are given by the system of equations

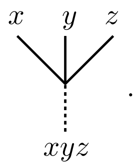
$$e\big(e - (d-2)^2\big) = 0,$$
$$(d-2)(d-8)(d-e) = 0.$$

The only possible values of the pair (d, e) of invariants are

$$\{(0,0),\ (1,1),\ (2,0),\ (4,4),\ (8,0),\ (8,36)\}.$$

In dimension 8 there are essentially two types of symmetric composition algebras (see [KMRT98, Theorem (34.37)]). One type is characterized by

the relation $e = (d-2)^2$ and the other by $e = 0$. Hence, the invariant e distinguishes the two types. In the next sections we investigate the possible dimensions of some special cases of symmetric composition algebras. The case of symmetric compositions arising from associative algebras is of particular interest.
In Chapter V we consider J-algebras, a class of Jordan algebras introduced by Springer. They are of two types: J-algebras of quadratic type (with any possible dimension ≥ 2) and Freudenthal algebras (with possible dimensions $d = 6, 9, 15, 27$). Since these two types satisfy the same tensor identities, it is not possible to have a unique relation determining the values of the dimension. However, the invariant e plays a fundamental role in detecting the two classes of algebras. We compute graphically and algebraically the values of e. We find that e is zero for Freudenthal algebras, and $e = (d-2)^2$ for J-algebras of quadratic type.
Until now, we only considered algebras where the bilinear form was associative with respect to the multiplication. The theory of ribbon graphs is extended in Chapter VI to include the treatment of 3-vector products. A 3-vector product on a d-dimensional vector space V over F is a trilinear map $P_3 \colon V \times V \times V \to V$ satisfying certain tensor identities. We aim to use a quadrivalent vertex to graphically represent the 3-product. Due to the fact that for 3-vector products the bilinear form is no longer associative, we have to generalize the concept of a category of ribbon graphs $\mathrm{Rib}_{\mathcal{C}}$ over a ribbon category $\mathcal{C}$. We introduce bicolored ribbon graphs: a graph, already colored by $\mathcal{C}$, has an additional color, with values $+1$ or -1. If the value is positive, the band is depicted with a normal line; for a negative color the band is depicted with a dotted line. This bicoloring allows to distinguish the elements x, y, z and the outgo of the multiplication $P_3(x, y, z) = xyz$:



This makes possible the use of a quadrivalent vertex to represent the multiplication. With this extension of the theory of ribbon graphs we get a new diagrammatic proof that 3-vector products only exist in $\dim_F V = 1, 2, 4, 8$.
In the last chapter, we present another extension of the theory of ribbon graphs, allowing to consider unital algebras. Simple examples where the bilinear form is a trace form include Jordan algebras of degree 2 and unital symmetric composition algebras. The last sections are dedicated to unital composition algebras and a diagrammatic proof of Hurwitz Theorem. The main difficulty here is that the bilinear form is not a trace form, so that, as for the 3-vector product, we need bicolored graphs to represent the multiplication by a trivalent vertex.

Parts of the thesis were published in [CKR05].

Chapter I

A tensor categorical approach

In this chapter we first introduce ribbon categories, which play a fundamental role in this work. These are categories with a tensor product endowed with braiding, twist and duality. Following the book of Turaev (see [Tur94, Chapter I]), we then describe the construction of the category of ribbon graphs associated with a given ribbon category. This allows to represent morphisms in a ribbon category by graphs and to develop a graphical calculus for these morphisms.
For details and proofs we refer to the books of Turaev ([Tur94, Chapter I]), Kassel ([Kas95, Part Three, Chapters X-XIV]), Bakalov and Kirillov ([BK01, Chapters 1 and 2]), and to the articles [Yet90], [RT90], [JS91], [JS93] and [Shu94].

1 Categories and functors

Definition 1.1. A *category* $\mathcal{C}$ consists of

1. a class $\mathrm{Obj}(\mathcal{C})$, whose elements are called *objects* of the category,

2. for each pair of objects $(U, V) \in \mathrm{Obj}(\mathcal{C})$ a set $\mathrm{Mor}_{\mathcal{C}}(U, V)$, whose elements are called *morphisms* from U to V, where $\mathrm{Mor}_{\mathcal{C}}(U, V) \cap \mathrm{Mor}_{\mathcal{C}}(U', V') = \emptyset$ if $U \neq U'$ or $V \neq V'$. If f is in $\mathrm{Mor}_{\mathcal{C}}(U, V)$, we write $f \colon U \to V$, $U = s(f)$ is the *source* and $V = b(f)$ the *target* of f,

3. a composition map

$$\begin{aligned} \mathrm{Mor}_{\mathcal{C}}(U, V) \times \mathrm{Mor}_{\mathcal{C}}(V, W) &\to \mathrm{Mor}_{\mathcal{C}}(U, W), \\ (f, g) &\mapsto g \circ f, \end{aligned}$$

$\big($the composition of two morphisms is defined whenever $b(f) = s(g)\big)$, such that the following axioms hold:

(A1) (associativity of the composition) for $f \in \mathrm{Mor}_{\mathcal{C}}(U,V)$, $g \in \mathrm{Mor}_{\mathcal{C}}(V,W)$ and $h \in \mathrm{Mor}_{\mathcal{C}}(W,Z)$, we have

$$h \circ (g \circ f) = (h \circ g) \circ f,$$

(A2) (identity) for every object U there exists a morphism $1_U \in \mathrm{Mor}(U,U)$, called the *identity morphism* for U, such that $1_U \circ f = f$ and $g \circ 1_U = g$ for all $f \in \mathrm{Mor}_{\mathcal{C}}(V,U)$, $g \in \mathrm{Mor}_{\mathcal{C}}(U,V)$.

From these axioms, one can prove that there is exactly one identity morphism for every object.

Example 1.2. A group is a category with one object in which all morphisms are isomorphisms.

Definition 1.3. Let $\mathcal{C}, \mathcal{D}$ be two categories. We call a map $F\colon \mathcal{C} \to \mathcal{D}$ a *covariant functor* if it associates to each object $U \in \mathrm{Obj}(\mathcal{C})$ an object $F(U) \in \mathrm{Obj}(\mathcal{D})$, and to each morphism $f \in \mathrm{Mor}_{\mathcal{C}}(U,V)$ a morphism $F(f) \in \mathrm{Mor}_{\mathcal{C}}\big(F(U), F(V)\big)$, such that the following properties hold:

1. for any $U \in \mathrm{Obj}(\mathcal{C})$: $F(1_U) = 1_{F(U)}$,
2. for any morphism $f \in \mathrm{Mor}_{\mathcal{C}}(U,V)$, we have $s\big(F(f)\big) = F\big(s(f)\big)$ and $b\big(F(f)\big) = F\big(b(f)\big)$,
3. $F(g \circ f) = F(g) \circ F(f)$ for all morphisms $f \in \mathrm{Mor}_{\mathcal{C}}(U,V)$, $g \in \mathrm{Mor}_{\mathcal{C}}(V,W)$.

We will call a covariant functor simply *functor*. Two important consequences of the functor axioms are: a functor F transforms each commutative diagram in $\mathcal{C}$ into a commutative diagram in $\mathcal{D}$, and if f is an isomorphism in $\mathcal{C}$, then $F(f)$ is an isomorphism in $\mathcal{D}$.

Recall that for two functors $F, G\colon \mathcal{C} \to \mathcal{C}'$ a *functorial morphism* $\phi\colon F \to G$ is a collection of morphisms

$$\phi_U\colon F(U) \to G(U), \qquad U \in \mathrm{Obj}(\mathcal{C}),$$

such that for every $f \in \mathrm{Mor}_{\mathcal{C}}(U,V)$ the following diagram is commutative:

$$\begin{array}{ccc} F(U) & \xrightarrow{F(f)} & F(V) \\ \phi_U \downarrow & & \downarrow \phi_V \\ G(U) & \xrightarrow[G(f)]{} & G(V) \end{array}$$

Functorial morphisms are also called *natural transformations* or *canonical morphisms*. If, furthermore, ϕ_U is an isomorphism of $\mathcal{C}'$ for any object U in $\mathcal{C}$, we say that ϕ is a *functorial isomorphism* (or *natural isomorphism*).

Example 1.4. (The category $\mathcal{V}ec_f(F)$ of finite-dimensional vector spaces over a field). Let F be a field. We call $\mathcal{V}ec_f(F)$ the category of finite-dimensional vector spaces, where morphisms are linear maps. In $\mathcal{V}ec_f(F)$ there exists a functorial isomorphism between a vector space V and its double dual, $V \simeq V^{**}$, but there is no functorial isomorphism between V and V^*.

2 Tensor categories

In this section we summarize without proofs some basic results from the theory of tensor categories (see [JS93], [Shu94] and [Tur94, Part I, I.1.]). Tensor categories are categories enriched with an associative (up to coherent isomorphism) tensor product. They have been studied and used extensively. In the literature, they are also called *monoidal categories*.

Definition 2.1. A *tensor product* in a category $\mathcal{C}$ is a covariant functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which associates to each pair of objects V, W of $\mathcal{C}$ an object $V \otimes W$ in $\mathcal{C}$ and to each pair of morphisms $f\colon V \to V', g\colon W \to W'$ a morphism $f \otimes g\colon V \otimes W \to V' \otimes W'$.

The following identities hold for the covariant functor $\otimes$:

$$(f \circ f') \otimes (g \circ g') = (f \otimes g) \circ (f' \otimes g'), \tag{2.2}$$

$$1_{V \otimes W} = 1_V \otimes 1_W, \tag{2.3}$$

whenever the composites $f \circ f'$ and $g \circ g'$ are defined. Relation (2.2) implies that

$$f \otimes g = (f \otimes 1_{W'}) \circ (1_V \otimes g) = (1_{V'} \otimes g) \circ (f \otimes 1_W). \tag{2.4}$$

Definition 2.5. A functor $F\colon \mathcal{C} \to \mathcal{D}$ *preserves* the tensor product $\otimes$ if for all $V, W \in \mathrm{Obj}(\mathcal{C})$ and all morphisms f, g, as above, we have

$$F(1_{\mathcal{C}}) = 1_{\mathcal{D}}, \quad F(V \otimes W) = F(V) \otimes F(W), \quad F(f \otimes g) = F(f) \otimes F(g).$$

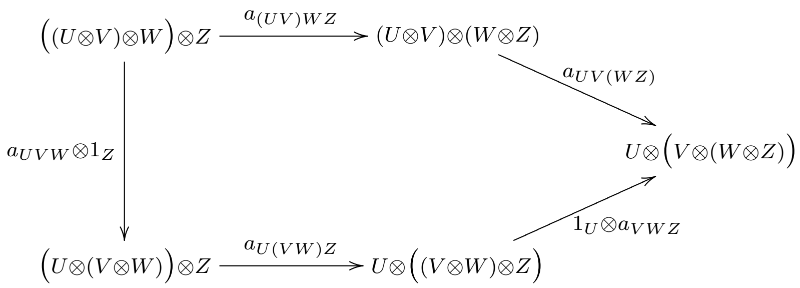
Given a category $\mathcal{C}$, we are interested in what kind of associativity should be required for a functor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. We cannot require the equality $(U \otimes V) \otimes W = U \otimes (V \otimes W)$ for all U, V, W, because this is not true even in the category of finite-dimensional vector spaces $\mathcal{V}ec_f(F)$. We may ask instead $(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$, but then the condition is too weak; for example, in $\mathcal{V}ec_f(F)$ every two vector spaces of equal dimension are isomorphic. The correct definition of associativity arises from natural isomorphisms:

Definition 2.6. A *tensor category* $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{I}, a, l, r)$ is a category $\mathcal{C}$ equipped with a tensor product $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, a unit object $\mathbb{I} \in \mathrm{Obj}(\mathcal{C})$, and natural isomorphisms

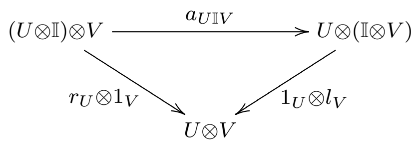
$$\begin{aligned} a = a_{UVW} \colon (U \otimes V) \otimes W &\xrightarrow{\sim} U \otimes (V \otimes W), \\ l = l_V \colon \mathbb{I} \otimes V &\xrightarrow{\sim} V, \\ r = r_V \colon V \otimes \mathbb{I} &\xrightarrow{\sim} V, \end{aligned}$$

(called the *associativity, left unit, right unit constraints*, respectively), such that for all objects $U, V, W, Z \in \mathrm{Obj}(\mathcal{C})$ the following diagrams commute:

1. **Associativity pentagon.**



2. **Triangle for unit.**



Given objects $V_1, \ldots, V_n$ in the category $\mathcal{C}$, we have many ways to define the tensor product $V_1 \otimes \cdots \otimes V_n$, depending on the choice of parentheses we use. If we have an associativity constraint a (see Definition 2.6), the various choices of parentheses on $V_1 \otimes \cdots \otimes V_n$ lead to isomorphic objects. For instance, there are five ways of putting parentheses around the tensor product of four objects U, V, W, Z: the associativity pentagon diagram shows all five of them, as well as the connecting morphisms that are expressible in terms of the identity morphisms and the constraint a. In particular, we see that there are a priori two different isomorphisms from $\big((U \otimes V) \otimes W\big) \otimes Z$ to $U \otimes \big(V \otimes (W \otimes Z)\big)$. It is natural to require that these two isomorphisms are the same or, equivalently, that the pentagon diagram commutes. When the associativity constraint a satisfies this condition, then, between two ways of putting parentheses on $U \otimes V \otimes W \otimes Z$, there exists a unique natural isomorphism expressible in terms of a.

Definition 2.7. A tensor category is *strict* when all the constraints a_{UVW}, l_V, r_V are identity morphisms.

We have

$$
\begin{aligned}
(U \otimes V) \otimes W &= U \otimes (V \otimes W), \\
(f \otimes g) \otimes h &= f \otimes (g \otimes h), \\
V \otimes \mathbb{I} = \; & V \; = \mathbb{I} \otimes V, \\
f \otimes 1_{\mathbb{I}} = \; & f \; = 1_{\mathbb{I}} \otimes f,
\end{aligned} \tag{2.8}
$$

for all objects U, V, W and all morphisms f, g, h in a strict category.

Example 2.9. The category $\mathcal{V}ec_f(F)$ is not a strict tensor category because the isomorphisms a, l and r are not identity morphisms.

It is clear that in a strict tensor category one can write multiple tensor products $V_1 \otimes \cdots \otimes V_n$ without employing brackets. MacLane's coherence theorem establishes that every tensor category is equivalent to a strict one (for details see [Mac98, XI.3.]). It follows that in a tensor category we can use the canonical isomorphisms to write the tensor product without brackets, in the same way we write the tensor product of vector spaces. Therefore, we can also think of $V_1 \otimes \cdots \otimes V_n$ as an object of $\mathcal{C}$ by identifying all expressions with brackets. In each formula it is also possible to omit parentheses, associativity and unit isomorphisms, without the loss of generality.

2.a Braidings and braided tensor categories

In the category of finite-dimensional vector spaces $\mathcal{V}ec_f(F)$, we have a commutativity isomorphism τ, called *switch isomorphism*:

$$\tau_{V,W} \colon V \otimes W \xrightarrow{\sim} W \otimes V, \qquad \text{for any } V, W \in \operatorname{Obj}\big(\mathcal{V}ec_f(F)\big).$$

The switch transforms any vector $v \otimes w$ into $w \otimes v$ and extends to $V \otimes W$ by linearity. The system of switch isomorphisms is compatible with the tensor product in the obvious way: for any three objects U, V, W, we have

$$\tau_{U,V \otimes W} = (1_V \otimes \tau_{U,W}) \circ (\tau_{U,V} \otimes 1_W),$$

$$\tau_{U \otimes V,W} = (\tau_{U,W} \otimes 1_V) \circ (1_U \otimes \tau_{V,W}).$$

Moreover, it is involutive in the sense that $\tau^2 = 1$. We would like to axiomatize this kind of structure; however, we want to allow that $\tau^2 \neq 1$, since this happens in many interesting examples.

Definition 2.10. A *braiding* in a tensor category $\mathcal{C}$ consists of a natural family of isomorphisms

$$c = \{c_{V,W} \colon V \otimes W \to W \otimes V\}, \tag{2.11}$$

where V, W run over all objects of $\mathcal{C}$, such that for any three objects U, V, W, we have

$$c_{U,V \otimes W} = (1_V \otimes c_{U,W}) \circ (c_{U,V} \otimes 1_W), \tag{2.12}$$

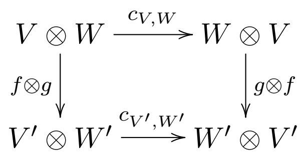
$$c_{U \otimes V,W} = (c_{U,W} \otimes 1_V) \circ (1_U \otimes c_{V,W}). \tag{2.13}$$

According to some authors, a braiding is also called *commutativity isomorphism*.

The naturality of the isomorphisms (2.11) means that for any morphisms $f\colon V \to V'$, $g\colon W \to W'$, we have

$$(g \otimes f) \circ c_{V,W} = c_{V',W'} \circ (f \otimes g). \tag{2.14}$$

In other words, this diagram commutes

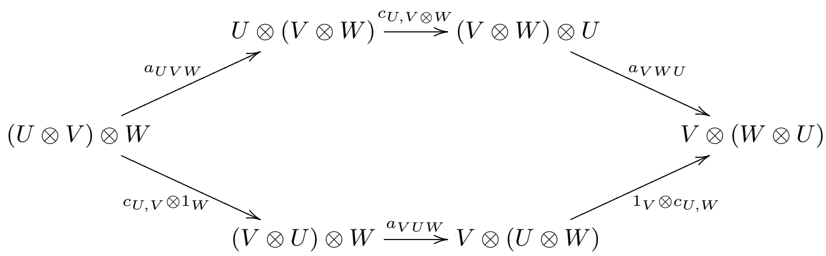


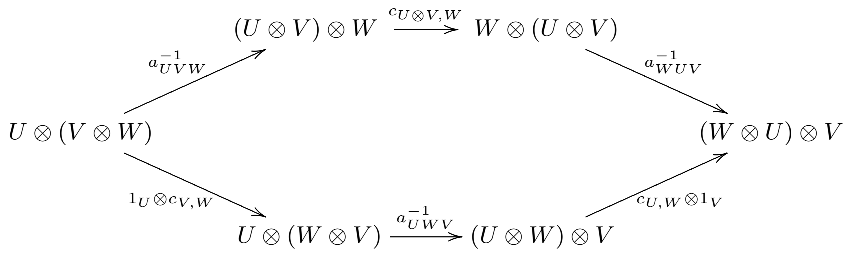
Applying (2.12) and (2.13) to $V = W = \mathbb{I}$ and $U = V = \mathbb{I}$ and using the invertibility of $c_{V,\mathbb{I}}, c_{\mathbb{I},V}$ we get

$$c_{V,\mathbb{I}} = c_{\mathbb{I},V} = 1_V \tag{2.15}$$

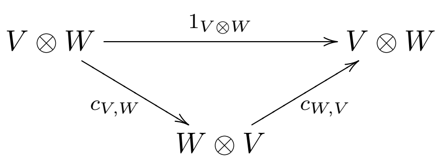
for any object V of $\mathcal{C}$ (for a proof, see [Ste98, Lemme I.3.2.]).

Definition 2.16. A *braided tensor category* is a tensor category $\mathcal{C}$ with a braiding c such that the following two hexagon diagrams commute:





Definition 2.17. If a braiding c in a braided tensor category $\mathcal{C}$ satisfies the extra condition $c_{W,V} \circ c_{V,W} = 1_{V\otimes W}$, we call $\mathcal{C}$ a *symmetric tensor category*:



If c is a braiding then c', given by $c'_{V,W} = (c_{W,V})^{-1}$, is also a braiding, since the second hexagon diagram is obtained from the first by replacing c with c'. Note that $\mathcal{C}$ is symmetric if $c = c'$.

Example 2.18. The category $\mathcal{V}ec_f(F)$ is a symmetric tensor category.

Now we state an important property of a braided tensor category. It is the categorical version of the Yang-Baxter equation.

Proposition 2.19. (Yang-Baxter identity). *Let U, V, W be objects of a strict braided tensor category. Then*

$$(c_{V,W} \otimes 1_U)(1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W) = (1_W \otimes c_{U,V})(c_{U,W} \otimes 1_V)(1_U \otimes c_{V,W}).$$

Proof. Because of the length of the formulas we have omitted the composition symbol $\circ$. We have

$$\begin{aligned}(c_{V,W} \otimes 1_U)(1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W) &= (c_{V,W} \otimes 1_U)c_{U,V\otimes W} \\ &= c_{U,W\otimes V}(1_U \otimes c_{V,W}) \\ &= (1_W \otimes c_{U,V})(c_{U,W} \otimes 1_V)(1_U \otimes c_{V,W}).\end{aligned}$$

The first and the last equalities follow from (2.12), the second one from (2.14) with $f = 1_U$ and $g = c_{V,W}$. □

Finally, we recall the definition of functor between braided tensor categories:

Definition 2.20. Let $\mathcal{C}_1, \mathcal{C}_2$ be braided tensor categories. A *tensor functor* from $\mathcal{C}_1$ to $\mathcal{C}_2$ is a pair (F, J), where F is a functor $F\colon \mathcal{C}_1 \to \mathcal{C}_2$ and J is a functorial isomorphism

$$J_{U,V}\colon F(U \otimes V) \xrightarrow{\sim} F(U) \otimes F(V)$$

such that:

1. $F(a_1) = a_2$, where a_1, a_2 are the associativity isomorphisms in $\mathcal{C}_1, \mathcal{C}_2$, respectively, and

   $$F(a_1)\colon F\big((U \otimes V) \otimes W\big) \to F\big(U \otimes (V \otimes W)\big)$$

   is considered as an operator

   $$\big(F(U) \otimes F(V)\big) \otimes F(W) \to F(U) \otimes \big(F(V) \otimes F(W)\big)$$

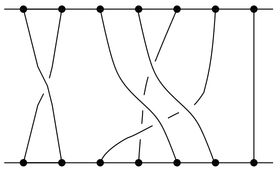
   using J.

2. $F(c_1) = c_2$, where c_1, c_2 are the braidings in $\mathcal{C}_1, \mathcal{C}_2$.

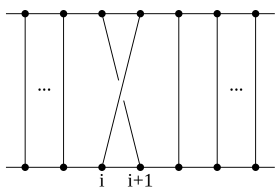
2.b An example: the braid category

We dedicate this section to a non-trivial example of particular interest: the braid category. This example, which will not be used later, has the purpose of clarifying some definitions. For more details, we refer the reader to [JS93, Example 2.1], [Kas95, Section X.6] and [BK01, Section 1.2.].

A *braid* in n strands is an isotopy class of a union of n nonintersecting segments of smooth curves in $\mathbb{R}^3$ with start points $\{1, \ldots, n\} \times \{0\} \times \{1\}$ and end points $\{1, \ldots, n\} \times \{0\} \times \{0\}$, such that for each of these strands the third coordinate z is strictly decreasing from 1 to 0 (so strands are considered "going down"). An example of a braid in 7 strands is illustrated below:


.

Let s_i be the braid depicted by

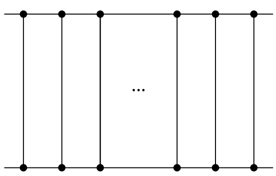
$s_i =$ 

,

then the Artin braid group on n strands $\mathcal{B}_n$ is given by the generators $s_1, \ldots, s_{n-1}$ subject to the relations

$$
\begin{aligned}
s_i s_j &= s_j s_i && \text{for } 1 \leq i < j - 1 \leq n - 2, \\
s_{i+1} s_i s_{i+1} &= s_i s_{i+1} s_i && \text{for } 1 \leq i \leq n - 2.
\end{aligned}
$$

The set $\mathcal{B}_0$ has one single element, namely the empty braid. The unit 1_n is represented by

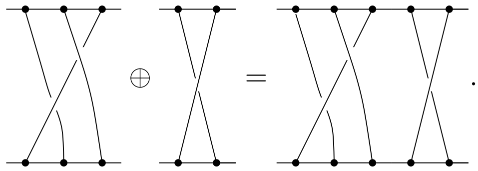


.

Composition of braids is just multiplication in this group, represented diagrammatically by vertical stacking of braids with the same number of strings. A strict tensor product of braids is defined by

$$\oplus \colon \mathcal{B}_n \times \mathcal{B}_m \to \mathcal{B}_{n+m}$$

and algebraically described by the equation $s_i \oplus s_j = s_i s_{n+j} (= s_{n+j} s_i)$. It is illustrated as in the following example:

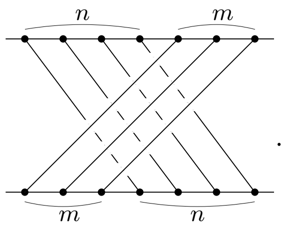


In other words, one adds the number of strings by placing one braid next to the other longitudinally.

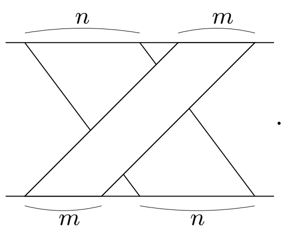
The *braid category* $\mathcal{B}$ is the disjoint union of the $\mathcal{B}_n$. Therefore, the objects of $\mathcal{B}$ are the natural numbers $\mathbb{N}$ and the morphisms are given by

$$\mathcal{B}(n, m) = \begin{cases} \mathcal{B}_n, & \text{when } n = m, \\ \emptyset, & \text{otherwise.} \end{cases}$$

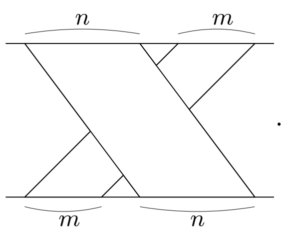
For any couple (n, m) of non-negative integers, a braiding $c_{n,m} \colon n \oplus m \to m \oplus n$ for $\mathcal{B}$ is given by crossing the first n strings under the remaining m as illustrated below:



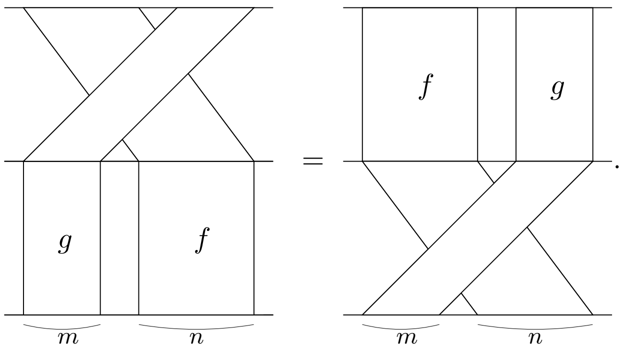
Note that we have $c_{0,n} = c_{n,0} = 1_n$. For simplicity, one can also depict $c_{n,m}$ as follows:



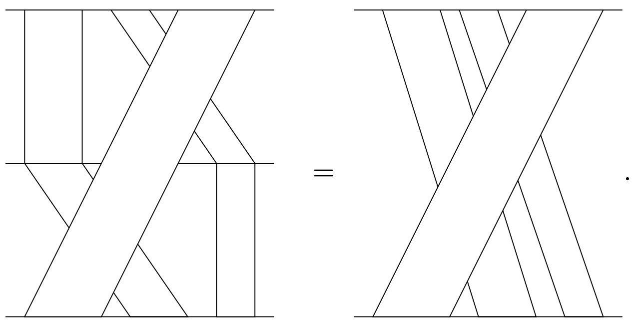
Then, $c'_{n,m} = (c_{m,n})^{-1}$ is represented by



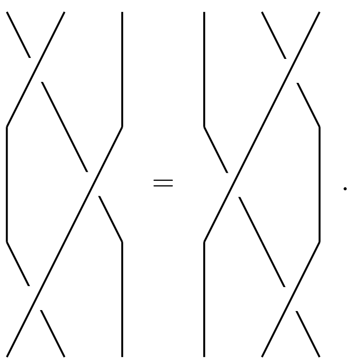
For all $f \in \mathcal{B}_n$ and $g \in \mathcal{B}_m$, naturalness of $c_{n,m}$ is proved pictorially by the equality



The second hexagon diagram is proved pictorially by the following two illustrations: the left and the right one represent, respectively, the lower and the upper way of the diagram.



Example 2.21. The Yang-Baxter identity can be drawn with the aid of $\mathcal{B}_3$ as the identity:



3 Ribbon categories

In this section we introduce the notion of a *ribbon category*: it is a braided tensor category provided with two additional structures (the terminology of *rigid balanced braided tensor category* is also used). Every object has a dual satisfying some natural properties, and there are functorial isomorphisms

$V \xrightarrow{\sim} V^{**}$ compatible with the tensor product. This kind of category will be useful to develop graphical calculus in the sequel.

The category $\mathcal{C}$ will always be assumed to be a braided tensor category. As mentioned in the previous section, it would be too restrictive to require $c_{W,V} \circ c_{V,W} = 1_{V\otimes W}$ for the braiding. This suggests to introduce the notion of a twist as follows:

Definition 3.1. A *twist* (or balancing isomorphism) in a tensor category $\mathcal{C}$ with a braiding c consists of a natural family of isomorphisms

$$\theta = \{\theta_V : V \to V\},$$

where V runs over all objects of $\mathcal{C}$, such that for any two objects V, W of $\mathcal{C}$, we have

$$\theta_{V\otimes W} = c_{W,V} \circ c_{V,W} \circ (\theta_V \otimes \theta_W). \tag{3.2}$$

The naturality of θ means that for any morphism $f : U \to V$ in $\mathcal{C}$, we have $\theta_V \circ f = f \circ \theta_U$. Then, we may rewrite the last equation as follows:

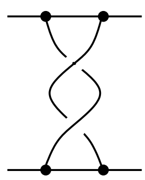
$$\theta_{V\otimes W} = c_{W,V} \circ (\theta_W \otimes \theta_V) \circ c_{V,W} = (\theta_V \otimes \theta_W) \circ c_{W,V} \circ c_{V,W}.$$

Note that $\theta_{\mathbb{I}} = 1_{\mathbb{I}}$. This follows from the invertibility of $\theta_{\mathbb{I}}$ and the formula

$$(\theta_{\mathbb{I}})^2 = (\theta_{\mathbb{I}} \otimes 1_{\mathbb{I}}) \circ (1_{\mathbb{I}} \otimes \theta_{\mathbb{I}}) = \theta_{\mathbb{I}} \otimes \theta_{\mathbb{I}} = \theta_{\mathbb{I}}.$$

These equalities follow respectively from (2.8), (2.2), ($A2$), (3.2) and (2.15), where we substitute $V = W = \mathbb{I}$.

Example 3.3. (The braid category). The twist $\theta_n : n \to n$ is obtained by taking n vertical parallel strings with ends tied to two horizontal parallel rigid rods, and rotating the top rod through a full 360° twist in the right-hand screw direction with thumb pointing upwards. Then θ_0, θ_1 are identities, while $\theta_2 = (c_{1,1})^2 = (s_1)^2 : 2 \to 2$ and is represented by



Let $\mathcal{C}$ be a tensor category. Assume that to each object V of $\mathcal{C}$ there are associated an object V^* of $\mathcal{C}$ and two morphisms

$$i_V : \mathbb{I} \to V \otimes V^*,$$
$$e_V : V^* \otimes V \to \mathbb{I}.$$

Definition 3.4. The rule $V \mapsto (V^*, i_V, e_V)$ is called a *duality* in $\mathcal{C}$ if the composition

$$V \xrightarrow{i_V \otimes 1_V} V \otimes V^* \otimes V \xrightarrow{1_V \otimes e_V} V$$

is equal to 1_V, and the composition

$$V^* \xrightarrow{1_{V^*} \otimes i_V} V^* \otimes V \otimes V^* \xrightarrow{e_V \otimes 1_{V^*}} V^*$$

is equal to 1_{V^*}.

Note that we do not require $(V^*)^* = V$. We also skip the canonical associativity and left unit constraints in our formulas; otherwise, we would have to write, for example, the map $i_V \otimes 1_V$ as

$$V \xrightarrow{l_V^{-1}} \mathbb{I} \otimes V \xrightarrow{i_V \otimes 1_V} (V \otimes V^*) \otimes V \xrightarrow{a} V \otimes (V^* \otimes V).$$

Definition 3.5. The identities

$$(1_V \otimes e_V) \circ (i_V \otimes 1_V) = 1_V, \tag{3.6}$$

$$(e_V \otimes 1_{V^*}) \circ (1_{V^*} \otimes i_V) = 1_{V^*}, \tag{3.7}$$

defining a duality are sometimes called *rigidity axioms*.

We need one axiom relating the duality morphisms i_V, e_V with braiding and twist.

Definition 3.8. Let $\mathcal{C}$ be a category equipped with a braiding c, a twist θ and a duality $(*, i, e)$. We say that the duality is *compatible* with the braiding and the twist of $\mathcal{C}$ if

$$(\theta_V \otimes 1_{V^*}) \circ i_V = (1_V \otimes \theta_{V^*}) \circ i_V \tag{3.9}$$

holds for every object V of $\mathcal{C}$. In this case we say that $\mathcal{C}$ is a *ribbon category*. A ribbon category is called *strict* if its underlying tensor category is strict.

Examples of ribbon categories are provided by the theory of quantum groups, for example finite-dimensional representations of a quantum group form a ribbon category. The term *ribbon* was introduced by Reshetikhin and Turaev in [RT90]; in the literature they are also called *tortile tensor categories* (see [Shu94]).

The compatibility relation (3.9) leads to a number of implications pertaining to duality. In particular, Turaev showed that any duality in $\mathcal{C}$ compatible with braiding and twist is involutive in the sense that $V^{**} = (V^*)^*$ is canonically isomorphic to V ([Tur94, Chapter I, Corollary 2.6.1.]): in every ribbon category there exists a functorial isomorphism

$$\delta_V \colon V \xrightarrow{\sim} V^{**} \tag{3.10}$$

with the properties $\delta_{V\otimes W} = \delta_V \otimes \delta_W$, $\delta_{\mathbb{I}} = 1_{\mathbb{I}}$, and $\delta_{V^*} = (\delta_V^*)^{-1}$ (see also [BK01, Section 2.2.]).

Ribbon categories admit a consistent theory of traces of morphisms and dimensions of objects. This is one of the most important features of ribbon categories, sharply distinguishing them from tensor categories (see Section 5.b).

MacLane's coherence theorem, that establishes equivalence of any tensor category to a strict tensor category, works in the setting of ribbon categories as well. This enables us to focus attention on strict ribbon categories: all results reported below for these categories directly extend to arbitrary ribbon categories.

Remark 3.11. (cf. [Tur94, Remark XI.1.4.]). There is a general procedure transforming any tensor category $\mathcal{R}$ into a strict tensor category $\mathcal{C} = \mathcal{C}(\mathcal{R})$. The objects of $\mathcal{C}$ are finite sequences $(V_1, \dots, V_k)$ of objects of $\mathcal{R}$, including the empty sequence. The morphisms from $(V_1, \dots, V_k)$ to $(W_1, \dots, W_l)$ are $\mathcal{R}$-morphisms $V_1 \otimes \cdots \otimes V_k \to W_1 \otimes \cdots \otimes W_l$. The tensor product of objects of $\mathcal{C}$ is the juxtaposition of sequences, the tensor product of morphisms is obtained by the obvious application of the tensor product in $\mathcal{R}$. It is easy to check that $\mathcal{C}$ is a strict tensor category with the unit object being the empty sequence. There is a covariant inclusion functor $\mathcal{R} \to \mathcal{C}$ assigning to any object V of $\mathcal{R}$ the 1-term sequence V and assigning to any morphism $f\colon V \to W$ the same morphism in $\mathcal{C}$. This inclusion is an equivalence of categories. It may be verified that any braiding (resp. twist, duality) in $\mathcal{R}$ induces a braiding (resp. twist, duality) in $\mathcal{C}$. In particular, $(V_1, \dots, V_k)^* = (V_k^*, \dots, V_1^*)$. In this way any ribbon category gives rise to a strict ribbon category. An exhaustive construction turning tensor categories into strict ones is also given in [Kas95, XI.5].

With the help of the morphisms i_V and e_V one defines the *dual* $f^*\colon W^* \to V^*$ of a morphism $f\colon V \to W$ in $\mathcal{C}$ to be the morphism

$$f^* = (e_W \otimes 1_{V^*}) \circ (1_{W^*} \otimes f \otimes 1_{V^*}) \circ (1_{W^*} \otimes i_V). \tag{3.12}$$

If $f\colon V \to W$ and $g\colon U \to V$ are morphisms of $\mathcal{C}$, then we have $(f \circ g)^* = g^* \circ f^*$ and $(1_V)^* = 1_{V^*}$ for any object V.

Example 3.13. (The category $\mathcal{V}ec_f(F)$). For every finite-dimensional vector space V, there is a dual vector space V^* and natural morphisms

$$i_V\colon F \to V \otimes V^*,$$
$$e_V\colon V^* \otimes V \to F.$$

The morphism e_V is the evaluation map, i.e., $e_V(f^i \otimes e_j) = f^i(e_j) = \delta_{ij}$, and $i_V(1_F) := \sum e_i \otimes f^i$, where $\{e_i\}$ and $\{f^i\}$ are dual bases in V and V^*. Via the isomorphism

$$V \otimes V^* \xrightarrow{\sim} \mathrm{End}_F(V)\ , \quad x \otimes g \mapsto h : y \mapsto h(y) = x \cdot g(y),$$

the element $i_V(1_F)$ corresponds to 1_V. The duality in the category $\mathcal{V}ec_f(F)$ satisfies the rigidity axioms: for example (3.6) is equivalent to the well-known identity $\sum_i e_i(f^i, v) = v$ for $v \in V$.

4 The category of ribbon graphs

In [Tur94] Turaev relates the theory of ribbon categories to the theory of links in the Euclidean space $\mathbb{R}^3$. More precisely he associates to a ribbon category a category of ribbon graphs in such a way that morphisms in the ribbon category are represented by graphs. This leads to the diagrammatic calculus used in Section 5. Section 4 is basically a summary of [Tur94, Chapter II, Section 2] and many of the definitions and results are taken from there. Only the orientation of the ribbon graphs has been reversed.

4.a Ribbon graphs and their diagrams

Ribbon graphs are oriented compact surfaces in $\mathbb{R}^3$ decomposed into elementary pieces, called bands, coupons and annuli.

Definition 4.1. A *band* is a square $[0,1] \times [0,1]$ or a homeomorphic image of this square. The images of the intervals $[0,1] \times \{0\}$ and $[0,1] \times \{1\}$ are called *bases of the band*; the image of the interval $\{\frac{1}{2}\} \times [0,1]$ is called the *core* of the band. An *annulus* is a cylinder $S^1 \times [0,1]$ or a homeomorphic image of this cylinder; the image of the circle $S^1 \times \{\frac{1}{2}\}$ is called the *core* of the annulus. A band or an annulus is *directed* if its core is oriented. The orientation of the core is called the *direction* of the band (annulus).

Definition 4.2. A *coupon* is a band with a distinguished base. This distinguished base is called the *top base* of the coupon, the opposite base is said to be the *bottom one*.

Given k, l non-negative integers, we define ribbon graphs with k inputs and l outputs or, briefly, ribbon (k,l)-graphs:

Definition 4.3. A *ribbon (k,l)-graph* in $\mathbb{R}^3$ is an oriented surface Ω embedded in the strip $\mathbb{R}^2 \times [0,1]$ and decomposed into a union of a finite number of annuli, bands and coupons such that

1. Ω meets the planes $\mathbb{R}^2 \times \{0\}$, $\mathbb{R}^2 \times \{1\}$ orthogonally along the following segments which are bases of certain bands of Ω:

   $\{[i - \frac{1}{10}, i + \frac{1}{10}] \times \{0\} \times \{0\} \mid i = 1, \ldots, l\}$ "bottom boundary intervals"

   $\{[j - \frac{1}{10}, j + \frac{1}{10}] \times \{0\} \times \{1\} \mid j = 1, \ldots, k\}$ "top boundary intervals"

   In the points of these segments, the orientation of Ω is determined by the pair of vectors $(1,0,0)$, $(0,0,1)$ tangent to Ω.

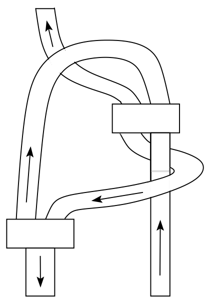
2. Other bases of bands lie on the bases of coupons; otherwise the bands, coupons and annuli are disjoint.

3. The bands and annuli of Ω are directed.

The surface Ω with the splitting into annuli, bands, and coupons forgotten is called the *surface* of the ribbon (k,l)-graph Ω. Condition 1. fixes the right-handed orientation in $\mathbb{R}^3$: near the boundary intervals the preferred side of Ω is the one turned up, i.e. towards the reader.

By *isotopy* of ribbon graphs we mean isotopy in the strip $\mathbb{R}^2 \times [0,1]$ constant on the boundary intervals and preserving the splitting into bands, annuli and coupons, as well as preserving the directions of bands and annuli, and the orientation of the graph surface.

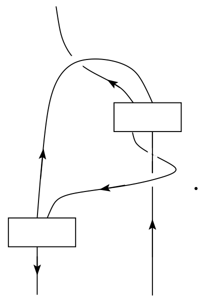
Each band should be thought of as a narrow strip or ribbon with small bases. Each coupon should be thought of as lying in $\mathbb{R}^2 \times (0,1)$ as a small rectangle with a distinguished base.
The following picture represents a ribbon $(1,2)$-graph:



In [Tur94, pp. 32-33], Turaev describes a technique enabling the representation of ribbon graphs by plane pictures, generalizing the standard knot diagrams. The main idea is to put a ribbon graph by isotopy into a "standard position" close to the plane $\mathbb{R} \times \{0\} \times \mathbb{R}$:

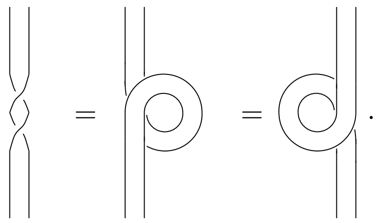
- coupons are plane rectangles parallel to $\mathbb{R} \times \{0\} \times \mathbb{R}$,
- the bases of coupons are parallel to the horizontal line $\mathbb{R} \times \{0\} \times \{0\}$,
- the top base of each coupon lies higher than the bottom one,
- the orientation of coupons induced by the orientation of Ω should correspond to the counterclockwise orientation in $\mathbb{R} \times \{0\} \times \mathbb{R}$,
- the bands and annuli of the graph should go close and "parallel" to this plane,
- the projections of the cores of bands and annuli in the plane $\mathbb{R}\times\{0\}\times\mathbb{R}$ should have only double transversal crossings and should not overlap with the projections of coupons.

After having deformed the graph in such a position, one can draw the projections of the coupons and the cores of the bands and annuli in $\mathbb{R} \times \{0\} \times \mathbb{R}$ taking into account the overcrossings and undercrossings of the cores. The projections of the cores of bands and annuli are oriented in accordance with their directions. The resulting picture is called the *diagram of the ribbon graph*, or shorter *graph diagram*. The following graph diagram represents the above ribbon graph



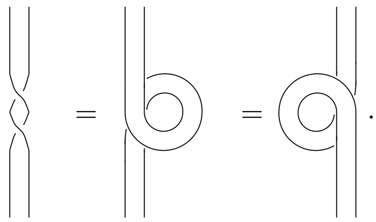
The technique of graph diagrams suffices to represent ribbon graphs. Any ribbon graph is isotopic to a ribbon graph lying in a standard position and therefore presented by a graph diagram. From such a diagram it is possible to reconstruct the original ribbon graph, up to isotopy, just by allowing the bands and annuli to be "parallel" to the plane of the picture along their cores. The only problem one may encounter is that the bands may be twisted several times around their cores; however, both positive and negative twists in a band are isotopic to cirls which go "parallel" to the plane. Hence, denoting isotopies by equalities, we have that:

(4.4)



The same holds for:

(4.5)



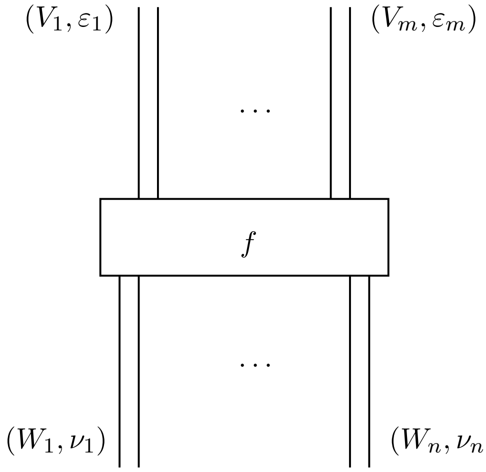
4.b Ribbon graphs over $\mathcal{C}$

Let $\mathcal{C}$ be a strict tensor category with duality.

Definition 4.6. A ribbon graph is said to be *colored* (over $\mathcal{C}$) if each band and each annulus of the graph is labelled by an object of $\mathcal{C}$. This object is called the *color* of the band or annulus.

Coupons of a ribbon graph will be colored by morphisms in $\mathcal{C}$. For this we need to distinguish between bottom and top bases of coupons. Let T be a coupon of a colored ribbon graph Ω. Let $V_1, \dots, V_m$ be the colors of the bands of Ω incident to the top base of T and encountered in the order induced by the orientation of Ω restricted to T. Let $\varepsilon_1, \dots, \varepsilon_m \in \{+1, -1\}$ be the numbers determined by the directions of these bands: we set $\varepsilon_i = +1$ if the band is directed "in" the coupon and $\varepsilon_i = -1$ in the opposite case.

(4.7)



Let $W_1, \dots, W_n$ be the colors of the bands of Ω incident to the bottom base of T and encountered in the order induced by the opposite orientation of T. Let $\nu_1, \dots, \nu_n \in \{+1, -1\}$ be the numbers determined by the directions of the bands W_j: $\nu_j = +1$ if the band is directed "out" of the coupon and $\nu_j = -1$ in the opposite case. In other words, the band is directed downwards if the corresponding number is $+1$ and upwards if the number is -1.
A *color* of the coupon T is then a morphism

$$f\colon V_1^{\varepsilon_1} \otimes \cdots \otimes V_m^{\varepsilon_m} \to W_1^{\nu_1} \otimes \cdots \otimes W_n^{\nu_n},$$

where for an object V of $\mathcal{C}$, we set

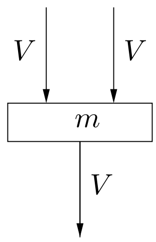
$$V^{+1} = V \qquad \text{and} \qquad V^{-1} = V^*.$$

Definition 4.8. A ribbon graph is $\mathcal{C}$-*colored* if it is colored and all its coupons are provided with colors as above.

By isotopy of $\mathcal{C}$-colored ribbon graphs, we mean color-preserving isotopy. The technique of diagrams readily extends to $\mathcal{C}$-colored ribbon graphs. In order to represent a colored ribbon graph by a graph diagram, we attach objects of $\mathcal{C}$ to the cores of bands and annuli, and we assign colors to all coupons.

Example 4.9. Figure (4.7) presents a $\mathcal{C}$-colored ribbon (m,n)-graph containing only one coupon and $m{+}n$ vertical untwisted unlinked bands incident to this coupon, and no annuli. We call a $\mathcal{C}$-colored ribbon graph with only one coupon an *elementary $\mathcal{C}$-colored ribbon graph*.

Example 4.10. Let $\mathcal{V}ec_f(F)$ be the category of finite-dimensional vector spaces. The picture



is the graph diagram of a $\mathcal{V}ec_F(F)$-colored ribbon $(2,1)$-graph. The color of the bands is the vector space V, and the color of the coupon represents the morphism $m\colon V\otimes V\to V$, since all ε_i and ν_i are equal to $+1$.

4.c The category of ribbon graphs over $\mathcal{C}$

Let $\mathcal{C}$ be a strict tensor category with duality. The $\mathcal{C}$-colored ribbon graphs over $\mathcal{C}$ may be organized into a strict tensor category denoted by $\mathrm{Rib}_{\mathcal{C}}$ and called the *category of ribbon graphs over $\mathcal{C}$*. The objects of the category $\mathrm{Rib}_{\mathcal{C}}$ are finite sequences $\big((V_1,\varepsilon_1),\ldots,(V_m,\varepsilon_m)\big)$, where $V_1,\ldots, V_m$ are objects of $\mathcal{C}$ and $\varepsilon_1,\ldots,\varepsilon_m\in\{+1,-1\}$. The empty sequence is also considered as an object of $\mathrm{Rib}_{\mathcal{C}}$ and is denoted by $\emptyset$.

A morphism $\nu\to\nu'$ in $\mathrm{Rib}_{\mathcal{C}}$ is the isotopy type of a $\mathcal{C}$-colored ribbon graph such that ν (resp. ν') is the sequence of colors and directions of those bands which hit the top (resp. bottom) boundary intervals. As usual, $\varepsilon=+1$ corresponds to the downward direction near the corresponding boundary interval and $\varepsilon=-1$ corresponds to the band directed upwards. For example, the ribbon graph drawn in Figure (4.7) represents a morphism $f\colon\big((V_1,\varepsilon_1),\ldots,(V_m,\varepsilon_m)\big)\to\big((W_1,\varepsilon_1),\ldots,(W_n,\varepsilon_n)\big)$.
Let $f\colon\nu\to\nu'$ and $g\colon\nu'\to\nu''$ be two morphisms, represented by $\mathcal{C}$-colored ribbon graphs F and G. The composition $g\circ f$ is obtained by placing G on the bottom of the ribbon graph F, gluing the corresponding ends, and compressing the result into $\mathbb{R}^2\times[0,1]$. The identity morphisms are represented by ribbon graphs which have no coupons, no annuli and consist of untwisted unlinked vertical bands. The identity endomorphism of the empty sequence is represented by the empty ribbon graph.

We next provide $\mathrm{Rib}_{\mathcal{C}}$ with a tensor multiplication $\square$. The tensor product of objects ν and ν' is their juxtaposition $\nu\square\nu'$. The tensor product of morphisms f,g represented by graphs F,G is obtained by placing the graph F

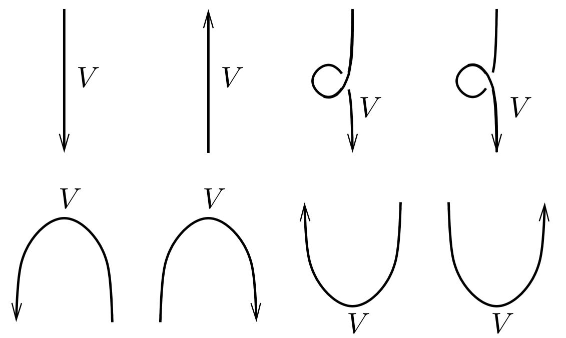
to the left of G so that there is no mutual linking or intersection, and is denoted by $F\Box G$. It is obvious that this tensor multiplication makes $\mathrm{Rib}_{\mathcal{C}}$ a strict tensor category.

Remark 4.11. Note that the same technique for defining composition and tensor multiplication was used for the braid category.

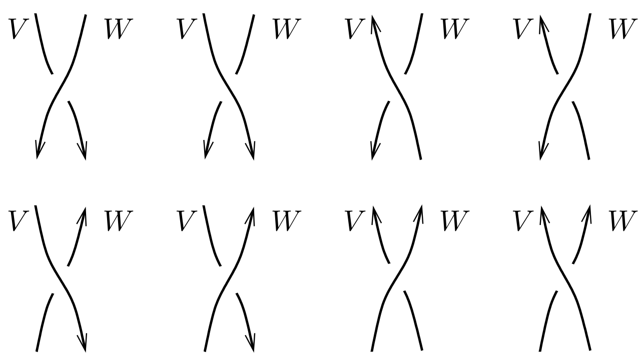
The category $\mathrm{Rib}_{\mathcal{C}}$ admits a natural braiding, a twist and a duality, and becomes in this way a ribbon category. We shall not use these structures nor discuss them here.

Definition 4.12. A ribbon graph over $\mathcal{C}$ without coupons is called a *ribbon tangle* over $\mathcal{C}$ (or simply *tangle* if the context is clear).

It is clear that ribbon tangles form a subcategory of $\mathrm{Rib}_{\mathcal{C}}$ which has the same objects as $\mathrm{Rib}_{\mathcal{C}}$ but less morphisms. This subcategory is called the *category of colored ribbon tangles* and is a strict tensor category under the same tensor product. There exist particular ribbon tangles which generate through composition and tensor product all ribbon tangles. This *alphabet* of tangles consists of the tangles



denoted $\downarrow_V$, $\uparrow_V$, φ_V, φ'_V, $\cap_V$, $\cap_V^-$, $\cup_V$, $\cup_V^-$, and of the tangles



denoted $X^+_{V,W}$, $X^-_{V,W}$, $Y^+_{V,W}$, $Y^-_{V,W}$, $Z^+_{V,W}$, $Z^-_{V,W}$, $T^+_{V,W}$, $T^-_{V,W}$, where V, W run over objects of $\mathcal{C}$ (see [Tur94, Chapter I, Figures 2.5 and 2.6]). In comparison to Turaev's alphabet, note that the orientations of some tangles have been reversed. This is due to the drawing of graphs from top to bottom. However, the tangles have the same algebraic meaning.

4.d The functor $\mathcal{R}$

The following Theorem 4.13 plays a fundamental role in the theory of ribbon categories. It relates a ribbon category with its category of ribbon graphs and can be viewed as a machine representing morphisms in the ribbon category by graphs.

Theorem 4.13. (Reshetikhin-Turaev, [Tur94, Chapter I, Theorem 2.5.]). *Let $\mathcal{C}$ be a strict ribbon category with braiding c, twist θ, and compatible duality $(*, i, e)$. There exists a unique covariant functor $\mathcal{R} = \mathcal{R}_{\mathcal{C}} \colon \mathrm{Rib}_{\mathcal{C}} \to \mathcal{C}$ preserving the tensor product and satisfying the following conditions:*

1. *$\mathcal{R}(\emptyset) = \mathbb{I}$, and $\mathcal{R}$ transforms any object $(V, +1)$ into V and any object $(V, -1)$ into V^*;*
2. *$\mathcal{R}(\downarrow_V) = 1_V$, $\mathcal{R}(\uparrow_V) = 1_{V^*}$;*
3. *for any objects V, W of $\mathcal{C}$ we have*
$$\mathcal{R}(X^+_{V,W}) = c_{V,W}, \quad \mathcal{R}(\varphi_V) = \theta_V, \quad \mathcal{R}(\cap_V) = i_V, \quad \mathcal{R}(\cup_V) = e_V;$$
4. *for any elementary $\mathcal{C}$-colored ribbon graph G, we have $\mathcal{R}(G) = g$, where g is the color of the only coupon of G;*
5. *for any objects ν, ν' or $\mathcal{C}$-colored ribbon graphs G, G' of $\mathrm{Rib}_{\mathcal{C}}$ we have: $\mathcal{R}(\nu\Box\nu') = \mathcal{R}(\nu) \otimes \mathcal{R}(\nu')$ and $\mathcal{R}(G\Box G') = \mathcal{R}(G) \otimes \mathcal{R}(G')$;*
6. *if G and G' are composable, we have $\mathcal{R}(G \circ G') = \mathcal{R}(G) \circ \mathcal{R}(G')$.*

The functor $\mathcal{R}$ has the following properties:

$$\mathcal{R}(\varphi'_V) = (\theta_V)^{-1}, \quad \mathcal{R}(X^-_{V,W}) = (c_{W,V})^{-1}, \quad \mathcal{R}(Y^+_{V,W}) = (c_{W,V^*})^{-1},$$

$$\mathcal{R}(Y^-_{V,W}) = c_{V^*,W}, \quad \mathcal{R}(Z^+_{V,W}) = (c_{W^*,V})^{-1}, \quad \mathcal{R}(Z^-_{V,W}) = c_{V,W^*},$$

$$\mathcal{R}(T^+_{V,W}) = c_{V^*,W^*}, \quad \mathcal{R}(T^-_{V,W}) = (c_{W^*,V^*})^{-1}.$$

For a proof of Theorem 4.13 we refer to [Tur94, Chapter 1, Theorem 2.5.] and to the original paper [RT90, Theorem 5.1.]. The idea of the proof is the following. One can express any ribbon graph through composition and tensor product in $\mathrm{Rib}_{\mathcal{C}}$ in terms of the ribbon graphs occuring in items 3. and 4. of the theorem. One would like to use such an expression to define the value of $\mathcal{R}$ for any ribbon graph. Such an expression is not unique. However two different expressions for the same ribbon graph can be obtained from each other by Redemeister moves and some other local transformations. To show that $\mathcal{R}$ is well defined, one has to verify the invariance of $\mathcal{R}$ under these moves.

Let G and G' be $\mathcal{C}$-colored ribbon graphs. We write $G \doteq G'$ if the corresponding morphisms $\mathcal{R}(G)$ and $\mathcal{R}(G')$ are equal in $\mathcal{C}$. Similarly, for a $\mathcal{C}$-colored ribbon graph G and a morphism f in $\mathcal{C}$, we write $G \doteq f$ whenever $f = \mathcal{R}(G)$. For example, $X^+_{V,W} \doteq c_{V,W}$ and $\varphi_V \doteq \theta_V$.

Lemma 4.14. *The image under* $\mathcal{R}$ *of any (non empty)* $\mathcal{C}$*-colored ribbon* $(0,0)$*-graph is in* $\mathrm{End}(\mathbb{I})$.

Proof. A ribbon $(0,0)$-graph G is a graph without free ends, hence it is necessarily a closed graph (or a union of closed graphs) and represents a morphism $\emptyset \to \emptyset$. Since $\mathcal{R}(\emptyset) = \mathbb{I}$, $\mathcal{R}(G)$ is a morphism in $\mathrm{End}(\mathbb{I})$. □

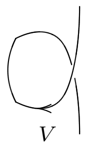
5 Diagrammatic calculus for morphisms

Using the category $\mathrm{Rib}_{\mathcal{C}}$ and the functor $\mathcal{R}$ constructed in the previous section one can represent any morphism in a strict ribbon category by a ribbon graph. This leads to a diagrammatic calculus which is developed in this section.

The projections of coupons and cores of a ribbon graph, deformed into a standard position, give rise to the *graph diagram* (as explained on page 15). In the sequel we will represent all ribbon graphs with their graph diagrams.

Remark 5.1. The twist $\theta_V : V \to V$ is represented by the graph diagram

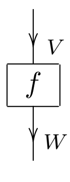
$\varphi_V =$



which is obviously isotopic to the graph diagram $\downarrow_V$, but $\mathcal{R}(\varphi_V) \neq \mathcal{R}(\downarrow_V)$, or $\theta_V \neq 1_V$, unless the category $\mathcal{C}$ is symmetric. This becomes clear considering the Figures (4.4) and (4.5): to count twists, one has to consider ribbon graphs instead of the diagrams. Hence, isotopy must always be referred to as ribbon graphs (and not to their diagrams).

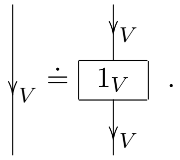
5.a Graphical calculus

The basic idea is that we can represent a morphism $f : V \to W$ in the category $\mathcal{C}$ by a graph diagram in $\mathrm{Rib}_{\mathcal{C}}$:

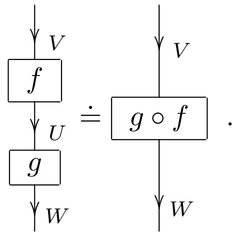


.

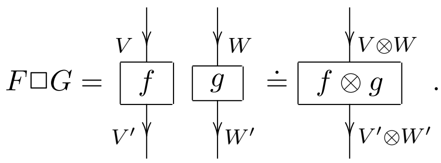
When f is the identity morphism, the box will be omitted: 1_V is represented by a vertical arrow directed downwards and colored with V:



Since $\mathbb{I} \otimes V \simeq V \otimes \mathbb{I} \simeq V$ for any object V, we can add arrows labeled with $\mathbb{I}$'s to any picture without changing the morphism it represents. Similarly, vertical arrows colored with $\mathbb{I}$ may be effaced from any picture. Hence, we agree that the empty picture represents the identity endomorphism of $\mathbb{I}$.
The composition $g \circ f$ of two morphisms is obtained by placing the picture of f on the top of that of g and gluing the corresponding free ends of the arrows. Of course, this procedure may be applied only when the numbers of the arrows, as well as their directions and colors, are compatible. For $f: V \to U$ and $g: U \to W$, we represent $g \circ f$ by



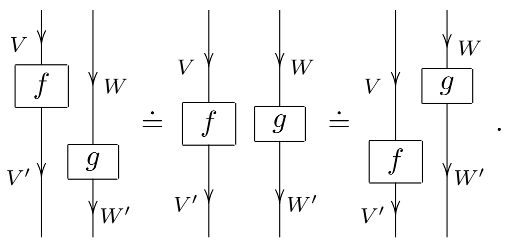
The tensor product $f \otimes g$ of two morphisms $f: V \to V'$ and $g: W \to W'$ is depicted by placing the diagram F of f to the left of the diagram G of g as follows:



Example 5.2. For any morphisms $f: V \to V'$ and $g: W \to W'$, the identities

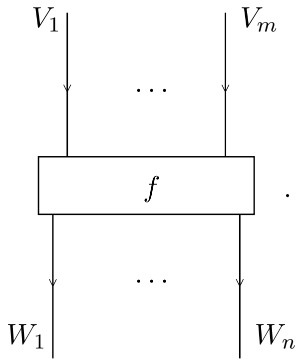
$$(1_{V'} \otimes g) \circ (f \otimes 1_W) = f \otimes g = (f \otimes 1_{W'}) \circ (1_V \otimes g)$$

have the graphical incarnation:

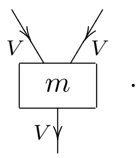


This leads to a useful isotopy principle: for any figure presenting a morphism of $\mathcal{C}$, the part of the figure lying to the left (or to the right) of a vertical line may be pushed up or down without changing the corresponding morphism in $\mathcal{C}$. We shall use this principle frequently and without any further explanation in the sequel.

A morphism $f\colon V_1 \otimes \cdots \otimes V_m \to W_1 \otimes \cdots \otimes W_n$ is depicted as

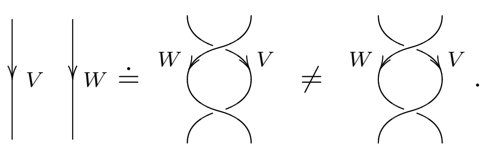


Example 5.3. (Bilinear product). Let V a vector space of $\mathcal{V}ec_f(F)$. We represent a bilinear product $m\colon V \otimes V \to V$, $x \otimes y \mapsto xy$, as a graph



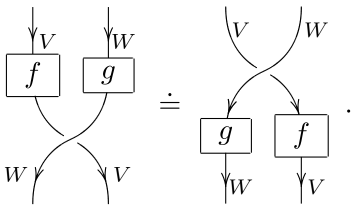
Assume now that $\mathcal{C}$ is braided with braiding c. For any pair V, W of objects we represent $c_{V,W} = \mathcal{R}(X^+_{V,W})$ and its inverse $(c_{V,W})^{-1} = \mathcal{R}(X^-_{W,V})$ by the graph diagrams

Remark 5.4. Note that $(c_{V,W})^{-1} \neq c_{W,V}$, since $1_{V\otimes W} = (c_{V,W})^{-1} \circ c_{V,W} \neq c_{W,V} \circ c_{V,W}$. Diagrammatically:

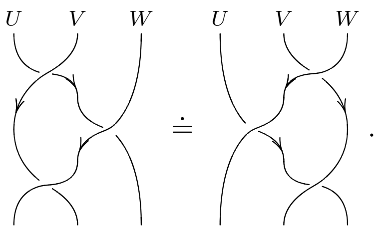


The first equality is called “the second Reidemeister move” in knot theory.

The functoriality of c, i.e. $c_{V,W} \circ (f \otimes g) = (g \otimes f) \circ c_{V,W}$, has a graphical incarnation:



Remark 5.5. Replacing f by $c_{U,V} : U \otimes V \to V \otimes U$ and g by 1_W in the last relation, we obtain the identity known as "the third Reidemeister move". This relates two different ways to go from $U \otimes V \otimes W$ to $W \otimes V \otimes U$ by repeated braiding:



Example 5.6. Let $1 \le i, k \le n$. For $U_i \in \mathrm{Obj}(\mathcal{C})$ we consider the "switch" morphism

$$p_n : U_1 \otimes \cdots \otimes U_n \to U_n \otimes \cdots \otimes U_1$$

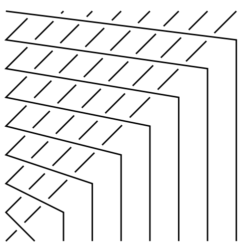
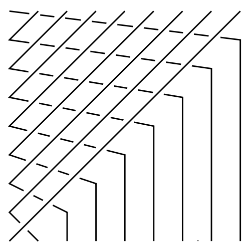
defined by the formula

$$\begin{aligned} p_n \;=\; & \big((c_{n,n-1})^{-1} \otimes 1_{n-2} \otimes \cdots \otimes 1_2 \otimes 1_1\big) \circ \cdots \circ \big((c_{4\ldots n,3})^{-1} \otimes 1_2 \otimes 1_1\big) \\ & \circ\big((c_{3\ldots n,2})^{-1} \otimes 1_1\big) \circ (c_{2\ldots n,1})^{-1} \end{aligned}$$

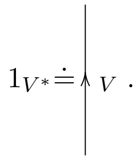
where the object U_i is labeled by i, $U_i \otimes U_{i+1} \otimes \cdots \otimes U_n$ by $i \ldots n$ and $U_k \otimes U_{k-1} \otimes \cdots \otimes U_1$ by $k \ldots 1$. The inverse $(p_n)^{-1} : U_n \otimes \cdots \otimes U_1 \to U_1 \otimes \cdots \otimes U_n$ is the morphism

$$\begin{aligned} (p_n)^{-1} \;=\; & (c_{2,1} \otimes 1_3 \otimes \cdots \otimes 1_n) \circ (c_{3,21} \otimes 1_4 \otimes \cdots \otimes 1_n) \circ \cdots \circ \\ & (c_{n-2,n-3\ldots 1} \otimes 1_{n-1} \otimes 1_n) \circ (c_{n-1,n-2\ldots 1} \otimes 1_n) \circ (c_{n,n-1\ldots 1}). \end{aligned}$$

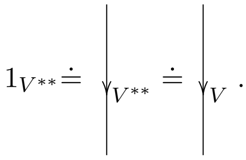
We call P_n the graph diagram in $\mathrm{Rib}_{\mathcal{C}}$ representing the morphism p_n. For example, for $n = 9$ we have:

$P_9 =$  , $(P_9)^{-1} =$  .

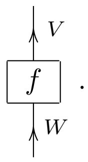
A vertical arrow directed upwards and colored with V represents the identity endomorphism of V^*:



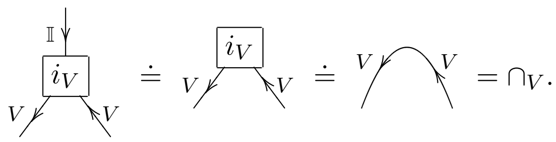
Also we identify V^{**} with V via δ_V $\big($see (3.10)$\big)$:



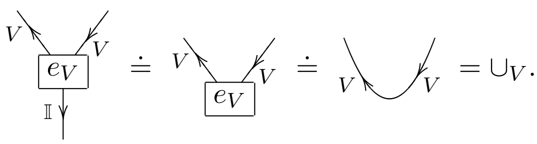
A morphism $f\colon V^* \to W^*$ may be represented as



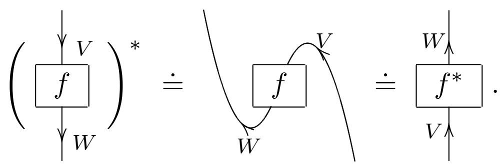
The morphism $i_V\colon \mathbb{I} \to V \otimes V^*$ corresponds to the alphabet graph $\cap_V$:



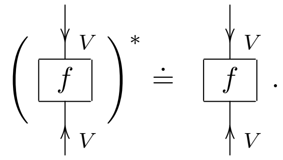
Similarly, $e_V\colon V^* \otimes V \to \mathbb{I}$ corresponds to $\cup_V$:



By using these pictures as building blocks of the alphabet of ribbon graphs, one can represent any morphism in $\mathcal{C}$. For example, with our graphical conventions, the dual $f^*\colon W^* \to V^*$ of a morphism $f\colon V \to W$, defined by $f^* = (e_W \otimes 1_{V^*}) \circ (1_{W^*} \otimes f \otimes 1_{V^*}) \circ (1_{W^*} \otimes i_V)$ as in (3.12), is represented by

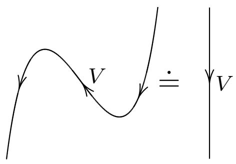


Hence, to draw duals one has to rotate the diagram by 180°. Note that we do not need to write the asterisk on the label to denote the dual space because the direction of the arrow does this for us automatically. Conversely, we need to put the asterisk in the label of the coupon because the direction of the arrows may not be sufficient to distinguish f from f^*. Consider an operator $f\colon V \to V^*$. The adjoint is $f^*\colon V \to V^*$ and, because the operator f does not need to be self-adjoint, not putting the asterisk would lead to an ambiguous diagram

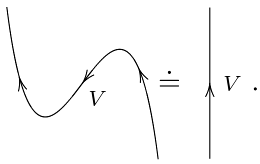


Remark 5.7. The diagrams of i_V and e_V are not dual, since they are obtained from one another by reflection and not by rotation.

Example 5.8. (Rigidity axioms). The rigidity axioms (3.6) and (3.7) take the graphical form

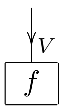


and

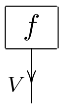


Thus we can equate the left diagrams to the identity on V, resp. V^*, by "straightening out" the diagrams. In the category $\mathcal{V}ec_f(F)$ of finite dimensional vector spaces over F, the first relation means that $\sum_i e_i(f^i, v) = v$, for $v \in V$, $\{e_i\}$ and $\{f^i\}$ bases of V and V^* (V a finite-dimensional vector space).

Example 5.9. (Degenerate cases). The pictures



and



represent two degenerate cases, in the sense that one of the involved objects is $\mathbb{I}$. The first is a representation of a morphism $f\colon V \to \mathbb{I}$ and the second of a morphism $f\colon \mathbb{I} \to V$. Finally, we interpret a morphism f from $\mathbb{I}$ to $\mathbb{I}$ as the diagram

.

5.b Trace and dimension

Let $\mathcal{C}$ be a ribbon category. Denote by $E_{\mathcal{C}}$ the semigroup $\mathrm{End}(\mathbb{I})$ with multiplication induced by the composition of morphisms and with the unit element $1_{\mathbb{I}}$. The semigroup $E_{\mathcal{C}}$ is commutative because for any morphisms $k, k'\colon \mathbb{I} \to \mathbb{I}$, we have $k \circ k' = (k \otimes 1_{\mathbb{I}}) \circ (1_{\mathbb{I}} \otimes k') = k \otimes k' = (1_{\mathbb{I}} \otimes k') \circ (k \otimes 1_{\mathbb{I}}) = k' \circ k$.

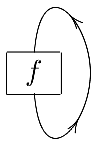
Definition 5.10. Let V be an object in a ribbon category $\mathcal{C}$. For any endomorphism $f\colon V \to V$, we define the *trace* $\mathrm{Tr}(f)\colon \mathbb{I} \to \mathbb{I}$ of f to be the composition:

$$\mathrm{Tr}(f) = e_V \circ c_{V,V^*} \circ (\theta_V f \otimes 1_{V^*}) \circ i_V,$$

i.e., the composition of the morphisms

$$\mathbb{I} \xrightarrow{i_V} V \otimes V^* \xrightarrow{\theta_V f \otimes 1_{V^*}} V \otimes V^* \xrightarrow{c_{V,V^*}} V^* \otimes V \xrightarrow{e_V} \mathbb{I}.$$

Proposition 5.11. ([Tur94, Corollary 2.7.1.]). *Let f be an endomorphism of an object V of $\mathcal{C}$. Let G_f be the ribbon $(0,0)$-graph consisting of one f-colored coupon and one V-colored band, presented by the diagram below. Then $\mathcal{R}(G_f) = \mathrm{Tr}(f)$.*

$G_f =$ 

The main properties of the trace are collected in the following lemma.

Lemma 5.12. ([Tur94, Lemma 1.5.1.]). *For $f\colon V \to W$ and $g\colon W \to V$ in a ribbon category $\mathcal{C}$ we have $\mathrm{Tr}(f \circ g) = \mathrm{Tr}(g \circ f)$. Moreover, $\mathrm{Tr}(f \otimes g) = \mathrm{Tr}(f)\,\mathrm{Tr}(g)$ for any endomorphisms f, g of objects of $\mathcal{C}$ and for any morphism $k\colon \mathbb{I} \to \mathbb{I}$, we have $\mathrm{Tr}(k) = k$.*

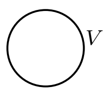
The first claim of this lemma implies the naturality of the trace: for any isomorphism $g\colon W \to V$ and any $f \in \mathrm{End}(V)$, we have $\mathrm{Tr}(g^{-1} \circ f \circ g) = \mathrm{Tr}(f)$. For a graphical proof we refer to [Kas95, Theorem XIV.4.2.].

Definition 5.13. We define the *dimension* of an object V as the trace of the identity morphism 1_V:

$$\mathrm{dim}V = \mathrm{Tr}(1_V) = e_V \circ c_{V,V^*} \circ (\theta_V \otimes 1_{V^*}) \circ i_V.$$

The last lemma implies fundamental properties of the dimension. Isomorphic objects have equal dimensions, since for $f\colon V \to W$ and $g\colon W \to V$ with $g \circ f = 1_V$ and $f \circ g = 1_W$, we have $\mathrm{dim}V = \mathrm{Tr}(1_V) = \mathrm{Tr}(g \circ f) = \mathrm{Tr}(f \circ g) = \mathrm{Tr}(1_W) = \mathrm{dim}W$. For any objects V, W, we have that $\mathrm{dim}(V \otimes W) = \mathrm{Tr}(1_{V \otimes W}) = \mathrm{Tr}(1_V \otimes 1_W) = \mathrm{Tr}(1_V) \otimes \mathrm{Tr}(1_W) = \mathrm{dim}V\mathrm{dim}W$ and $\mathrm{dim}(\mathbb{I}) = \mathrm{Tr}(1_{\mathbb{I}}) = 1_{\mathbb{I}}$.

The dimension of V is depicted by an oriented circle. Since $\mathrm{dim}V = \mathrm{dim}V^*$ (see [Tur94, Corollary 2.7.1.]), the orientation is irrelevant and we represent $\mathrm{dim}V$ by the diagram:

(5.14) $\mathrm{dim}V \doteq$ 

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Chapter II

Graph categorical approach for algebras

In this chapter we associate ribbon categories to a certain type of algebraic structures. This allows to give a categorical description of these structures and to use the diagrammatic calculus introduced in Chapter I. We consider structures defined on a vector space by multilinear identities. Moreover we will always assume the existence of a symmetric bilinear form on the space.

6 Bilinear forms

Let F denote an arbitrary field of characteristic different from 2 and let V be a finite-dimensional vector space over F of dimension d. Let $q\colon V \to F$ be a quadratic form. The *polar* of q is the symmetric bilinear form b defined by

$$b(x,y) = \tfrac{1}{2}\big(q(x+y) - q(x) - q(y)\big).$$

Note that b is the unique symmetric bilinear form satisfying $b(x,x) = q(x)$. We view b as a tensor $b\colon V \otimes V \to F$ and symmetry can be written as the tensor identity $b = b \circ \tau$, where τ denotes the switch $\tau(x \otimes y) = y \otimes x$, see Section 2.a. We assume that the bilinear form b is nonsingular, i.e. for each vector $x \neq 0$ there is a y with $b(x,y) \neq 0$. In other words, b induces an isomorphism

$$\hat{b}\colon V \xrightarrow{\sim} V^* = \mathrm{Hom}_F(V,F),\ x \mapsto \hat{b}_x,$$

defined by $\hat{b}_x(y) = b(x,y)$ for $x, y \in V$.

Let $V^{\otimes 0} = F$ and $V^{\otimes n} = V \otimes \cdots \otimes V$ be the tensor product of n copies of the vector space V. For any linear map $\varphi\colon V^{\otimes n} \to V^{\otimes m}$ let the map $\varphi^*\colon (V^{\otimes m})^* \to (V^{\otimes n})^*$ be its *dual*. By identifying the spaces $\big(V^{\otimes n}\big)^*$ with

$(V^*)^{\otimes n}$ through the canonical map $(V^*)^{\otimes n} \to \big(V^{\otimes n}\big)^*$ given by

$$(g_1 \otimes \cdots \otimes g_n)(v_1 \otimes \cdots \otimes v_n) = g_1(v_1) \cdots g_n(v_n)$$

for $g_i \in V^*$ and $v_i \in V$, one can identify the dual φ^* of φ with a map

$$\varphi^\vee \colon (V^*)^{\otimes m} \to (V^*)^{\otimes n}.$$

Definition 6.1. We denote by $\varphi^t \colon V^{\otimes m} \to V^{\otimes n}$ the linear map given by $(\hat{b}^{\otimes n})^{-1} \circ \varphi^\vee \circ \hat{b}^{\otimes m}$ and call it the *transpose* of φ.

By extending the base field F if necessary, we will assume that the symmetric bilinear form b always admits an orthonormal basis $(e_1, \dots, e_d)$ of V, i.e. $b(e_i, e_j) = \delta_{ij}$ for $1 \le i, j \le d$. We fix $f_i = b(e_i, -)$, $i = 1, \dots, d$, as a dual basis of V^*, so that the isomorphism $\hat{b} \colon V \to V^*$ is given by $e_i \mapsto f_i$.
The bilinear form b can be viewed as a contraction $b \colon V \otimes V \to F$. Its transpose $b^t \colon F \to V \otimes V$ is given by

$$b^t(1_F) = \sum_{i=1}^{d} e_i \otimes e_i.$$

We can extend b to a bilinear form b_k on $V^{\otimes k}$, $k \ge 1$: for $u_i, v_i \in V$ the map $b_k \colon V^{\otimes k} \otimes V^{\otimes k} \to F$ is defined as:

$$b_k\big((u_1 \otimes \cdots \otimes u_k) \otimes (v_1 \otimes \cdots \otimes v_k)\big) = \prod_{i=1}^{k} b(u_i, v_i).$$

The form b_k can be expressed as a composition and tensor product of linear maps from the alphabet $\{1_V,\ b,\ \tau = c_{V,V}\}$. Numbering the i-th copy of V in $V^{\otimes k}$ by i, we have:

$$\begin{aligned} b_k = {} & b \circ (1_k \otimes b \otimes 1_k) \circ \cdots \circ (1_k \otimes \cdots \otimes 1_3 \otimes b \otimes 1_3 \otimes \cdots \otimes 1_k) \\ & \circ (1_k \otimes \cdots \otimes 1_2 \otimes b \otimes 1_2 \otimes \cdots \otimes 1_k) \circ (p_k \otimes 1_1 \otimes \cdots \otimes 1_k) \end{aligned} \tag{6.2}$$

and the claim follows from the fact that the morphism p_k of Example 5.6 admits such an expression. Similarly, we have

$$\begin{aligned} b_k^t = {} & \big((p_k)^{-1} \otimes 1_1 \otimes \cdots \otimes 1_k\big) \circ (1_k \otimes \cdots \otimes 1_2 \otimes b^t \otimes 1_2 \otimes \cdots \otimes 1_k) \\ & \circ (1_k \otimes \cdots \otimes 1_3 \otimes b^t \otimes 1_3 \otimes \cdots \otimes 1_k) \circ \cdots \circ (1_k \otimes b^t \otimes 1_k) \circ b^t, \end{aligned} \tag{6.3}$$

and

$$b_k^t(1_F) = \sum_{i_k=1}^{d} \cdots \sum_{i_2=1}^{d} \sum_{i_1=1}^{d} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \otimes e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}.$$

Remark 6.4. Let $\dim_F V = d$. It easily follows that

$$(b \circ b^t)(1_F) = \sum_{i=1}^{d} b(e_i, e_i) = d,$$

and $(b_k \circ b_k^t)(1_F) = d^k$.

7 The ribbon category $\mathcal{V}$

Let V be a finite-dimensional vector space over a field F of characteristic different from 2 and let $b\colon V^{\otimes 2} \to V$ be a nonsingular symmetric bilinear form. We associate to (V, b) a ribbon category $\mathcal{V}$. The objects of the category $\mathcal{V}$ are all tensor products $V^{\otimes n} = V \otimes \cdots \otimes V$, $n \in \mathbb{N}$, with the convention $V^{\otimes 0} = F$, and the morphisms for two objects $V^{\otimes n}$ and $V^{\otimes m}$ are F-linear maps $f\colon V^{\otimes n} \to V^{\otimes m}$. We also call elements of $\mathrm{Hom}_F(V^{\otimes n}, V^{\otimes m})$, for arbitrary n and m, *tensors*.
A braiding in $\mathcal{V}$ is given by the switch morphism of vector spaces $\tau = \tau_{U,W}\colon U \otimes W \to W \otimes U$, $\tau(u \otimes w) = w \otimes u$. Moreover, the category is symmetric, since $\tau_{W,U} \circ \tau_{U,W} = 1_{U,W}$, and is equipped with the trivial twist $\theta_U = 1_U$. Symmetry of τ leads to $(\tau_{U,W})^{-1} = \tau_{W,U}$. Obviously, the unit object $\mathbb{I}$ is the ground field $F = V^{\otimes 0}$.

We next define the structure of a ribbon category on $\mathcal{V}$. Let $\hat{b}\colon V \xrightarrow{\sim} V^*$ be the isomorphism induced by the bilinear form b (see the previous section). The diagrams

$$\begin{array}{ccc} V^*\otimes V & \xrightarrow{e_V} & F \\ \big\downarrow{\scriptstyle \hat b^{-1}\otimes 1_V} & & \| \\ V\otimes V & \xrightarrow{b} & F\end{array} \qquad\qquad \begin{array}{ccc} F & \xrightarrow{i_V} & V\otimes V^* \\ \| & & \big\downarrow{\scriptstyle 1_V\otimes \hat b^{-1}} \\ F & \xrightarrow{b^t} & V\otimes V \end{array} \tag{7.1}$$

are commutative and the rule $V \mapsto (V, b^t, b)$ satisfies the properties of a duality for the object V, with $V^* = V$, as defined in Definition 3.4. For example, fixing an orthogonal basis $(e_1, \ldots, e_d)$ of V, we have:

$$\begin{aligned} (1_V \otimes b) \circ (b^t \otimes 1_V)(v) &= (1_V \otimes b)\big(\textstyle\sum_i e_i \otimes e_i \otimes v\big) \\ &= \textstyle\sum_i e_i \otimes b(e_i, v) = \sum_i b(v, e_i) e_i = v. \end{aligned}$$

The diagrams (7.1) can be extended with the help of formulas (6.2) and (6.3) to any object $V^{\otimes k}$ of $\mathcal{V}$ and one proves that $V^{\otimes k} \mapsto (V^{\otimes k}, b_k^t, b_k)$ is a duality for the category $\mathcal{V}$. Thus $\mathcal{V}$ is a tensor category equipped with a braiding τ, a twist (the identity morphism) and a compatible duality $(1, b_k^t, b_k)$. In other words, $\mathcal{V}$ is a ribbon category. We say that $\mathcal{V}$ is the *ribbon category generated by the space* (V, b).

The dual of a morphism $f\colon V^{\otimes n} \to V^{\otimes m}$ in $\mathcal{V}$ is the transpose $f^t\colon V^{\otimes m} \to V^{\otimes n}$ as defined in Definition 6.1,

$$f^t = (b_m \otimes 1_{V^{\otimes n}}) \circ (1_{V^{\otimes m}} \otimes f \otimes 1_{V^{\otimes n}}) \circ (1_{V^{\otimes m}} \otimes b_n^t). \tag{7.2}$$

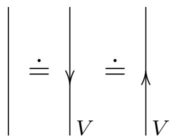
For any object $V^{\otimes k}$ we have $b_k \circ \tau_{V^{\otimes k}, V^{\otimes k}} = b_k$, and the trace of an endomorphism $f\colon V^{\otimes k} \to V^{\otimes k}$ is given by the formula

$$\mathrm{Tr}(f) = b_k \circ (f \otimes 1_{V^{\otimes k}}) \circ b_k^t.$$

Note that $\mathcal{V}$ is not a strict ribbon category, since for example $(U \otimes W) \otimes Z$ is not equal to $U \otimes (W \otimes Z)$. By MacLane's coherence theorem, $\mathcal{V}$ is equivalent to a strict ribbon category (see Remark 3.11). Thus we can apply Theorem 4.13 to $\mathcal{V}$ and use the diagrammatic calculus of Section 5.

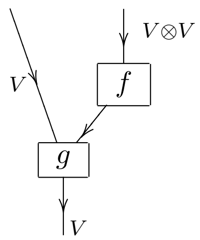
8 The category of ribbon graphs $\mathrm{Rib}_{\mathcal{V}}$

Using the general formalism of Section 4 we associate to $\mathcal{V}$ a category $\mathrm{Rib}_{\mathcal{V}}$ of ribbon graphs. The properties of $\mathcal{V}$ simplify in a nice way the construction of $\mathrm{Rib}_{\mathcal{V}}$. First of all, in the finite sequences representing the objects of $\mathrm{Rib}_{\mathcal{V}}$ the ε_i (determining the directions of the bands) are superfluous, since $V^+ = V \simeq V^* = V^-$. The directions of the bands can also be eliminated since

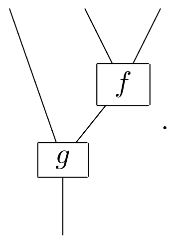


(a graph without arrows means that all arrows are pointing downwards). Similarly the tangles $\cap_V$, $\cap_V^-$ can be represented by the same diagram $\cap$ and the tangles $\cup_V$, $\cup_V^-$ by $\cap$. By convention, we will draw a morphism $f\colon V^{\otimes n} \to V^{\otimes m}$ by representing each V with a distinguished band; in other words, by a coupon labeled with f, with n incoming and m outgoing bands. In this way, we do not need to label the bands of the graph anymore since each band is labeled with V. An object of $\mathrm{Rib}_{\mathcal{V}}$ is then written as a sequence $(V_1, \ldots, V_n)$, where V_i denotes a copy of V.

Example 8.1. For $f\colon V^{\otimes 2} \to V$ and $g\colon V \otimes V \to V$, the diagram expressing



will simply be drawn as



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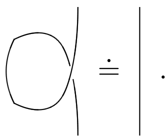
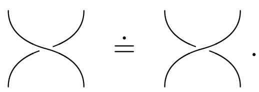
Since $\theta_V = 1_V$, we do not care about twists: this enables us to replace ribbon graphs with graph diagrams (see Remark 5.1). Hence

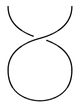


Moreover, the symmetry of the category implies

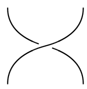
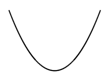
(8.2)



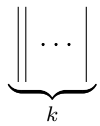
Since $e_V = b$ the tangle $\cup$ represents the bilinear form b and the symmetry of b, $b(x, y) = b(y, x) = b \circ \tau(x, y)$, is depicted as

 $\doteq$  .

The symmetry of b also makes the choice of an alphabet for graphs easier: since labels and arrows have been eliminated, only a small set of tangles remains, namely

(8.3) $I =$, $\tau =$  , $\beta =$  , $\beta^t =$  .

Let $k \in \mathbb{N}$. By $I_k = I \Box \cdots \Box I$ (k times) we understand the diagram

$I_k =$  .

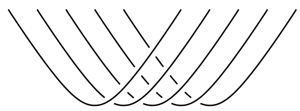
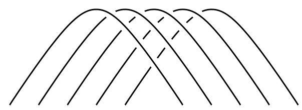
We define the ribbon graphs β_k and β_k^t as the compositions

$$\beta_k \;=\; \beta \circ (I \Box \beta \Box I) \circ \cdots \circ (I_{k-2} \Box \beta \Box I_{k-2}) \circ (I_{k-1} \Box \beta \Box I_{k-1}) \circ (P_k \Box I_k)$$

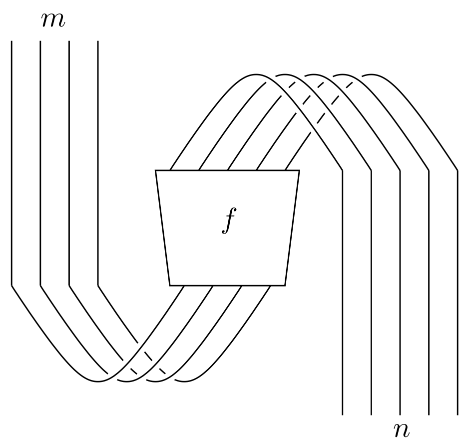
and

$$\beta_k^t = \big((P_k)^{-1} \Box I_k)\big) \circ (I_{k-1} \Box \beta^t \Box I_{k-1}) \circ \cdots \circ (I_2 \Box \beta^t \Box I_2) \circ (I \Box \beta^t \Box I) \circ \beta^t.$$

For example, the pictorial interpretations of β_5 and β_5^t are

$\beta_5 =$  $\beta_5^t =$  .

In $\mathrm{Rib}_{\mathcal{V}}$ the transpose $f^t \colon V^{\otimes m} \to V^{\otimes n}$ of a morphism $f \colon V^{\otimes n} \to V^{\otimes m}$, defined in (7.2), is pictured by



.

Informally, the bottom bands are moved up and the top bands are moved down.

Remark 8.4. Observe that in $\mathrm{Rib}_{\mathcal{V}}$ the transpose of a morphism is graphically depicted by a reflection along the horizontal axis, and not by a rotation of 180°. This would be the case if the n bands colored with V were combined to a unique band with color $V^{\otimes n}$.

9 Updown transformations

The transpose of a ribbon (n, m)-graph is depicted as ribbon (m, n)-graph. Let $m \geq n$. There is another interesting way of inverting the number of bottom and top bands: it consists in "moving" the rightmost $m-n$ bottom bands up to the horizontal line of the top ones. For $\rho \in \mathrm{Hom}_F(V^{\otimes n}, V^{\otimes m})$, the new morphism generated from this move is given by

$$\rho' = (1_{V^{\otimes n}} \otimes b_{m-n}) \circ (\rho \otimes 1_{V^{\otimes(m-n)}}) \in \mathrm{Hom}_F(V^{\otimes m}, V^{\otimes n}).$$

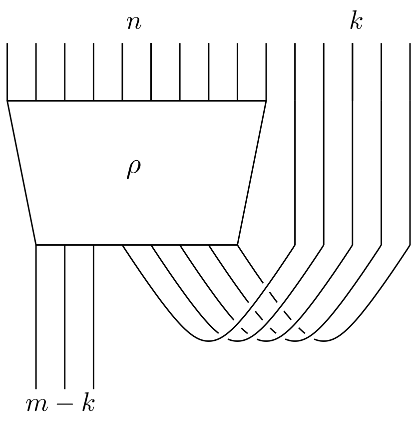
This is a special case of a bigger family of operations in the category $\mathcal{V}$. Let $n, m, k \in \mathbb{N}$ arbitrary and $m \geq k$, then we have a bijection

$$\mathrm{Hom}_F(V^{\otimes n}, V^{\otimes m}) \xrightarrow{\sim} \mathrm{Hom}_F(V^{\otimes(n+k)}, V^{\otimes(m-k)}),\ \rho \mapsto \rho^{[-k]}, \tag{9.1}$$

where $\rho^{[-k]}$ is the composite

$$\underbrace{V \otimes \cdots \otimes V}_{n+k \text{ factors}} \xrightarrow{\rho \otimes 1_{V^{\otimes k}}} \underbrace{V \otimes \cdots \otimes V}_{m+k \text{ factors}} \xrightarrow{1_{V^{\otimes(m-k)}} \otimes b_k} \underbrace{V \otimes \cdots \otimes V}_{m-k \text{ factors}}.$$

The operation $\rho \mapsto \rho^{[-k]}$ can obviously also be realised on the level of the category $\mathrm{Rib}_{\mathcal{V}}$. Pictorially,

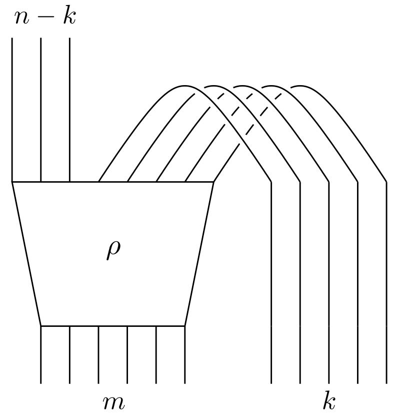
$\rho^{[-k]} \doteq$  .

The isomorphisms (9.1) are a sort of "shift up" of bottom bands on the right. It is also possible to "shift down" the right top bands. Let $k \leq n$ and let $\rho \mapsto \rho^{[k]}$ denote the inverse of $\rho \mapsto \rho^{[-k]}$. Then $\rho^{[0]} = \rho$ and

$$\begin{array}{rcl} \mathrm{Hom}_F(V^{\otimes n}, V^{\otimes m}) & \longrightarrow & \mathrm{Hom}_F(V^{\otimes(n-k)}, V^{\otimes(m+k)}) \\ \rho & \longmapsto & \rho^{[k]} = (\rho \otimes 1_{V^{\otimes k}}) \circ (1_{V^{\otimes(n-k)}} \otimes {b_k}^t) \end{array}$$

is represented by

$\rho^{[k]} \doteq$

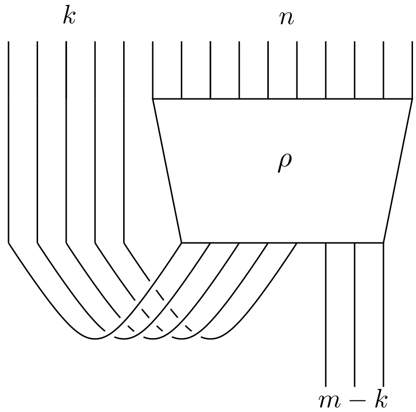


.

In an analogous way it is possible to define operations which “shift up” left bands from the bottom:

$$\begin{array}{ccc} \mathrm{Hom}_F(V^{\otimes n}, V^{\otimes m}) & \longrightarrow & \mathrm{Hom}_F(V^{\otimes(n+k)}, V^{\otimes(m-k)}) \\ \rho & \longmapsto & {}^{[-k]}\rho = (b_k \otimes 1_{V^{\otimes(m-k)}}) \circ (1_{V^{\otimes k}} \otimes \rho). \end{array}$$

${}^{[-k]}\rho \doteq$

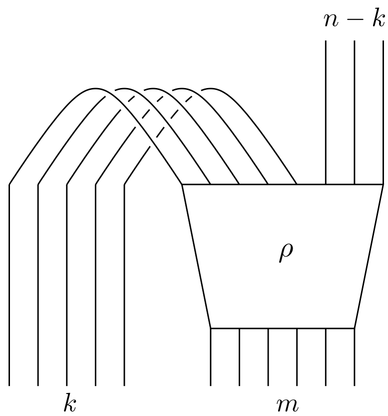


.

The inverse of the previous operation is given by

$$\begin{array}{ccc} \mathrm{Hom}_F(V^{\otimes n}, V^{\otimes m}) & \longrightarrow & \mathrm{Hom}_F(V^{\otimes(n-k)}, V^{\otimes(m+k)}) \\ \rho & \longmapsto & {}^{[k]}\rho = (1_{V^{\otimes k}} \otimes \rho) \circ (b_k^t \otimes 1_{V^{\otimes(n-k)}}). \end{array}$$

${}^{[k]}\rho \doteq$



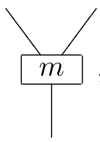
.

We shall refer to these operations in graphs as *updown transformations*. Of particular interest is a composition of these operations which yields to an alternative definition of the transpose of $\rho \in \mathrm{Hom}_F(V^{\otimes n}, V^{\otimes m})$:

$$\rho^t = \ ({}^{[-m]}\rho)^{[n]} = {}^{[-m]}(\rho^{[n]}) \in \mathrm{Hom}_F(V^{\otimes m}, V^{\otimes n}).$$

10 Algebras of tensor type

Definition 10.1. Let F be a field. An *algebra* A is a pair (V, m) where V is a finite-dimensional vector space over F and $m\colon V \otimes V \to V$ is a bilinear multiplication. From now on we assume that F has characteristic different from 2 and that the underlying space V carries a nonsingular symmetric bilinear form b. The multiplication m is then a morphism in the ribbon category $\mathcal{V}$ generated by (V, b), hence it defines a coupon in $\mathrm{Rib}_{\mathcal{V}}$, colored by m:

$\mu =$ 

,

and we may view m as a ribbon $(2, 1)$-graph.

For simplicity, we will denote the algebra by $A = (V, b, m)$ and the product $m(x \otimes y)$ by xy. In most interesting examples the product satisfies a set of tensor identities. In this work we will focus our attention on algebras whose identities can be expressed with the help of the morphisms $1_V, \tau, b, b^t, m$.

Definition 10.2. For any algebra $A = (V, b, m)$, we call *alphabet* the set of tensors

$$Alph = \{1_V, \tau, b, b^t, m\}.$$

Let $\mathfrak{C} = \{c_i \in \mathrm{Hom}_F(V^{\otimes n_i}, V^{\otimes m_i})\}$ be a set of tensors generated through compositions, $\mathbb{Z}$-linear combinations and tensor products of elements of *Alph*. We say that the algebra A is *of tensor type* $\mathfrak{C}$ if the c_i are identities for A, i.e. if $c_i(x_1 \otimes \cdots \otimes x_{n_i}) = 0$ for all i and all $x_k \in V$.

Example 10.3. The vector product in $\mathbb{R}^3$, $m\colon\ x \otimes y \mapsto x \times y$, satisfies the identities

$$\begin{aligned} 0 &= x \times y + y \times x, \\ 0 &= \langle x \times y, z\rangle - \langle x, y \times z\rangle, \\ 0 &= x \times (y \times z) - \langle x, z\rangle y + \langle x, y\rangle z, \end{aligned} \tag{10.4}$$

for x, y, $z \in V$, where $\langle x, y\rangle = b(x, y)$ is the scalar product. The tensors are $m + m\circ\tau$, $b\circ(m\otimes 1_V) - b\circ(1_V\otimes m)$ and $m\circ(1_V\otimes m) - (b\otimes 1_V)\circ(1_V\otimes\tau) + b\otimes 1_V$ in $\mathrm{Hom}_F(V^{\otimes k}, V)$, $k = 2, 3$.

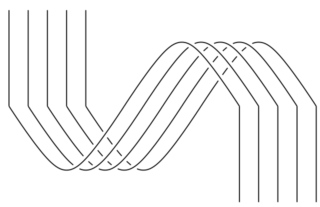
Remark 10.5. Note that the decomposition of a morphism in elements of *Alph* is not unique.

To describe the algebra $A = (V, b, m)$, we do not need the whole category $\mathcal{V}$ since the only morphisms needed are those induced by the alphabet *Alph*. We look for a smaller category endowing A: it appears natural to consider the ribbon subcategory $\mathcal{V}_1$ of $\mathcal{V}$, where $\mathrm{Obj}(\mathcal{V}_1) = \mathrm{Obj}(\mathcal{V})$ and the sets of morphisms are the sets of morphisms generated through composition and tensor product of elements in *Alph*. The category $\mathcal{V}_1$ is not strict, but there is an equivalent strict category in view of MacLane's coherence theorem (cf. Remark 3.11). Hence, we may apply Theorem 4.13 to $\mathcal{V}_1$ without restrictions.

Theorem 10.6. *There exists a unique covariant functor $\mathcal{R} = \mathcal{R}_{\mathcal{V}_1} : \mathrm{Rib}_{\mathcal{V}_1} \to \mathcal{V}_1$ preserving the tensor product and satisfying the following conditions:*

1. *$\mathcal{R}(\emptyset) = F$, $\mathcal{R}\big((V_1, \dots, V_n)\big) = V^{\otimes n}$ (V_i denotes a copy of V);*
2. *$\mathcal{R}(I) = 1_V$, $\mathcal{R}(\tau) = \tau$, $\mathcal{R}(\beta) = b$, $\mathcal{R}(\mu) = m$;*
3. *for any objects ν, ν' or $\mathcal{V}_1$-colored ribbon graphs G, G' of $\mathrm{Rib}_{\mathcal{V}_1}$ we have: $\mathcal{R}(\nu \Box \nu') = \mathcal{R}(\nu) \otimes \mathcal{R}(\nu')$ and $\mathcal{R}(G \Box G') = \mathcal{R}(G) \otimes \mathcal{R}(G')$;*
4. *if G and G' are composable ribbon graphs, we have $\mathcal{R}(G \circ G') = \mathcal{R}(G) \circ \mathcal{R}(G')$;*
5. *$\mathcal{R}(G^t) = \big(\mathcal{R}(G)\big)^t$ for any $\mathcal{V}_1$-colored ribbon (n, m)-graph G, where*
$$G^t = (\beta_m \Box I_n) \circ (I_m \Box G \Box I_n) \circ (I_m \Box \beta_n^t);$$
6. *the functor $\mathcal{R}$ is compatible with the updown transformations $\rho \mapsto {}^{[k]}\rho$ and $\rho \mapsto \rho^{[\ell]}$ defined in Section 9.*

Proof. Items *1.*, *2.*, *3.* and *4.* follow directly from Theorem 4.13. It remains to prove that $\mathcal{R}(G^t) = \big(\mathcal{R}(G)\big)^t$. The fact that $\mathcal{R}(\beta_k) = b_k$ and $\mathcal{R}(\beta_k^t) = b_k^t$ follows trivially from the definitions (6.2) and (6.3). The picture



shows that I_k^t is isotopic to I_k, hence $\mathcal{R}(I_k^t) = \mathcal{R}(I_k) = \mathcal{R}(I) \otimes \cdots \otimes \mathcal{R}(I) = (1_V)^{\otimes k} = 1_{V^{\otimes k}}$. Finally,

$$\begin{aligned}\mathcal{R}(G^t) &= \mathcal{R}\big((\beta_m \Box I_n) \circ (I_m \Box G \Box I_n) \circ (I_m \Box \beta_n^t)\big)\\ &= \big(\mathcal{R}(\beta_m) \otimes \mathcal{R}(I_n)\big) \circ \big(\mathcal{R}(I_m) \otimes \mathcal{R}(G) \otimes \mathcal{R}(I_n)\big) \circ \big(\mathcal{R}(I_m) \otimes \mathcal{R}(\beta_n^t)\big)\\ &= (b_m \otimes 1_{V^{\otimes n}}) \circ (1_{V^{\otimes m}} \otimes \mathcal{R}(G) \otimes 1_{V^{\otimes n}}) \circ (1_{V^{\otimes m}} \otimes b_n^t) = \big(\mathcal{R}(G)\big)^t.\end{aligned}$$

□

Definition 10.7. We call the set of ribbon graphs $Alph^* = \{I, \tau, \beta, \beta^t, \mu\}$ the alphabet of *basic ribbon graphs*. We have $\mathcal{R}(Alph^*) = Alph$.

11 Categories associated to tensor relations

The categories $\mathcal{V}$ and $\mathcal{V}_1$ are *pre-additive* categories. This means that for any pair of objects U, W the set $\mathrm{Hom}_F(U, W)$ is an additive abelian group and the composition of morphisms and tensor product are $\mathbb{Z}$-bilinear, i.e. for all $f, g, h \in \mathcal{V}$:

$$(f+g)\circ h = f\circ h + g\circ h, \qquad f\circ(g+h) = f\circ g + f\circ h,$$

$$(f+g)\otimes h = f\otimes h + g\otimes h, \qquad f\otimes(g+h) = f\otimes g + f\otimes h.$$

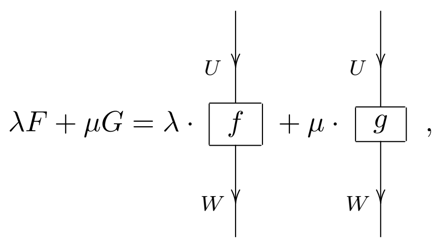
For $k \in \mathbb{Z}$ we have

$$(kf)\circ g = (k\otimes f)\circ(1_F\otimes g) = k\otimes(f\circ g) = k(f\circ g)$$

and similarly $f\circ(kg) = k(f\circ g)$.

However, in the category $\mathrm{Rib}_{\mathcal{V}_1}$, the sets $\mathrm{Mor}_{\mathrm{Rib}_{\mathcal{V}_1}}(X, Y)$ do not possess an additive structure. Let R be a fixed associative and commutative ring with 1, which admits a homomorphism $h\colon R \to F$. The ring R will be called the *coefficient ring*. In most applications we choose $R = \mathbb{Q}$ for simplicity, assuming that the base field F has characteristic zero.
We associate to the category $\mathrm{Rib}_{\mathcal{V}_1}$ its *R-linearization* $\mathrm{Rib}^{+}_{\mathcal{V}_1}$. This category has the same objects as $\mathrm{Rib}_{\mathcal{V}_1}$ and for two objects X, Y in $\mathrm{Rib}_{\mathcal{V}_1}$, $\mathrm{Mor}_{\mathrm{Rib}^{+}_{\mathcal{V}_1}}(X, Y)$ is the free R-module $R^{\mathrm{Mor}_{\mathrm{Rib}_{\mathcal{V}_1}}(X,Y)}$ generated by the set $\mathrm{Mor}_{\mathrm{Rib}_{\mathcal{V}_1}}(X, Y)$. Thus a morphism of $\mathrm{Rib}^{+}_{\mathcal{V}_1}$ is a formal R-linear combination of ribbon graphs of a fixed type (m, n). We also call such a linear combination a ribbon (m, n)-graph.

Diagrammatically, we draw an R-linear combination of graphs in $\mathrm{Rib}^{+}_{\mathcal{V}_1}$ as



where f, g morphisms of $\mathcal{V}_1$ and $\lambda, \mu \in R$.

For $F = \sum_i a_i F_i$ and $G = \sum_j b_j G_j$, $1 \le i, j < \infty$, $a_i, b_j \in R$, F_i ribbon (n, m)-graphs and G_j ribbon (m, q)-graphs in $\mathrm{Rib}_{\mathcal{V}_1}$ we define the composition in $\mathrm{Rib}^{+}_{\mathcal{V}_1}$ to be the ribbon (n, q)-graph

$$G\circ F = \sum_{i,j} a_i b_j (G_j \circ F_i).$$

Moreover, the tensor product extends by linearity:

$$F \square H = \sum_{i,j} a_i b_j (F_i \square H_j)$$

for $F_i, H_j \in \mathrm{Rib}_{\mathcal{V}_1}$. The identity for the composition is $1_F \cdot I_k$ and for the tensor product $\square$ the identity is $1_F \cdot \emptyset$. Moreover the updown transformations defined in Section 9 clearly also extends to $\mathrm{Rib}^+_{\mathcal{V}_1}$. Composition and tensor product coincide on $\mathrm{Mor}_{\mathrm{Rib}^+_{\mathcal{V}_1}}(\emptyset, \emptyset)$ and endow $\mathrm{Mor}_{\mathrm{Rib}^+_{\mathcal{V}_1}}(\emptyset, \emptyset)$ with the structure of a commutative R-algebra with 1.

We can extend Theorem 10.6 by R-linearity to the category $\mathrm{Rib}^+_{\mathcal{V}_1}$:

Theorem 11.1. *There exists a unique functor* $\mathcal{R}^+ \colon \mathrm{Rib}^+_{\mathcal{V}_1} \to \mathcal{V}_1$ *such that*

1. $\mathcal{R}^+ = \mathcal{R}$ *in* $\mathrm{Rib}_{\mathcal{V}_1}$*;*
2. $\mathcal{R}^+(G) = \sum_i h(a_i)\mathcal{R}(G_i)$ *for* $G = \sum_i a_i G_i$*,* $a_i \in R$*,* $G_i \in \mathrm{Rib}_{\mathcal{V}_1}$*;*
3. $\mathcal{R}^+(F \square G) = \mathcal{R}^+(F) \otimes \mathcal{R}^+(G)$*, and* $\mathcal{R}^+(F \circ G) = \mathcal{R}^+(F) \circ \mathcal{R}^+(G)$ *for* $F, G \in \mathrm{Rib}^+_{\mathcal{V}_1}$*;*
4. $\mathcal{R}^+$ *is compatible with updown transformations.*

Let now A be an F-algebra of tensor type, defined by a set of tensors $\mathfrak{C} = \{c_i \in \mathrm{Hom}_F(V^{\otimes n_i}, V^{\otimes m_i})\}$. Recall that the tensors c_i are generated by elements of *Alph* through compositions, $\mathbb{Z}$-linear combinations and tensor products. The category $\mathrm{Rib}^+_{\mathcal{V}_1}$ does not reflect the identities of A. The next step is to impose relations in $\mathrm{Rib}^+_{\mathcal{V}_1}$, in order to obtain the "right" category associated with A.
Let

$$\Gamma = \{\gamma_i \in \mathrm{Mor}_{\mathrm{Rib}^+_{\mathcal{V}_1}}(V^{\otimes n_i}, V^{\otimes m_i})\}$$

be a set of morphisms in $\mathrm{Rib}^+_{\mathcal{V}_1}$ such that $\mathcal{R}^+(\gamma_i) = c_i$ for all $c_i \in \mathfrak{C}$. The elements γ_i are represented by ribbon (n_i, m_i)-graphs and Γ represents pictorially the relations defining the algebra A. The set Γ generates an *ideal* I of the category $\mathrm{Rib}^+_{\mathcal{V}_1}$. For each pair of objects (X, Y) of $\mathrm{Rib}^+_{\mathcal{V}_1}$, this ideal is given by the R-submodule $I(X, Y)$ of $\mathrm{Mor}_{\mathrm{Rib}^+_{\mathcal{V}_1}}(X, Y)$ generated through compositions and tensor products by the γ_i. Let Rib_Γ be the *factor category* $\mathrm{Rib}^+_{\mathcal{V}_1}/I$. By definition we have $\mathrm{Obj}(\mathrm{Rib}_\Gamma) = \mathrm{Obj}(\mathrm{Rib}_{\mathcal{V}_1})$ and

$$\mathrm{Mor}_{\mathrm{Rib}_\Gamma}(X, Y) = \mathrm{Mor}_{\mathrm{Rib}^+_{\mathcal{V}_1}}(X, Y)/I(X, Y).$$

For any $\rho \in \mathrm{Mor}_{\mathrm{Rib}^+_{\mathcal{V}_1}}(X, Y)$, let $[\rho]$ be its class in $\mathrm{Mor}_{\mathrm{Rib}_\Gamma}(X, Y)$. We get a projection functor

$$\mathcal{P} \colon \mathrm{Rib}^+_{\mathcal{V}_1} \to \mathrm{Rib}_\Gamma$$

such that $\mathcal{P}(X) = X$ for $X \in \mathrm{Obj}(\mathrm{Rib}^+_{\mathcal{V}_1})$ and $\mathcal{P}(\rho) = [\rho]$ for ρ any morphism. In particular we have $\mathcal{P}(\gamma_i) = 0$ for each $\gamma_i \in \Gamma$.

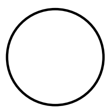
The category Rib_Γ is a category of ribbon graphs. We say that the γ_i are *relations defining* Rib_Γ and we call Rib_Γ the *ribbon graph category associated with the set of relations* Γ. Classes of ribbon graphs in Rib_Γ will also be called ribbon graphs. The R-algebra $\mathrm{Mor}_{\mathrm{Rib}^+_{\mathcal{V}_1}}(\emptyset, \emptyset)$ projects onto $E_\Gamma = \mathrm{Mor}_{\mathrm{Rib}_\Gamma}(\emptyset, \emptyset)$. We say that E_Γ is the *ring of numerical invariants* of the category Rib_Γ, and we call the elements of E_Γ *closed graphs*. The computation of E_Γ for a series of algebras of tensor type is the main feature of this dissertation.

An immediate consequence of Theorem 11.1 is:

Theorem 11.2. *Let $A = (V, b, m)$ be an algebra of tensor type $\mathfrak{C}$ for a set of tensors $\mathfrak{C} = \{c_i\}$. Let $\Gamma = \{\gamma_i\}$ be a set of relations in $\mathrm{Rib}^+_{\mathcal{V}_1}$ such that $\mathcal{R}^+(\gamma_i) = c_i$ for all $c_i \in \mathfrak{C}$, and let Rib_Γ be the corresponding ribbon category of type Γ. Then there is a unique functor $\mathcal{R}_\Gamma \colon \mathrm{Rib}_\Gamma \to \mathcal{V}$ of ribbon categories such that $\mathcal{R}^+ = \mathcal{R}_\Gamma \circ \mathcal{P}$. Moreover $\mathcal{R}_\Gamma$ is compatible with updown transformations.*

For ribbon graphs F, G, we write from now on $F \doteq G$ if $\mathcal{R}_\Gamma(F) = \mathcal{R}_\Gamma(G)$.

Example 11.3. The simplest nontrivial element of E_Γ is the circle, denoted by d:

$$d = \beta \circ \beta^t =$$

.

Since

$$\mathcal{R}_\Gamma(\beta \circ \beta^t)(1_F) = (b \circ b^t)(1_F) = \sum_i b(e_i, e_i) = \dim_F V,$$

the image under $\mathcal{R}_\Gamma$ is the dimension of V; for this reason it is convenient to use the same symbol d for both circle and dimension.

Corollary 11.4. *A necessary condition for the existence of an algebra of tensor type $\mathfrak{C} = \{c_i\}$, is the existence of an R-algebra homomorphism $\Phi \colon E_\Gamma \to F$, where $\Gamma = \{\gamma_i\}$ is a set of relations in $\mathrm{Rib}^+_{\mathcal{V}_1}$ such that $\mathcal{R}^+(\gamma_i) = c_i$ for all $c_i \in \mathfrak{C}$.*

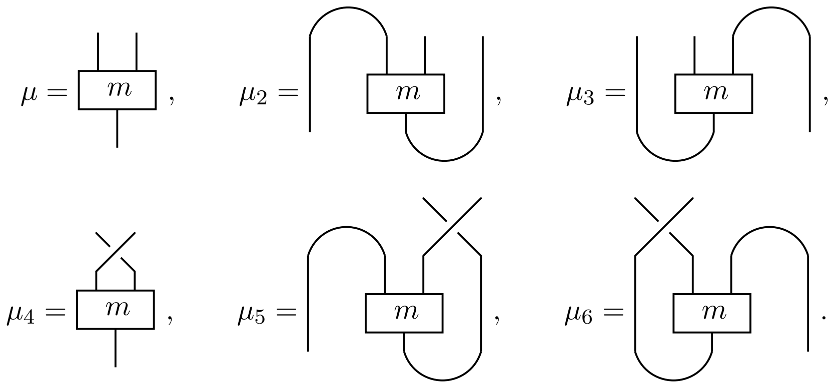
Proof. If such an algebra exists, the functor $\mathcal{R}_\Gamma$ induces a homomorphism $E_\Gamma = \mathrm{Mor}_{\mathrm{Rib}_\Gamma}(\emptyset, \emptyset) \to \mathrm{Hom}_F(F, F) = F$. □

Remark 11.5. Let $A = (V, b, m)$ be an algebra of tensor type. The category Rib_Γ associated to A is the same as the one attached to $A_L = A \otimes L$ for any field extension L of F. In some sense Rib_Γ is a "universal category" reflecting the tensorial structure of A.

12 Associative bilinear forms

Applying updown transformations and twist to the ribbon $(2,1)$-graph μ representing the multiplication of the algebra, we get a set of six ribbon $(2,1)$-graphs, $\mu_1 = \mu$, $\mu_2 = {}^{[1]}\mu^{[-1]}$, $\mu_3 = {}^{[-1]}\mu^{[1]}$, $\mu_4 = \mu_1 \circ \tau$, $\mu_5 = \mu_2 \circ \tau$ and $\mu_6 = \mu_3 \circ \tau$, which corresponds to the set of permutations of the three bands connected with the coupon colored by m:

(12.1)



These six ribbon graphs are not isotopic. However there are cases where we can identify some of these graphs in the category Rib_Γ.

Definition 12.2. Let $A = (V, b, m)$ be an F-algebra of tensor type. We say that the bilinear form b is *associative with respect to* m or is a *trace form* if

$$b(xy, z) = b(x, yz) \tag{12.3}$$

for all $x, y, z \in V$, equivalently, if the trilinear form

$$f(x, y, z) := b(xy, z) \tag{12.4}$$

is invariant under a cyclic permutation.

Let $m_1 = m$, $m_2 = {}^{[1]}m^{[-1]}$, $m_3 = {}^{[-1]}m^{[1]}$, $m_4 = m_1 \circ \tau$, $m_5 = m_2 \circ \tau$ and $m_6 = m_3 \circ \tau \in \mathrm{Hom}(V \otimes V, V)$, with the notations of Section 9.

Lemma 12.5. *If the bilinear form b is associative with respect to m we have $m_1 = m_2 = m_3$ and $m_4 = m_5 = m_6$.*

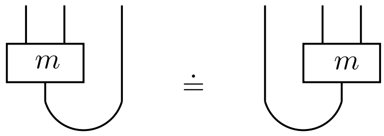
Proof. We only check $m_1 = m_2$. We have

$$\begin{aligned} {}^{[1]}m^{[-1]}(x \otimes y) &= (1_V \otimes b) \circ (1_V \otimes m \otimes 1_V) \circ (b^t \otimes 1_{V^{\otimes 2}})(x \otimes y) \\ &= (1_V \otimes b) \circ (1_V \otimes m \otimes 1_V)(\textstyle\sum_i e_i \otimes e_i \otimes x \otimes y) \\ &= (1_V \otimes b)(\textstyle\sum_i e_i \otimes e_i x \otimes y) \\ &= \textstyle\sum_i e_i \otimes b(e_i x, y) = \sum_i e_i \otimes b(e_i, xy) = xy. \end{aligned}$$

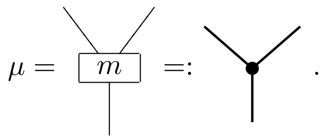
□

If (12.3) holds for A, it is a tensor identity and we assume that this identity belongs to the set $\mathfrak{C}$ of tensors identities defining the algebra A. Then the associativity of b is graphically represented by the relation

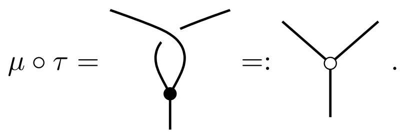
(12.6)



and Lemma 12.5 implies that $\mu_1 \doteq \mu_2 \doteq \mu_3$ and $\mu_4 \doteq \mu_5 \doteq \mu_6$. Observe that the two graphs in (12.6) are not isotopic, even if they are equal under $\mathcal{R}_\Gamma$. The same is true for the three graphs μ_1, μ_2, μ_3, resp. μ_4, μ_5, μ_6. This suggests to introduce a new notation for μ: we omit the coupon in the graph and replace it with a black point, as in the figure

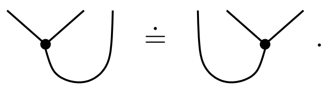


Thus the multiplication m is now represented by a black trivalent vertex which we still call μ. We use a white trivalent vertex to represent $m \circ \tau$:

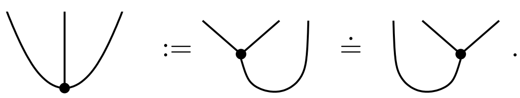


The associativity of the bilinear form b is depicted by

(12.7)



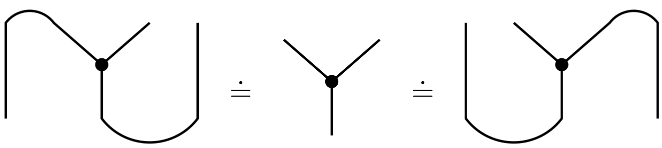
The two graphs now are isotopic; we can deform them to obtain a third isotopic graph:



This reflects the fact that the trilinear form

$$f(x_1, x_2, x_3) := b(x_1 x_2, x_3)$$

is invariant under cyclic permutations of the x_i. Moreover the equalities $\mu_2 \doteq \mu_1 \doteq \mu_3$ mean that



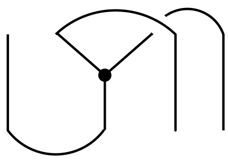
and we see that the graphs μ_2 and μ_3 are just rotations of the graph μ. In particular they are isotopic to μ. The same holds for $\mu_4 = \mu \circ \tau$, μ_5 and μ_6. We fix a clockwise (or positive) orientation on black trivalent vertices and, accordingly, a counterclockwise (or negative) orientation on white trivalent vertices. Then μ corresponds to a positive oriented trivalent vertex and $\mu \circ \tau$ to a negative oriented one. The replacement of the six ribbon $(2,1)$-graphs μ_i by two oriented trivalent vertices is a huge simplification and will be used in an essential way in the following chapters.

We conclude this section with some more properties of algebras with associative bilinear forms.

Example 12.8. Let $A = (V, b, m)$ be an algebra where b is a trace form. The image under $\mathcal{R}_\Gamma$ of the ribbon $(1,2)$-graph

$$\mu^t = (\beta \Box I_2) \circ (I \Box \mu \Box I_2) \circ (I \Box \beta_2^t),$$

depicted as

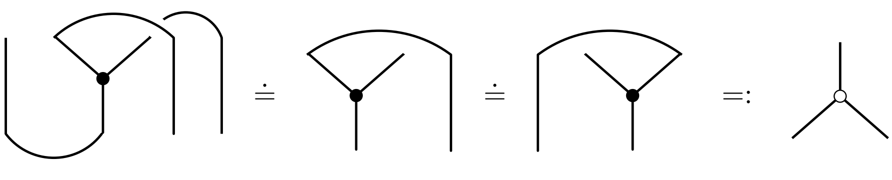


,

is the morphism $m^t \colon V \to V \otimes V$. Note that transposing changes the orientation of the trivalent vertex (μ^t is the result of a reflection of μ along the top line of the graph). There is a simple expression for $m^t \colon V \to V \otimes V$ (only valid for trace forms):

$$\begin{aligned} m^t(v) &= (b \otimes 1_{V^{\otimes 2}}) \circ (1_V \otimes m \otimes 1_{V^{\otimes 2}}) \circ (1_V \otimes b_2^t)(v) \\ &= (b \otimes 1_{V^{\otimes 2}}) \circ (1_V \otimes m \otimes 1_{V^{\otimes 2}}) \sum_{i,j} v \otimes e_i \otimes e_j \otimes e_i \otimes e_j \\ &= (b \otimes 1_{V^{\otimes 2}}) \sum_{i,j} v \otimes e_i e_j \otimes e_i \otimes e_j \\ &= \sum_{i,j} b(v, e_i e_j) e_i \otimes e_j = \sum_{i,j} b(e_j v, e_i) e_i \otimes e_j \\ &= \sum_j e_j v \otimes e_j. \end{aligned}$$

Hence, m^t corresponds to the morphism $(m \otimes 1_V) \circ (1_V \otimes \tau) \circ (b^t \otimes 1_V)$ for b associative. Similarly, modifying the last two steps in the above computation, we get the equivalent expression $m^t(v) = \sum_j e_j \otimes v e_j$. The graphical incarnations of these formulas are:



.

One can associate a *Casimir element* to any trace form:

Lemma 12.9. *Let $A = (V, b, m)$ be an algebra with a trace form b and let $(e_1, \dots, e_d)$ be an orthonormal basis of V. The* Casimir element

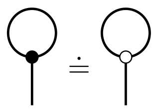
$$c = \sum_i e_i e_i \in V$$

is uniquely determined by the property $\mathrm{Tr}(l_x) = b(c, x)$*, where* $l_x \colon V \to V$ *is the linear map* $l_x(y) = xy$*. In particular, it is independent of the choice of the orthonormal basis.*

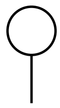
Proof. (cf. [CKR05, Lemma 1.5.]). Uniqueness follows from the fact that b is nonsingular. Let $xe_i = \sum_j x_{ij} e_j$, so that $\mathrm{Tr}(l_x) = \sum_i x_{ii}$. We have

$$\sum_i b(x, e_i e_i) = \sum_i b(xe_i, e_i) = \sum_{i,j} x_{ij} b(e_j, e_i) = \sum_i x_{ii}. \qquad \square$$

Tensorially, we have $m \circ b^t = m \circ \tau \circ b^t$ and both tensors map 1_F to the Casimir element $c = \sum_i e_i e_i$. Graphically c is represented by the *Casimir graph*

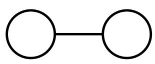
(12.10) $\mu \circ \beta^t =$  .

Hence the orientation of the trivalent vertex is superfluous in the Casimir graph, which will be depicted by

 .

If reversing the orientations at a specific trivalent vertex does not change the graph, we will systematically omit to mark the vertex.

Composing the Casimir graph with itself, we get the graph

$e =$  ,

which is another interesting element of the ring E_Γ. Its image under $\mathcal{R}_\Gamma$ is the scalar $b(c, c)$, where c is the Casimir element in V.

We conclude this section by a trivial example:

Example 12.11. Let (V, b, m) be an algebra with $m = 0$. Then $\Gamma = \{\mu\}$ and the circle d generates E_Γ. The ring E_Γ is isomorphic to the polynomial ring $R[d]$ and $\mathcal{R}_\Gamma(d) = \dim_F V$. Thus there is no obstruction to the existence of such an algebra in any dimension d (which is, of course, obvious!)

and by a less trivial one:

Example 12.12. The classical vector product $(\mathbb{R}^3, \times)$ satisfies the set of relations (10.4). In particular the bilinear form given by the scalar product is associative with respect to the product $\times$. With the techniques described in the next chapter, it can be shown that for the corresponding category Rib_Γ of relations, with $R = \mathbb{Q}$,

$$E_\Gamma \xrightarrow{\sim} \mathbb{Q}[d]/d(d-1)(d-3)$$

and it follows from Corollary 11.4 that a nontrivial vector product satisfying the identities (10.4) can only exist in dimension 3.

Chapter III

Vector product algebras

In this chapter we apply the formalism of Chapter II to vector products.

13 r-Vector products

Let V be a d-dimensional vector space over a field F of characteristic zero and let b be a nondegenerate, symmetric bilinear form on V.

Definition 13.1. An *r-vector product* in (V, b) is an r-linear map

$$P_r\colon V^r = V \times V \times \cdots \times V \longrightarrow V, \qquad 1 \le r \le d,$$

satisfying the following properties:

$$b\big(P_r(v_1, \ldots, v_r), v_i\big) = 0 \quad \text{for } v_i \in V,\ i = 1, \ldots, r \tag{13.2}$$

$$b\big(P_r(v_1, \ldots, v_r), P_r(v_1, \ldots, v_r)\big) = \det\big(b(v_i, v_j)\big)_{i,j} \tag{13.3}$$

It follows by linearizing (13.2) that the multilinear form

$$f(v_1, \ldots, v_r, v_{r+1}) = b\big(P_r(v_1, \ldots, v_r), v_{r+1}\big)$$

is alternating. Since the bilinear form b is nondegenerate the r-vector product is also alternating. In 1943 Eckmann defined a r-vector product on a d-dimensional real vector space V to be a continuous map $P_r\colon V^r \to V$ satisfying axioms (13.2) and (13.3) and proved with topological techniques that a vector product exists in precisely the following cases:

- d is even, $r = 1$,
- d is arbitrary, $r = d - 1$,
- $d = 3$ or 7, $r = 2$,
- $d = 4$ or 8, $r = 3$,

(see Eckmann [Eck43], Whitehead [Whi63]). The classification of multilinear vector products over arbitrary fields is given by Brown and Gray in [BG67]. Explicit formulas for these vector products are known in all cases; for further details we refer to the articles [Zve66], [Sha89], [Eck91] and [Eld96].

14 Vector product algebras

Definition 14.1. A *vector product algebra* $VPA = (V, b, \times)$ consists of a vector space V together with a nondegenerate symmetric bilinear form b on V and a 2-vector product P_2 in (V, b). We denote the product $P_2(x, y)$ with $x \times y$ and call it simply a vector product.

Setting $r = 2$ in (13.2) and (13.3), we see that the identities defining a vector product algebra are:

$$b(x \times y, x) = b(x \times y, y) = 0, \tag{14.2}$$

$$b(x \times y, x \times y) = b(x, x)b(y, y) - b(x, y)^2, \tag{14.3}$$

for all $x, y \in V$.

For later use we need another set of defining relations:

Lemma 14.4. *The set of identities* (14.2) *and* (14.3)*, defining a vector product algebra VPA, is equivalent to the set:*

$$b(x \times y, z) = b(x, y \times z) \quad \textit{and} \quad x \times y = -y \times x, \tag{14.5}$$

$$(x \times y) \times z + x \times (y \times z) = 2b(x, z)y - b(x, y)z - b(y, z)x, \tag{14.6}$$

for all $x, y, z \in VPA$. In particular, a vector product algebra admits an associative bilinear form b with respect to the 2-vector product $\times$.

Proof. Condition (14.2) is equivalent to $f(x, y, x) = b(x \times y, z)$ being alternating and this, in turn, is equivalent to (14.5). Linearizing (14.3) we get

$$b(x \times y, x \times z) = b(x, x)b(y, z) - b(x, y)b(x, z),$$

and the associativity of b yields to

$$b\big((x \times y) \times x, z\big) = b\big(b(x, x)y, z\big) - b\big(b(x, y)x, z\big).$$

Since b is nonsingular we have

$$(x \times y) \times x = b(x, x)y - b(x, y)x$$

and linearizing again gives (14.6). □

As a special case of the possible dimensions of r-vector products listed above we see that a vector product can only exist in dimension 3 or 7:

Theorem 14.7. *The dimension d of a vector product algebra satisfies the equation $d(d-1)(d-3)(d-7) = 0$.*

15 Vector product algebras and composition algebras

Vector algebras are related with composition algebras and Theorem 14.7 is a consequence of the classification of composition algebras with identity. Let (V, m) be an algebra, i.e., V is a finite dimensional vector space over a field F and $m\colon V \otimes V \to V$, $x \otimes y \mapsto x \cdot y$, is a F-bilinear multiplication.

Definition 15.1. A *composition algebra with identity* $C = (V, \cdot)$ is an algebra over a field F with an identity element 1_C and a nondegenerate quadratic form $q\colon V \to F$ such that:

$$q(x \cdot y) = q(x)q(y) \quad \text{for all } x, y \in V. \tag{15.2}$$

We say that a quadratic form satisfying (15.2) is *multiplicative*.

In characteristic different from 2, the notions of composition algebra with identity and that of vector product algebra are equivalent. Given any composition algebra C with identity, one can construct a vector product algebra VPA by taking $V = \langle 1_C \rangle^{\perp}$ and defining a vector product by

$$x \times y = \tfrac{1}{2}(x \cdot y - y \cdot x). \tag{15.3}$$

Conversely, given a vector product algebra $VPA = (V, b, \times)$, we get a composition algebra by putting $C = \langle 1_C \rangle \perp V$ and defining the product on C by

$$(a1_C + x) \cdot (b1_C + y) = \big(ab - b(x, y)\big)1_C + ay + bx + x \times y, \tag{15.4}$$

for $a, b \in F$, $x, y \in V$. In particular the dimension of a vector product algebra VPA associated to a composition algebra C with identity is related to the dimension of C by the formula

$$\dim_F VPA = \dim_F C - 1.$$

Thus Theorem 14.7 is an immediate consequence of the following structure theorem for composition algebras:

Theorem 15.5. (Hurwitz Theorem). *The following is a complete list of the composition algebras with identity over a field F: (I) F, (II) quadratic étale algebras, (III) quaternion algebras, (IV) Cayley algebras. In particular, they only exist in dimension* $1, 2, 4$ *and* 8.

Theorem 15.5 was proved by Hurwitz in 1898 for sums of squares over $\mathbb{R}$. A proof for composition algebras over arbitrary fields can be found in Jacobson [Jac58], or in [KMRT98, Chapter VIII].

Remark 15.6. A ribbon graph category can also be attached to composition algebras with identity and a direct diagrammatic proof of the fact that composition algebras with identity can only exist in dimension $1, 2, 4$ and 8 will be given in Chapter VII.

16 A diagrammatic proof of Theorem 14.7

Inspired by diagrammatic techniques, Markus Rost gave in 1996 a tensor categorical proof of Theorem 14.7 (see [Ros96], and also the two diploma theses of students of M. Rost, [Mau98] and [Boo98]). In 2002 Meyberg gave a modified version of the proof of Rost's dimension relation $d(d-1)(d-3)(d-7) = 0$ for vector product algebras using trace formulas ([Mey02]). In this section we present the diagrammatic proof of M. Rost given in [Ros95], using the formalism of Chapter II.

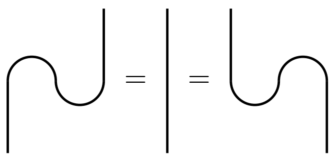
A vector product algebra $(V, b, \times)$ is an of algebra of tensor type with defining relations $\mathfrak{C} = \{c_1, c_2, c_3\}$, where:

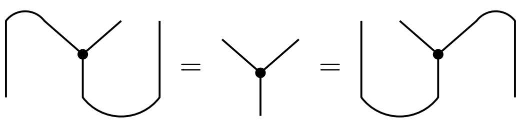
1. $c_1 = b \circ (m \otimes 1_V) - b \circ (1_V \otimes m)$,
2. $c_2 = m + m \circ \tau$,
3. $c_3 = m \circ (m \otimes 1_V) + m \circ (1_V \otimes m) - 2(b \otimes 1_V) \circ (1_V \otimes \tau) + b \otimes 1_V + 1_V \otimes b$.

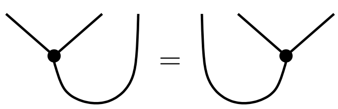
With the notations of Chapter II, we choose $\mathbb{Q}$ as ring of coefficients for the category $\mathrm{Rib}^+_{\mathcal{V}_1}$. If $\gamma_1, \gamma_2, \gamma_3$ in $\mathrm{Rib}^+_{\mathcal{V}_1}$ are ribbon graphs representing c_1, c_2 and c_3, we denote Rib_{VPA} the corresponding category of type $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. The functor $\mathcal{R}_\Gamma \colon \mathrm{Rib}_\Gamma \to \mathcal{V}$, where $\mathcal{V}$ is the tensor category generated by the vector space V (see Theorem 11.2), will be denoted $\mathcal{R}_{VPA}$. In view of the first relation, the bilinear form b is associative with respect to the product $\times$ and we are in the situation described in Section 12. In particular we shall represent the ribbon graph associated with the multiplication by an oriented trivalent vertex.

To simplify notations, we will denote equalities of graphs representing the same morphisms with the equality sign $=$, and no longer with $\doteq$, as in Chapter II.

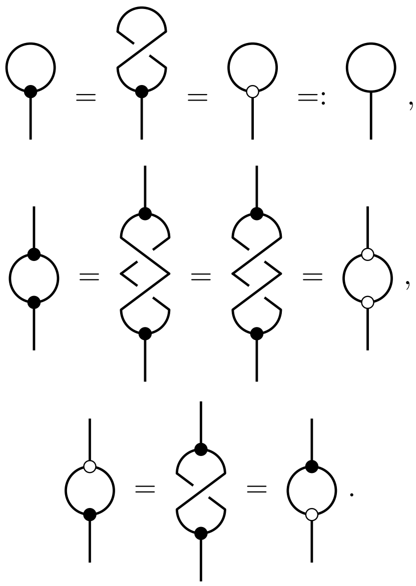
We have some basic identities in Rib_{VPA}:

 ,

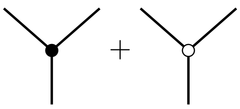
 ,

 ,

(16.1)

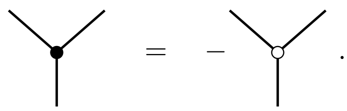


The first set is the pictorial version of the rigidity axioms, the second and the third follow from the associativity of b and have already been discussed in Section 12. The ribbon graph γ_2 is pictured by

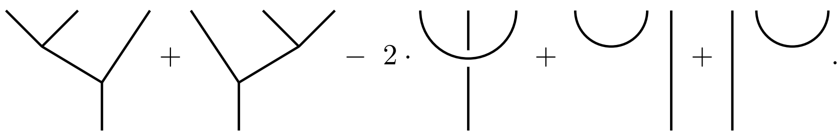
$\gamma_2 =$ 

and we have the relation

(16.2)

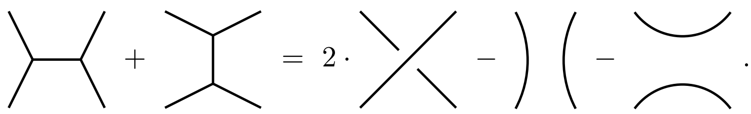


Relation (16.2) can be used to replace white trivalent vertices by black ones with a negative sign. Since we then have only black marks we can as well omit the mark and γ_3 is represented by

$\gamma_3 =$ 

Using updown transformations, we can shift down the leftmost band at the top of γ_3 and get the relation:

(16.3)



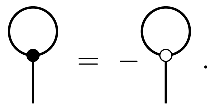
Let $E_{VPA} = \mathrm{Mor}_{\mathrm{Rib}_{VPA}}(\emptyset, \emptyset)$ be the ring of numerical invariants of the category Rib_{VPA}. The following result is due to M. Rost ([Ros95]):

Theorem 16.4. *Let J be the ideal of the polynomial ring $\mathbb{Q}[\bar{d}]$, generated by the element $p(\bar{d}) = \bar{d}(\bar{d}-1)(\bar{d}-3)(\bar{d}-7)$. Then the homomorphism $\Phi\colon \mathbb{Q}[\bar{d}] \to E_{VPA} = \mathrm{Mor}_{Rib_{VPA}}(\emptyset, \emptyset)$ given by $\Phi(\bar{d}) = d$, induces an isomorphism of rings*

$$\bar{\Phi}\colon \mathbb{Q}[\bar{d}]/J \xrightarrow{\sim} E_{VPA}.$$

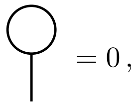
In particular the only possible values of the invariant d are 0, 1, 3 and 7.

Proof. Quoted identities in this work are indexed by a double index $(x.y)$, $x, y \in \mathbb{N}$. For G a ribbon graph, we use the notation $(x.y) \circ G$ to denote the identity $(x.y)$ composed with G. For example $(16.2) \circ \beta^t$ gives the identity



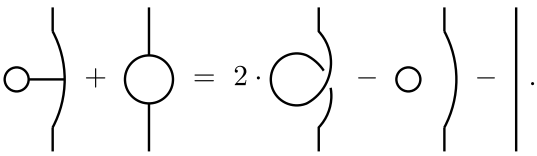
Together with (16.1) we deduce

(16.5)



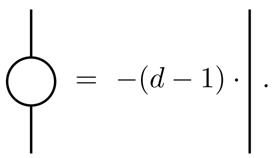
since, by assumption, 2 is invertible. It follows from (16.5) that any graph which contains a graph of type (16.5) is zero.

Next consider $(\beta\Box I) \circ \big(I\Box(16.3)\big) \circ (\beta^t\Box I)$:



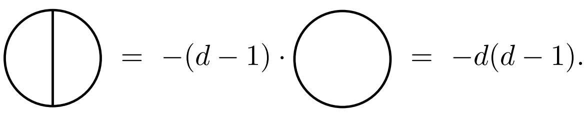
By the previous remark the first graph in the left sum is zero. On the right side the first graph is isotopic to a line, and the second one is the disjoint union of the circle d with a line. Hence

(16.6)

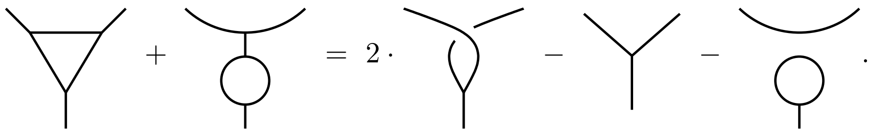


Gluing the top and the bottom bands together, that is $\beta \circ \big(I\Box(16.6)\big) \circ \beta^t$, yields immediately

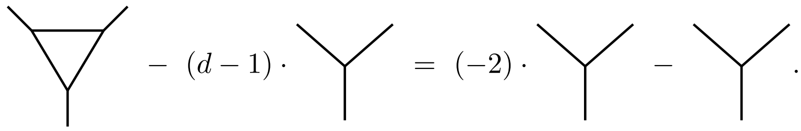
(16.7)



Relation $\mu \circ$ (16.3) gives

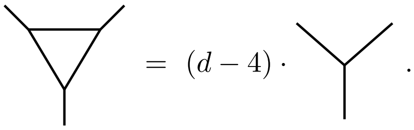


The last graph on the right is zero; the second on the left can be simplified by using (16.6), and the first graph on the right changes sign because of (16.2):



Then

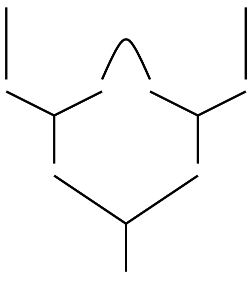
(16.8)



Remark 16.9. (Springer's Formula). The functor $\mathcal{R}_{VPA}$ associates to any diagrammatic identity a tensor identity. Identity (16.8) translates to the identity

$$\sum_{i=1}^{d}(u \times e_i) \times (e_i \times v) = (d-4)(u \times v)$$

due to Springer and which is valid in any vector product algebra. In fact, one can decompose the graph on the left as composition and disjoint union of elements in *Alph*:



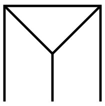
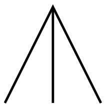
$$= \mu \circ (\mu \Box \mu) \circ (I \Box \beta^t \Box I).$$

We apply the functor $\mathcal{R}_{VPA}$ on both sides of equation (16.8). On the left we get

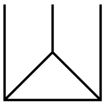
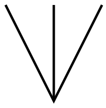
$$\begin{aligned}\mathcal{R}_{VPA}\big(\mu \circ (\mu\Box\mu) \circ (I\Box\beta^t\Box I)\big)(u,v) &= \sum_{i=1}^{d} \mathcal{R}_{VPA}\big(\mu \circ (\mu\Box\mu)\big)(u \otimes e_i \otimes e_i \otimes v)\\ &= \sum_{i=1}^{d}(u \times e_i) \times (e_i \times v),\end{aligned}$$

on the right $\mathcal{R}_{VPA}\big((d-4)\mu\big)(u,v) = (d-4)u \times v$.

The graphs in Springer's formula can be drawn as ribbon $(0,3)$-graphs via updown transformations:

(16.10)  $= (d-4)\cdot$ 

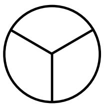
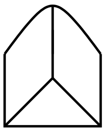
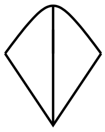
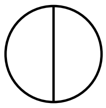
and also as ribbon $(3,0)$-graphs:

(16.11)  $= (d-4)\cdot$  .

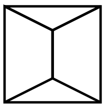
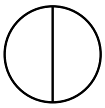
Let $\gamma = (I\Box\mu\Box I)\circ(\beta^t\Box\beta^t)$ be the graph

 .

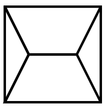
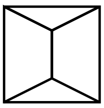
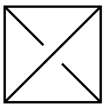
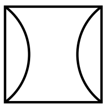
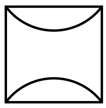
We close the three top bands together by gluing (16.11) with γ:

 $=$  $= (d-4)\cdot$  $= (d-4)\cdot$  .

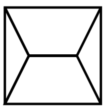
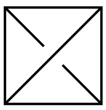
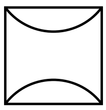
Composing (16.11) and (16.10), we find

(16.12)  $= (d-4)^2\cdot$  $\overset{(16.7)}{=} -d(d-1)(d-4)^2.$

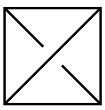
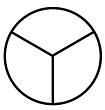
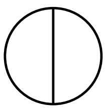
Now comes the final step. We compose $\beta\circ(\mu\Box\mu)\circ(I\Box(16.3)\Box I)\circ(\mu^t\Box\mu^t)\circ\beta^t$, which gives

 $+$  $= 2\cdot$  $-$  $-$  .

Note that μ^t is obtained by reflection along the horizontal axis, hence the orientation of the trivalent vertex is modified (see Example 12.8). This implies a change of sign. Since μ^t appears two times, the sign of the expression does not change. The two leftmost and the two rightmost graphs are isotopic. So, after dividing by 2, we have

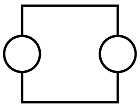
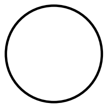
 $=$  $-$  .

For the middle term one finds

(16.13)  $=$  $= (d-4)\cdot$  $= -d(d-1)(d-4).$

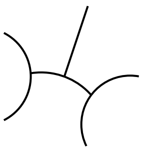
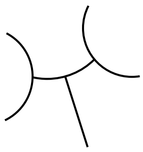
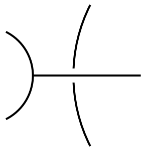
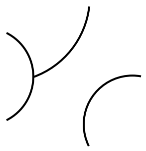
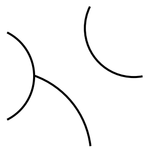
The first equality holds since both pictures are just different projections of the same graphs but with signs changed at two vertices; the other two equalities follow from (16.8) and (16.7).

To compute the rightmost graph, one applies formula (16.6) twice and finds

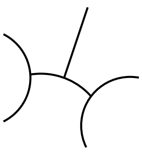
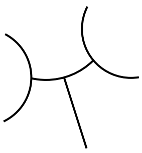
 $= (d-1)^2 \cdot$  $= d(d-1)^2.$

This formula, (16.12) and (16.13) give the relation $d(d-1)(d-3)(d-7) = 0$ in E_{VPA}. We next prove that the ring E_{VPA} is generated by d.

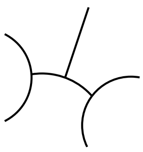
Performing (16.3) $\circ$ $(\mu \square I)$ we obtain the relation:

 $+$  $= 2 \cdot$  $-$  $-$  .

Hence

 $= -$  + sum of graphs with one trivalent vertex.

The two graphs differ by a sign and a rotation of order 5. Therefore, iterating 5 times yields

(16.14) $2 \cdot$  = sum of graphs with one trivalent vertex.

Consequently, any graph with three trivalent vertices can be replaced by a sum of graphs with only one trivalent vertex. In general one can reduce the number of trivalent vertices on a connected closed graph step by step by 2. Since closed graphs can only have an even number of trivalent vertices, they can all be reduced to sums and products of closed graphs without trivalent points, i.e., represented by the circle d. This observation, together with the computations of the previous section, yields to the conclusion that, for $k \in \mathbb{N}$, $k > 0$, any closed connected graph with, at most, $2k-2$ trivalent vertices can be expressed by a polynomial in d of degree $\leq k$. In particular E_{VPA} is generated by d. Thus we have a surjective homomorphism

$$\bar{\Phi} \colon \mathbb{Q}[\bar{d}]/J \to E_{VPA}.$$

By Corollary 11.4 this already implies that only the dimensions 0, 1, 3 and 7 are possible for nontrivial scalar products. The fact that $\bar{\Phi}$ is an isomorphism follows from the fact that all dimensions can be realized. $\square$

Chapter IV

Symmetric composition algebras

Symmetric composition algebras are a special class of composition algebras without identity. They were independently investigated by Petersson [Pet69], Okubo [Oku78], Faulkner [Fau88]. A complete classification was given by Elduque-Myung [EM93], see also Elduque-Pérez [EP96] and [KMRT98, Chapter VIII]. Symmetric compositions form a class of algebras of tensor type for which the techniques of Chapter II nicely work. The idea to apply diagrammatic calculus to symmetric compositions is due to M. Rost. We assume in all this chapter that F is a field of characteristic zero.

17 Invariants of symmetric composition algebras

Definition 17.1. A *symmetric composition algebra* is an algebra $A = (V, b, m)$ with a nonsingular associative bilinear form b such that for all $x, y, z \in V$:

$$q(x \star y) = q(x)q(y), \tag{17.2}$$

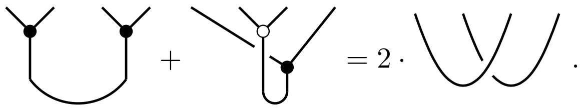
$$b(x \star y, z) = b(x, y \star z), \tag{17.3}$$

where $m(x, y) = x \star y$ and $q(x) = b(x, x)$.

Linearizing the first relation, we get the equivalent tensor identity

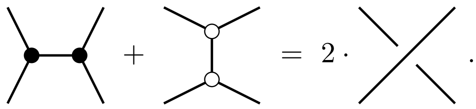
$$b(x \star y, z \star t) + b(x \star t, z \star y) = 2b(x, z)b(y, t) \tag{17.4}$$

for all $x, y, z, t \in V$. Thus symmetric compositions algebras are algebras of tensor type, with the two relations (17.3) and (17.4), and we can associate to the data $A = (V, b, m)$ a category Rib_{SCA} as in Section 11. We choose $R = \mathbb{Q}$ as coefficient ring. The identity (17.3) means that b is associative with respect to the multiplication $\star$. Thus we may replace the ribbon $(2,1)$-graph associated with the multiplication m by an oriented trivalent vertex, as in Section 12. The relation in Rib_{SCA} representing (17.4) is pictured by the graph



Applying the updown transformation $\rho \mapsto {}^{[1]}\big(\rho^{[1]}\big)$ to the above graph yields:

(17.5)



Let $E_{SCA} = \mathrm{Mor}_{\mathrm{Rib}_{SCA}}(\emptyset, \emptyset)$ be the ring of numerical invariants of the category Rib_{SCA}. As in Chapter II we denote

Theorem 17.6. *Let J be the ideal of the polynomial ring $\mathbb{Q}[\bar{d}, \bar{e}]$, generated by the elements*

(17.7)
$$p_1 = \bar{e}\big(\bar{e} - (2-\bar{d})^2\big),$$
$$p_2 = (\bar{d}-2)(\bar{d}-8)(\bar{d}-\bar{e}).$$

Then the homomorphism $\Phi\colon \mathbb{Q}[\bar{d}, \bar{e}] \to E_{SCA} = \mathrm{Mor}_{Rib_{SCA}}(\emptyset, \emptyset)$ given by $\Phi(\bar{d}) = d$, $\Phi(\bar{e}) = e$, induces a surjective homomorphism of rings

$$\bar{\Phi}\colon \mathbb{Q}[\bar{d}, \bar{e}]/J \to E_{SCA}.$$

In particular the only possible values of the pair (d,e) of invariants are

(17.8)
$$(d,e) = (0,0),\ (1,1),\ (2,0),\ (4,4),\ (8,0) \ \textit{and}\ (8,36).$$

Remark 17.9. We do not know whether the homomorphism $\bar{\Phi}$ in Theorem 17.6 is injective. There is an isomorphism

$$\mathbb{Q}[\bar{d}, \bar{e}]/J \ \simeq\ \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}[\epsilon]/(\epsilon^2) \times \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$$

given by

$$\bar{d} \mapsto (0,1,2,4,8,8), \qquad \bar{e} \mapsto (0,1,\epsilon,4,0,36).$$

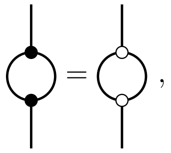
On the other hand, since the 6 pairs in (17.8) can be realized by a symmetric composition algebra over $\mathbb{Q}$ (see Proposition 17.34), there is a homomorphism $E_{SCA} \to \mathbb{Q}^6$ with

$$d \mapsto (0,1,2,4,8,8), \qquad e \mapsto (0,1,0,4,0,36).$$

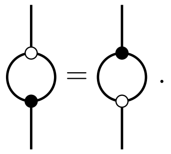
It follows that the kernel of $\bar{\Phi}$ is contained in the 1-dimensional radical of $\mathbb{Q}[\bar{d}, \bar{e}]/J$.

Proof. The proof of Theorem 17.6 is due to M. Rost (see [CKR05, pp. 46-58]). We keep the convention that if in a graph no orientation is indicated we understand that the graph does not depend on the choice of the orientation. We recall that the following identities are valid in $\mathrm{Rib}_{\mathcal{V}_1}$ (see page 51):

(17.10)

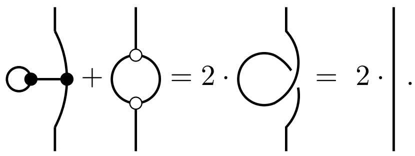


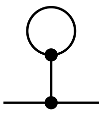
(17.11)



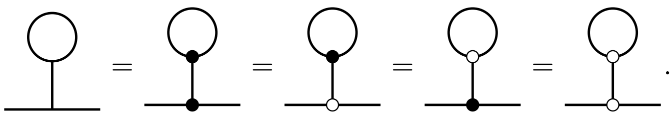
We first consider the following consequence of (17.5):

(17.12)



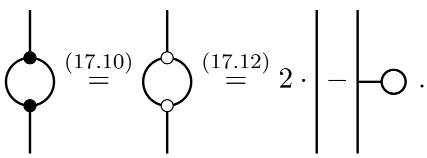
This and (17.10) show that  is invariant under a reflection. Together with the identity (12.10), this means that in such a graph the orientation at the two vertices is nonessential. Therefore, we may disregard them and write

(17.13)

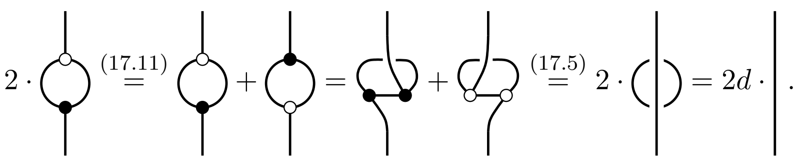


We have by (17.10) and (17.12)

(17.14)

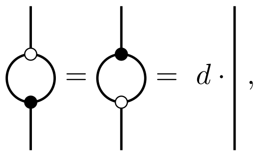


Identities (17.11) and (17.5) show



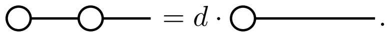
Hence we have

(17.15)

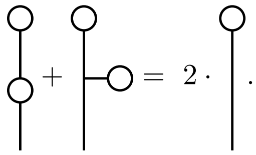


which implies

(17.16)

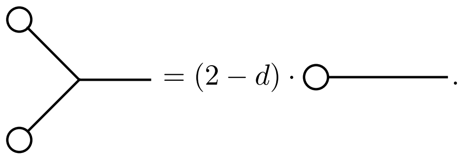


By (17.13) and (17.15) this equation is valid for any choice of orientation. Similarly, relation (17.14) implies

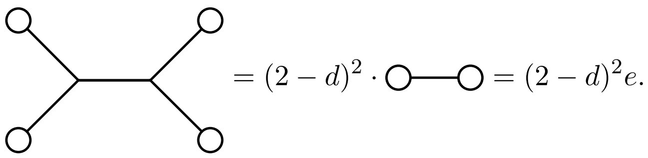


By (17.16) this gives

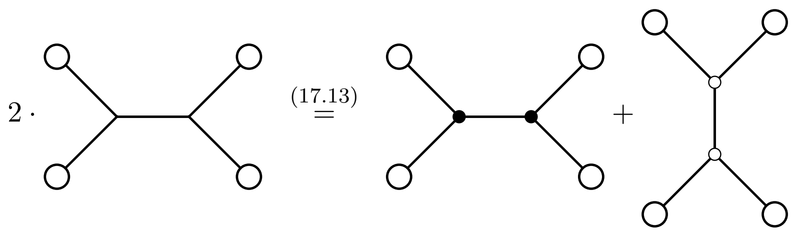
(17.17)

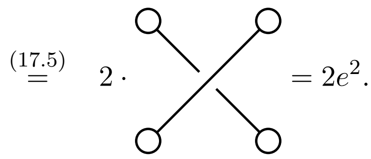


Composing both sides with each other yields



In the next computation, the second equation follows from (17.5):

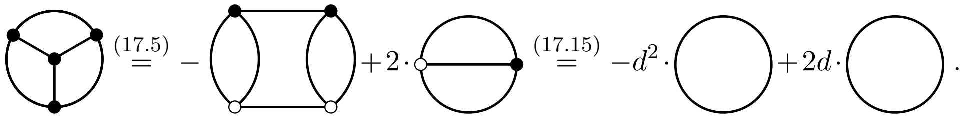




We conclude that $(2-d)^2 e = e^2$, and we obtain the first relation of Theorem 17.6. Now transforming (17.5) into

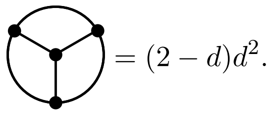
$$\beta \circ (I \Box \beta \Box I) \circ (I_2 \Box \beta \Box I_2) \circ \big((17.5) \Box \mu \Box \mu \Box I_2\big) \circ (I \Box \beta^t \Box I_2 \Box \beta^t \Box I) \circ (\beta^t \Box \beta^t)$$

gives



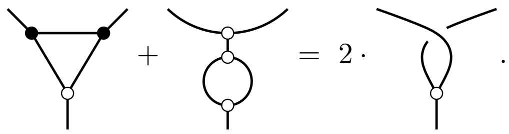
Hence

(17.18)

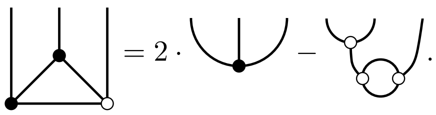


A further application of (17.5) gives

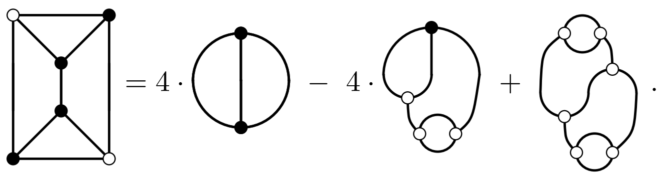
(17.19)



Hence

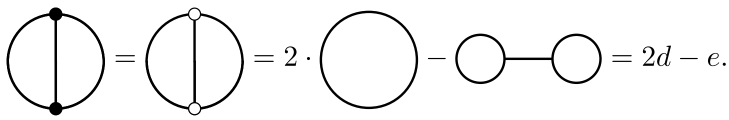


Composing this relation with itself shows

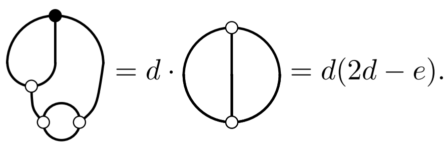


We compute the three terms on the right. Relation (17.14) gives

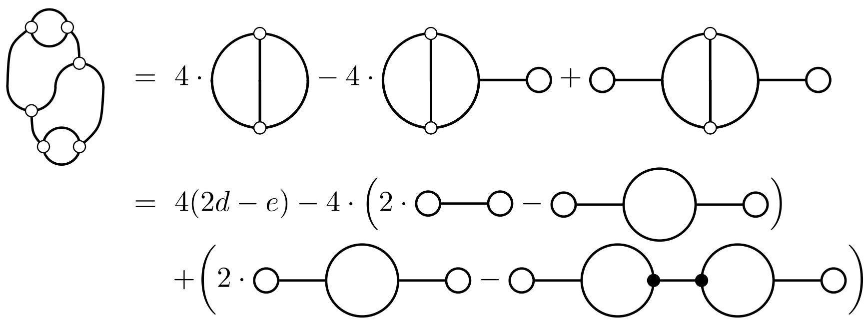
(17.20)



Then (17.15) and the last computation give

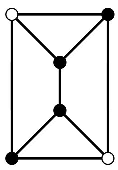


Applying (17.14) twice for the first equation, then (17.20), (17.14), (17.5) for the second equation and finally (17.16) for the third equation gives



$$= 4(2d-e) - 4(2e-de) + (2de - d^2e)$$
$$= 4(2d-e) + (2-d)(-4e+de).$$

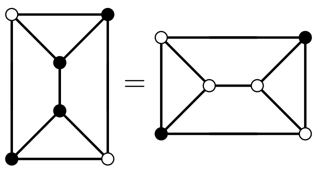
Thus far we have computed



$$= 4(2d-e) - 4d(2d-e) + 4(2d-e) + (2-d)(-4e+de)$$
$$= (2-d)(8d-8e+de).$$

The right hand side is invariant under orientation reversing. Hence

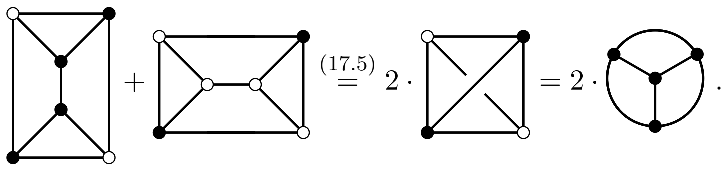
(17.21)



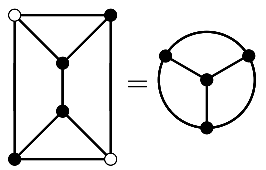
$$= (2-d)(8d-8e+de).$$

Next, observe that

(17.22)



For the second equality note that the two graphs are the same, they are just drawn in a different way. Relations (17.18), (17.21) and (17.22) give after dividing by 2



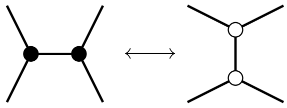
$$= (2-d)d^2.$$

Putting all together:

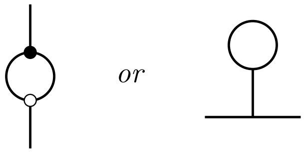
$$0 = (2-d)(8d-8e+de) - (2-d)d^2 = (d-2)(d-8)(d-e).$$

This is the second desired equality. To complete the proof of Theorem 17.6 we check that the two graphs d and e generate the ring of numerical invariants E_{SCA}.

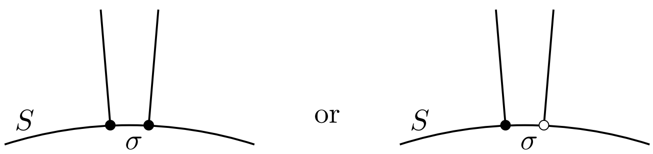
Definition 17.23. Two graphs are called *equivalent* if one can be obtained from the other by a sequence of moves



Proposition 17.24. *For any graph with at least one trivalent vertex there is an equivalent one which contains one of the portions*

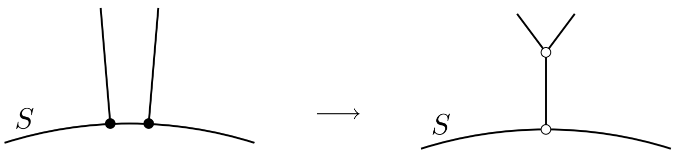


Proof. In a graph Ω we call *path* a graph with trivalent vertices $\{v_0, \ldots, v_k\}$ and edges (bands) connecting v_0 with v_1, v_1 with v_2, …, v_{k-1} with v_k, where the v_i are pairwise different, with the possible exception of $v_0 = v_k$. In the case $v_0 = v_k$ the path is closed and is called *cycle* in Ω. By length(S) we mean the number of trivalent vertices on the cycle S.
Fix some direction on S. Then at each vertex P we have an incoming edge, an outgoing edge and the third edge not belonging to S. This triple determines an orientation at the vertex P which may be different from the given orientation in P. We now divide the vertices on the cycle S into two classes: the set of vertices for which this new orientation coincides with the given orientation of the trivalent vertex and the others. If two vertices belong to the same class, we say they have the same orientation with respect to S.
We choose an isotopic graph to Ω which contains a cycle S of minimal length. If length$(S) = 1$, the proposition is clear. Suppose length$(S) \geq 2$. For an edge σ of S the two endpoints have either the same orientation or not:



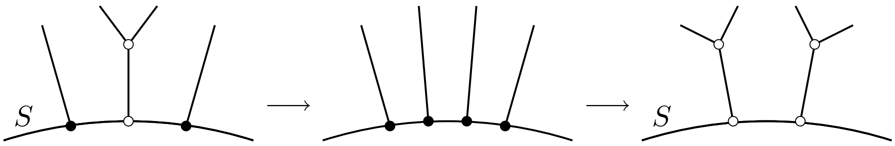
In the second case we say that σ is an alternating edge. In the first case we could make a move

(17.25)



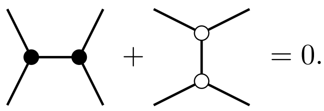
and find a cycle of smaller length in a graph equivalent to Ω. Since length(S) is minimal, any edge of S is alternating. Suppose that the endpoints P, Q

of an edge σ of $\Omega \setminus S$ lie both on S. Then the union of σ with one of the arcs on S between P and Q would have a smaller length than S, except when each of these arcs have length 1. But then $\operatorname{length}(S) = 2$. The last two conclusions show that in case $\operatorname{length}(S) \geq 3$ there is a portion as indicated by the first of the following pictures:



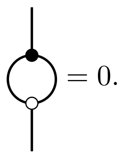
Now the two indicated moves would again give rise to a cycle of smaller length. So we know $\operatorname{length}(S) \leq 2$ and in case $\operatorname{length}(S) = 2$ the edges of S are alternating. □

We next consider graphs modulo the relation

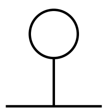
(17.26) 

Proposition 17.27. *The ring of closed graphs modulo the relation* (17.26) *is generated by the circle d and the closed graph e.*

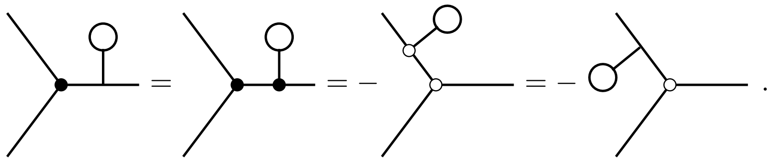
Proof. Relation (17.26) has the following consequence (see the proof of (17.15)):

(17.28) 

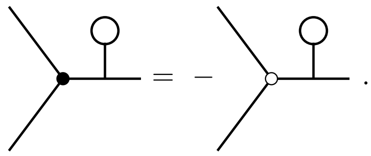
Let Ω be a closed graph. We must show that Ω is a polynomial in d and e. We may assume that Ω is connected and argue by induction on the number of vertices in Ω. By Proposition 17.24 Ω is equivalent to a graph Ω' containing one of the portions of Proposition 17.24. In the first case we may use (17.28) and reduce the number of vertices. Let us consider the second case, i.e., Ω contains a portion

$\eta =$  .

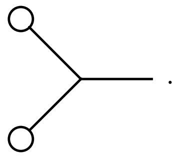
We may assume that Ω has more than two vertices because otherwise $\Omega = e$. Relation (17.26) gives

(17.29) 

Iterating this relation three times yields

(17.30) 

It follows that if we change the orientation of a vertex, we merely change the sign of Ω. This is clear if the vertex is near η as in (17.30). However, in general we may use (17.29) to move η to the vertex, then change the orientation and move back again using (17.29). If we are not concerned with orientations, then it is easy to use (17.26) to produce a further cycle of length 1. So Ω is equivalent to Ω', where Ω' contains a portion

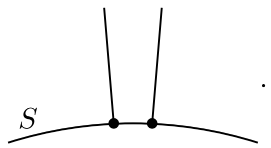


But this is equal to $(2-d)\cdot$ , see (17.17). $\square$

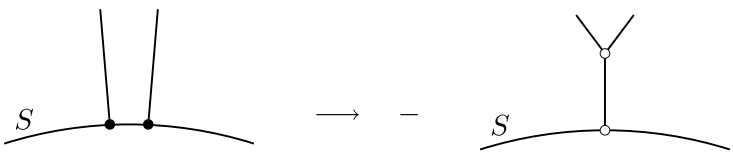
Theorem 17.6 follows now from the next corollary and Corollary 11.4. $\square$

Corollary 17.31. *The ring E_{SCA} of closed graphs is generated by d and e.*

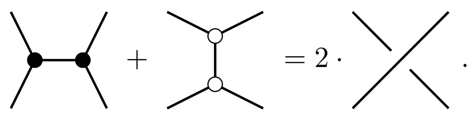
Proof. The claim follows from a filtration argument, where filtration is by the number of vertices. More precisely, consider for example the case of a cycle S in the closed graph Ω containing two black vertices:

(17.32) 

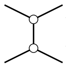
Instead of relation (17.26) which leads to steps



to reduce the length of S, we use the relation

(17.33) 

We claim that for S with minimal length, $\text{length}(S) \leq 2$. Assume that $\text{length}(S) \geq 3$. Let T be any cycle in Ω, different from S, but connected with

S through the two marked black vertices in (17.32). We have length$(T) \geq$ length(S). For each move induced by (17.33), T is replaced by a linear combination of a cycle of length length$(T) - 1$ (corresponding to the term ) and a cycle of length length$(T) +$ length$(S) - 4$ (corresponding to the term ). If length$(T) =$ length(S) we get in any case shorter cycles. If length$(T) >$ length(S), S is replaced by a linear combination of cycles which contains a shorter cycle. □

We show that all values in Theorem 17.6 can be realized.

Proposition 17.34. *Let F be an algebraically closed field of characteristic different from 2 and 3. For each value $(0, 0)$, $(1, 1)$, $(2, 0)$, $(4, 4)$, $(8, 0)$ and $(8, 36)$ of the pair (d, e), there exists up to isomorphisms exactly one symmetric composition algebra with the given value.*

Proof. Symmetric composition algebras are of two types (see [KMRT98, 34.A.], [OO81] or [EM93]). The first one is related to quadratic composition algebras with an identity element (also called *Hurwitz algebras*), which, by Hurwitz Theorem, occur in dimension 1, 2, 4 and 8. Over an algebraically closed field, Hurwitz algebras are split, and there is only one type up to isomorphism for each possible dimension. The polar form of the norm is not associative but satisfies identities of the form

$$b(xy, z) = b(x, z\bar{y}), \tag{17.35}$$

where $x \mapsto \bar{x}$ is the conjugation map. It readily follows from (17.35) that, for the new multiplication $x \star y = \bar{x}\bar{y}$, the bilinear form is associative. Thus we can associate to every Hurwitz algebra a symmetric composition algebra, which is usually called a *Para-Hurwitz algebra*. To compute the Casimir element c we may assume that the standard basis $(1, i, j, \dots)$ of quadratic extensions, resp. quaternions or octonions, is so normalized that $i^2 = j^2 = \dots = -1$. Then the standard basis is orthonormal. We get

$$c = 1 + (d - 1)(-1) = (2 - d).$$

Thus $e = b(c, c) = (d - 2)^2$ and the numerical invariants for Para-Hurwitz algebras are $(d, e) = (1, 1)$, $(2, 0)$, $(4, 4)$ and $(8, 36)$.

The other type of symmetric compositions is related to separable associative cubic algebras and was first considered by Okubo (see [Oku78] or [KMRT98, 34.C. and 34.D.]). These composition algebras are called *Okubo algebras*. We have three classes of cubic separable and associative algebras A over F: F, cubic étale F-algebras and central simple F-algebras of degree 3. We denote the multiplication of A by $(x, y) \mapsto xy$. Let

$$p_x(X) = X^3 - T(x)X^2 + S(x)X - N(x) \cdot 1$$

be the generic polynomial of A. The linear form T is the trace, S is a quadratic form and N is cubic. Let $A^0 = \{x \in A \mid T(x) = 0\}$ be the vector space of trace zero elements in A. Let $q = -\frac{1}{3}S$ and let b be the polar of the quadratic form q. Let ω be a primitive cubic root of 1. The bilinear form b is associative with respect to the multiplication defined on A^0 by $(x, y) \mapsto x \star y$ with

$$x \star y = \tfrac{1-\omega}{3}xy + \tfrac{2+\omega}{3}yx - \tfrac{1}{3}T(xy) \cdot 1 \tag{17.36}$$

and $q(x \star y) = q(x)q(y)$ for all $x \in A^0$ (see for example [KMRT98, (34.18)]). Thus $(A^0, q, \star)$ is a symmetric composition algebra. One verifies that in all three cases the Casimir element is trivial (see next sections), so that symmetric composition algebras associated with cubic algebras have invariants $(d, e) = (0, 0)$, $(2, 0)$ and $(8, 0)$. For algebraically closed fields, each class corresponds up to isomorphism to a unique separable associative algebra. The two symmetric compositions algebras in dimension 2 (and for which $e = 0$) are isomorphic over an algebraically closed field. In contrast, we have two types of symmetric compositions in dimension 8 which are never isomorphic, since they have different e-invariant. □

Remark 17.37. In dimension 8, one class of symmetric composition algebras ("type" G_2) is associated with octonions, and the second ("type" A_2) is associated with central simple algebras of degree 3. The existence of these two types is related to the existence of two non conjugate trialitarian actions of S_3 on Spin_8 (see [KMRT98, (35.35)] or [Hel01, Chap. X, Exercise E.2]). Observe that relation (17.7) has two branches, one ($e = 0$) corresponding to type A_2 and the other one ($e = (d-2)^2$) to type G_2. The fact that the type of a symmetric algebra can be easily detected through the Casimir element is new.

18 Algebraic computation of e in Okubo algebras

The computation of the invariant e is easy for Okubo algebras of dimension 2, see Section 19, and we consider in this section Okubo algebras of dimension 8.

18.a Via cyclic algebras

The idea behind this first computation consists of expressing a central simple associative algebra of degree 3 as a quotient of a twisted polynomial ring. This leads to a special base of the algebra. Some of the results reported in this section are taken from [Eld99].

We recall that the base field F has characteristic zero. We assume here that F contains a cubic primitive root ω of 1, but we do not have to assume that it is algebraically closed. Consider the twisted polynomial ring $F_\omega[X, Y]$.

This is a polynomial ring, however the variables X and Y do not commute but satisfy the relation $YX = \omega XY$. The center of $F_\omega[X,Y]$ is the subring generated by X^3 and Y^3. Given two nonzero scalars α and β in F, let $I_{\alpha,\beta}$ be the ideal generated by the central elements $X^3-\alpha$ and $Y^3-\beta$. Let x and y denote the classes of X and Y modulo $I_{\alpha,\beta}$, and denote by $F_\omega^{\alpha,\beta}[x,y]$ the quotient $F_\omega[X,Y]/I_{\alpha,\beta}$. Thus $F_\omega^{\alpha,\beta}[x,y]$ is the unital associative F-algebra generated by x and y and subject to the relations

$$x^3 = \alpha, \quad y^3 = \beta \quad \text{and} \quad yx = \omega xy.$$

In other terms the algebra $F_\omega^{\alpha,\beta}[x,y]$ is a cyclic algebra (see for example [Pie82, Chapter 15]). The next proposition is Wedderburn's theorem for central simple algebras of degree 3. A proof can be found in [Pie82, Chapter 15].

Theorem 18.1. *Let F be a field of characteristic not 3. If F contains a primitive cubic root ω of 1, then for any central simple associative algebra A of degree 3 over F there are nonzero scalars $\alpha, \beta \in F$ such that*

$$A \simeq F_\omega^{\alpha,\beta}[x,y].$$

Lemma 18.2. *For all x^iy^j, $0 \le i,j \le 2$, $(i,j) \ne (0,0)$, we have $T(x^iy^j) = T\big((x^iy^j)^2\big) = 0$.*

Proof. Note that the element x in $F_\omega^{\alpha,\beta}[x,y]$ verifies $T(x) = S(x) = 0$ since its minimum polynomial is of the form $X^3 - \alpha$. The same happens for y. Let $S(u,v)$ be the symmetric bilinear form such that $S(v,v) = S(v)$. It is known that the form is nonsingular. We have $T(vw) = T(v)T(w) - 2S(v,w)$. It immediately follows that $T(x^2) = -2S(x) = 0$ and $S(x^2) = -\frac{1}{2}T(x^4) = -\frac{1}{2}T(\alpha x) = 0$; hence $T(y^2) = S(y^2) = 0$. The rules $x^3 = \alpha$ and $y^3 = \beta$ imply $T(x^{-1}) = T(y^{-1}) = 0$.
We conclude by proving that mixed terms x^iy^j have trace zero. First note that $x^iy^j = \omega^{-ij}y^jx^i$, then $T(x^iy^j) = T(\omega^{-ij}y^jx^i) = \omega^{-ij}T(y^jx^i)$ since $T(xy) = T(yx)$. Now, if $\omega^{-ij} \ne 1$ the claim is clear, and if $\omega^{-ij} = 1$, which is equivalent to $i = 0$ or $j = 0$ (but not both), we have no mixed terms. The same arguments show that $T\big((x^iy^j)^2\big) = 0$. □

Let $F_\omega^{\alpha,\beta}[x,y]^0$ be the space of trace zero elements in $F_\omega^{\alpha,\beta}[x,y]$. In view of Lemma 18.2 the elements x^iy^j, $0 \le i,j \le 2$, $(i,j) \ne (0,0)$, lie in $F_\omega^{\alpha,\beta}[x,y]^0$ and clearly form a basis of $F_\omega^{\alpha,\beta}[x,y]^0$. Following [Eld99] we choose the elements

$$\{x_{i,j} := -\omega^{ij}x^iy^j \mid -1 \le i,j \le 1, (i,j) \ne (0,0)\}$$

as a special basis. Let $\mu = \frac{1-\omega}{3}$. On $F_\omega^{\alpha,\beta}[x,y]$ define the new product

$$u \diamond v = \mu uv + (1-\mu)vu.$$

The elements $\omega^{-ij}x^iy^j$ of $F_\omega^{\alpha,\beta}[x,y]$ multiply according to

$$\omega^{-ij}x^iy^j \diamond \omega^{-kl}x^ky^l = (1-\Delta)(\omega^{-(i+k)(j+l)}x^{(i+k)}y^{(j+l)}),$$

where $\Delta \equiv \begin{vmatrix} i & j \\ k & l \end{vmatrix} \bmod 3 \in \{0,1,2\}$.
The Okubo multiplication is:

$$u \star v = \mu uv + (1-\mu)vu - \frac{1}{3}T(uv)\cdot 1 = u \diamond v - 2b(u,v)\cdot 1.$$

The multiplication Table IV.1 for the Okubo multiplication in $F_\omega^{\alpha,\beta}[x,y]^0$ is in [Eld99].
To compute the Casimir element we need an orthonormal basis with respect to the norm form $q(u) = -\frac{1}{3}S(u) = \frac{1}{6}T(u^2)$. The basis used in the multiplication table is not orthonormal since, for example, $x_{1,0} = -x$ and $x_{0,1} = -y$ are isotropic.

Proposition 18.3. *Assume that F contains an element ι with $\iota^2 = -1$. The elements*

$$\begin{aligned} y_{1,0} &= x_{1,0} + x_{-1,0}, \\ y_{-1,0} &= \iota(x_{1,0} - x_{-1,0}), \\ y_{0,1} &= x_{0,1} + x_{0,-1}, \\ y_{0,-1} &= \iota(x_{0,1} - x_{0,-1}), \\ y_{1,1} &= \omega(x_{1,1} + x_{-1,-1}), \\ y_{-1,-1} &= \iota\omega(x_{1,1} - x_{-1,-1}), \\ y_{-1,1} &= \omega^2(x_{-1,1} + x_{1,-1}), \\ y_{1,-1} &= \iota\omega^2(x_{-1,1} - x_{1,-1}), \end{aligned}$$

form an orthonormal base of $F_\omega^{\alpha,\beta}[x,y]^0$.

Proof. A lenghty computation! □

Theorem 18.4. *For any Okubo algebra the Casimir element is zero. In particular the numerical invariant e of a symmetric composition of Okubo type is zero.*

Proof. The basis elements of the previous proposition can be grouped into four pairs: every pair $(y_{i,j}, y_{k,l})$ with $i+k = j+l = 0$ spans $< x_{i,j}, x_{k,l} >$. Let $p \in \{0,1,2\}$, then for any pair $(y_{i,j}, y_{k,l})$ we find:

$$\begin{aligned} y_{i,j} \star y_{i,j} &= \omega^p(x_{i,j} + x_{k,l}) \star \omega^p(x_{i,j} + x_{k,l}) \\ &= \omega^{2p}(x_{i,j}\star x_{i,j} + x_{i,j}\star x_{k,l} + x_{k,l}\star x_{i,j} + x_{k,l}\star x_{k,l}) \\ &\overset{\text{Table } IV.1}{=} \omega^{2p}(x_{i,j}\star x_{i,j} + x_{k,l}\star x_{k,l}), \end{aligned}$$

$\star$	$x_{1,0}$	$x_{-1,0}$	$x_{0,1}$	$x_{0,-1}$	$x_{1,1}$	$x_{-1,-1}$	$x_{-1,1}$	$x_{1,-1}$
$x_{1,0}$	$-\alpha x_{-1,0}$	0	0	$x_{1,-1}$	0	$x_{0,-1}$	0	$\alpha x_{-1,-1}$
$x_{-1,0}$	0	$-\alpha^{-1}x_{1,0}$	$x_{-1,1}$	0	$x_{0,1}$	0	$\alpha^{-1}x_{1,1}$	0
$x_{0,1}$	$x_{1,1}$	0	$-\beta x_{0,-1}$	0	$\beta x_{1,-1}$	0	0	$x_{1,0}$
$x_{0,-1}$	0	$x_{-1,-1}$	0	$-\beta^{-1}x_{0,1}$	0	$\beta^{-1}x_{-1,1}$	$x_{-1,0}$	0
$x_{1,1}$	$\alpha x_{-1,1}$	0	0	$x_{1,0}$	$-(\alpha\beta)x_{-1,-1}$	0	$\beta x_{0,-1}$	0
$x_{-1,-1}$	0	$\alpha^{-1}x_{1,-1}$	$x_{-1,0}$	0	0	$-(\alpha\beta)^{-1}x_{1,1}$	0	$\beta^{-1}x_{0,1}$
$x_{-1,1}$	$x_{0,1}$	0	$\beta x_{-1,-1}$	0	0	$\alpha^{-1}x_{1,0}$	$-\alpha^{-1}\beta x_{1,-1}$	0
$x_{1,-1}$	0	$x_{0,-1}$	0	$\beta^{-1}x_{1,1}$	$\alpha x_{-1,0}$	0	0	$-\alpha\beta^{-1}x_{-1,1}$

Table IV.1: Multiplication table

and

$$\begin{aligned} y_{k,l} \star y_{k,l} &= \iota\omega^p(x_{i,j} - x_{k,l}) \star \iota\omega^p(x_{i,j} - x_{k,l}) \\ &= -\omega^{2p}(x_{i,j} \star x_{i,j} - x_{i,j} \star x_{k,l} - x_{k,l} \star x_{i,j} + x_{k,l} \star x_{k,l}) \\ &\overset{\text{Table } IV.1}{=} -\omega^{2p}(x_{i,j} \star x_{i,j} + x_{k,l} \star x_{k,l}) \\ &= -y_{i,j} \star y_{i,j}. \end{aligned}$$

Hence, the sum of every pair gives zero. □

18.b Via a Gell-Mann basis

We now compute the Casimir element with the help of a Gell-Mann basis. Since we may assume that the base field is algebraically closed, we may take as a model of the Okubo composition the space $M_3(F)^0$ of trace zero elements in $M_3(F)$. The quadratic form S evaluated on a matrix $x = (x_{ij})$ is given by

$$S(x) = \sum_{i<j} x_{ii}x_{jj} - x_{ij}x_{ji}.$$

Let $q = -\frac{1}{3}S$ and let b be the associated polar form. We have

$$\begin{aligned} b(x,y) &= \tfrac{1}{2}\big(q(x+y) - q(x) - q(y)\big) = -\tfrac{1}{6}\big(S(x+y) - S(x) - S(y)\big) \\ &= -\tfrac{1}{6}\textstyle\sum_{i,j=1}^{3} x_{ii}y_{jj} - x_{ij}y_{ji} \\ &= -\tfrac{1}{6}(x_{11}y_{22} - x_{12}y_{21} + x_{11}y_{33} - x_{13}y_{31} + x_{22}y_{11} - x_{21}y_{12} \\ &\quad +x_{22}y_{33} - x_{23}y_{32} + x_{33}y_{11} - x_{31}y_{13} + x_{33}y_{22} - x_{32}y_{23}). \end{aligned}$$

We generate an orthonormal basis of $M_3(F)^0$ with respect to b by Gram-Schmidt orthonormalization, starting with the matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The following set of matrices $\{e_1, \dots, e_8\}$ defines an orthonormal basis of $M_3(F)^0$:

$$e_1 = \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & -\sqrt{3} & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$e_3 = \begin{pmatrix} 0 & \sqrt{3} & 0 \\ \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad e_4 = \begin{pmatrix} 0 & i\sqrt{3} & 0 \\ -i\sqrt{3} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$e_5 = \begin{pmatrix} 0 & 0 & \sqrt{3} \\ 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 \end{pmatrix} \quad e_6 = \begin{pmatrix} 0 & 0 & i\sqrt{3} \\ 0 & 0 & 0 \\ -i\sqrt{3} & 0 & 0 \end{pmatrix}$$

$$e_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & \sqrt{3} & 0 \end{pmatrix} \quad e_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i\sqrt{3} \\ 0 & -i\sqrt{3} & 0 \end{pmatrix}.$$

We recall that Okubo multiplication is obtained from the matrix multiplication $(x, y) \mapsto xy$ by the formula:

$$x \star y = \mu xy + (1-\mu)yx - \frac{1}{3}T(yx) \cdot 1,$$

where $\mu = \frac{1-\omega}{3}$ and ω is a primitive cubic root of unity. Then:

$$e_1 \star e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad e_2 \star e_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$e_3 \star e_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad e_4 \star e_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

$$e_5 \star e_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad e_6 \star e_6 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$e_7 \star e_7 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad e_8 \star e_8 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

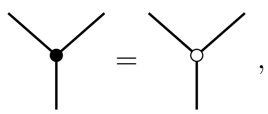
Thus the Casimir element $c = \sum_{i=1}^{8} e_i \star e_i$ is zero and the invariant $e = b(c, c)$ is also zero.

Remark 18.5. The basis used here are a modification of the Gell-Mann basis given in [EM90, p. 307].

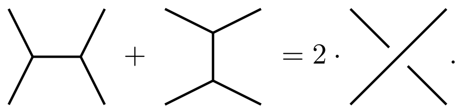
To give more illustrations of diagrammatic calculus we consider, in the next sections, commutative symmetric composition algebras, associative symmetric composition algebras and symmetric composition algebras associated with associative algebras.

19 Commutative symmetric composition algebras

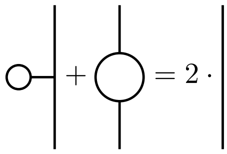
Commutativity for symmetric composition algebras means that



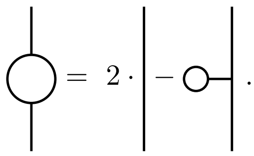
and we do not have to care about orientations of trivalent vertices. The graph relation (17.4) reduces to

(19.1) 

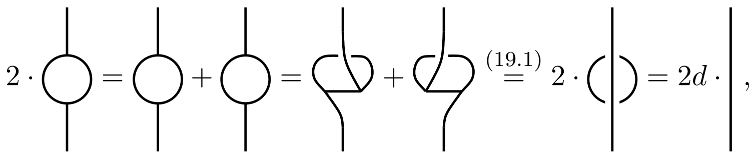
Closing on the left side we obtain



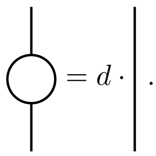
and hence

(19.2) 

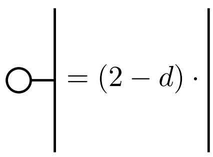
On the other hand,



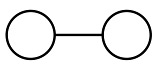
so that

(19.3) 

Comparing relations (19.2) and (19.3) we find



and finally

$e =$  $= (2-d)d.$

If we substitute this value of e in the formula $e\big(e-(2-d)^2\big)=0$, valid for arbitrary symmetric composition algebras, we get

$$\begin{aligned} 0 &= (2-d)d\big((2-d)d-(2-d)^2\big) = d(2-d)^2(d-2+d) \\ &= 2d(2-d)^2(d-1). \end{aligned}$$

Thus:

Theorem 19.4. *A symmetric composition algebra is commutative only if the dimension is equal to* 0, 1 *or* 2*; the possible values of the pair* (d, e) *are* $(0, 0)$, $(1, 1)$ *and* $(2, 0)$.

Example 19.5. Para-Hurwitz algebras of dimension 2 with orthonormal basis $\{e_1, e_2\}$ have the following multiplication table

$\star$	e_1	e_2
e_1	e_1	$-e_2$
e_2	$-e_2$	$-e_1$

For the Casimir element we have: $c = e_1^2 + e_2^2 = 0$. Hence $e = b(c, c) = 0$.

Example 19.6. Let $\dim_F S = 2$, for S a symmetric composition algebra which is not Para-Hurwitz. Let q be the norm on S and b the polar of q, $b(x, x) = q(x)$. Then S contains no idempotent and is isomorphic to an algebra A_λ with basis $\{u, v\}$, with $q(u) = q(v) = 1$ and $b(u, v) = \frac{\lambda}{2}$, and multiplication table

$\star$	u	v
u	v	u
v	u	$\lambda u - v$

,

where $\lambda \in F$ is chosen such that the polynomial $X^3 - 3X - \lambda$ is irreducible, see [Eld97, p. 298]. The basis

$$a = \frac{1}{\sqrt{2+\lambda}}(u + v) \quad \text{and} \quad b = \frac{1}{\sqrt{2-\lambda}}(u - v)$$

is orthonormal and we can compute the invariant e with the help of this basis. Since

$$\begin{aligned} a^2 &= \frac{1}{2+\lambda}(u+v)^2 = \frac{1}{2+\lambda}(u^2 + uv + vu + v^2) \\ &= \frac{1}{2+\lambda}(v + 2u + \lambda u - v) = \frac{1}{2+\lambda}(2+\lambda)u = u \end{aligned}$$

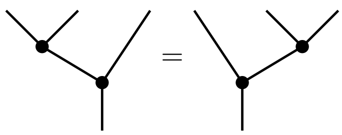
and, analogously,

$$b^2 = \frac{1}{2-\lambda}(u-v)^2 = \frac{1}{2-\lambda}(v - u - u + \lambda u - v) = \frac{1}{2-\lambda}(\lambda - 2)u = -u,$$

we find that $e = q(a^2 + b^2) = q(u - u) = 0$.

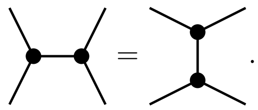
20 Associative symmetric composition algebras

Associativity of the algebra $A = (V, b, m)$ means that $(x \star y) \star z = x \star (y \star z)$ for all $x, y, z \in V$. Graphically we translate this relation as the identity for ribbon $(3, 1)$-graphs

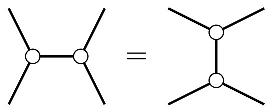


or, by applying updown transformations, as an identity of ribbon $(2,2)$-graphs

(20.1)

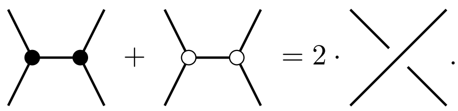


The relation $\tau \circ (20.1) \circ \tau$ give rise to a similar graph relation with white trivalent vertices:

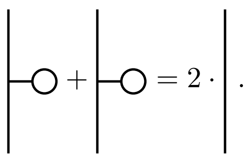


which can be used to replace the defining relation of symmetric composition algebras (17.5) by

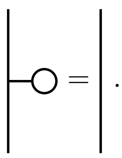
(20.2)



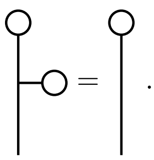
By gluing the upper and lower bands on the right we get



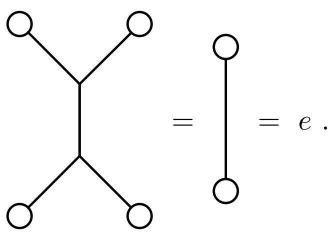
Hence,



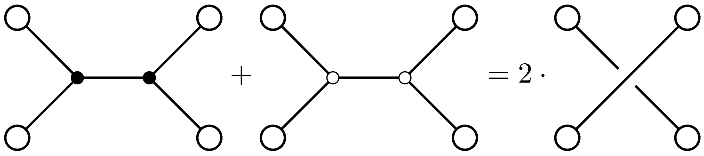
We attach a circle in the upper band to obtain a ribbon $(0,1)$-graph



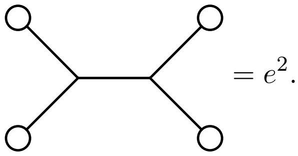
Transposing and collating gives



On the other hand, we get from (20.2) the identity



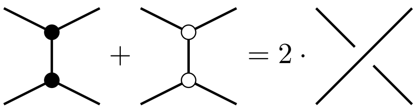
which yields:



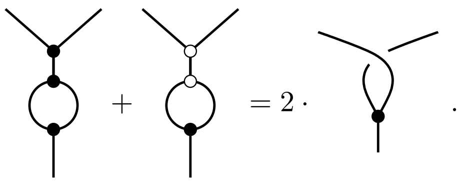
Hence, $e = e^2$. Moreover, by closing the two left bands and the two right bands of (20.2), we find that $2e = 2d$. Thus the only possible values of (17.8) are $(d, e) = (0, 0)$ and $(1, 1)$.

Theorem 20.3. *An nontrivial associative symmetric composition algebra has dimension* 1*.*

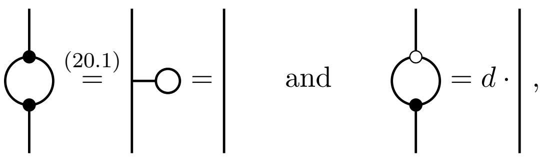
The fact that an associative symmetric composition algebra is commutative is trivial, but we show how it can be proven with the help of diagrammatic calculus. We first compose the equivalent version of (20.2)



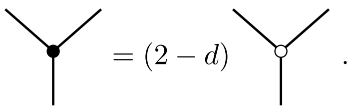
with the graph μ:



From



we have



Since $d = 1$, we get the commutativity identity.

21 Symmetric compositions and associative algebras

More interesting cases are symmetric compositions arising from associative algebras A. More precisely, we consider symmetric compositions whose multiplication is defined in terms of the multiplication of an associative algebra. This is the case of Okubo algebras, where the multiplication is defined through the multiplication of a cubic associative algebra, and the case of para-Hurwitz algebras of dimension ≤ 4. As we shall see, all these algebras admit a tensor identity which is not valid for para-Cayley compositions. We call such symmetric compositions *symmetric compositions of associative type*.

Lemma 21.1. *Any symmetric composition* $(V, b, \star)$ *of associative type satisfies the identities*

$$\begin{aligned}
&(21.2) & 2b(x,z)y &= (x\star y)\star z + (z\star y)\star x \\
&(21.3) & &= x\star(y\star z) + z\star(y\star x) \\
&(21.4) & 2b(x,y)z - 2b(z,y)x &= x\star(z\star y) + (y\star z)\star x - z\star(x\star y) \\
& & &\quad -(y\star x)\star z - (z\star y)\star x + z\star(y\star x)
\end{aligned}$$

Proof. The first two identities are well known and hold for arbitrary symmetric composition (see [KMRT98, Formula (34.2)]). They can be obtained graphically from the basic identity (17.4) by applying updown transformations. We first prove the last identity for Okubo compositions. Let A be a cubic associative algebra with multiplication $(x, y) \mapsto xy$. We recall that the Okubo multiplication on $A^0 = \{x \in A \mid T(x) = 0\}$ is given by

$$x \star y = \mu xy + (1-\mu)yx - \frac{1}{3}T(xy)\cdot 1,$$

where $\mu = \frac{1-\omega}{3}$ and ω is a primitive cubic root of unity. Then $(A^0, q, \star)$ is a symmetric composition algebra for the quadratic form $q(x) = -\frac{1}{3}S(x) = \frac{1}{6}T(x^2)$. Let b the polar of q. We compute $(x\star y)\star z - x\star(y\star z)$ in terms of the multiplication of A:

$$\begin{aligned}
(x\star y)\star z = \;&\mu^2(xy)z + \mu(1-\mu)(yx)z + \mu(1-\mu)z(xy) + (1-\mu)^2 z(yx) \\
&-\frac{1}{3}T(yx)z - \frac{1}{3}\mu T\big(z(xy)\big) - \frac{1}{3}(1-\mu)T\big(z(yx)\big) + \frac{1}{9}T(yx)T(z)
\end{aligned}$$

and analogously

$$\begin{aligned}
x\star(y\star z) = \;&\mu^2 x(yz) + \mu(1-\mu)x(zy) + \mu(1-\mu)(yz)x + (1-\mu)^2(zy)x \\
&-\frac{1}{3}T(zy)x - \frac{1}{3}\mu T\big((yz)x\big) - \frac{1}{3}(1-\mu)T\big((zy)x\big) + \frac{1}{9}T(zy)T(x).
\end{aligned}$$

Since $(xy)z = x(yz)$, $\mu(1-\mu) = \frac{1}{3}$, $T(x) = T(z) = 0$ and $T(xy) = -2S(x,y) = 6b(x,y)$, we get

$$(x\star y)\star z - x\star(y\star z) = \frac{1}{3}\big((yx)z + z(xy) - x(zy) - (yz)x\big) - 2b(y,x)z + 2b(z,y)x.$$

The terms on the right can be computed with the help of formula

$$\begin{aligned} z(xy) \;&=\; \omega z\star(x\star y) + z\star(y\star x) + (x\star y)\star z + \omega^2(y\star x)\star z \\ &\quad +2b(x,y)z + 2(1+\omega)b(z,x\star y) - 2\omega b(x,y\star z), \end{aligned}$$

which is a consequence of $xy = (1+\omega)x\star y - \omega y\star x + 2b(x,y)\cdot 1$, cfr. [KMRT98, (34.22)] (with a 2 factor). Then

$$\begin{aligned} (x\star y)\star z - x\star(y\star z) =& \frac{1}{3}\big(x\star(z\star y) + (y\star z)\star x - z\star(x\star y) - (y\star x)\star z\big) \\ &+\frac{2}{3}\big(z\star(y\star x) + (x\star y)\star z - x\star(y\star z)\big) \\ &-\frac{2}{3}\big((z\star y)\star x - b(z,y)x + b(x,y)z\big). \end{aligned} \tag{21.5}$$

The wanted identity for Okubo algebras then follows using the first two identities of Lemma 21.1:

$$\begin{aligned} 2b(x,y)z - 2b(z,y)x =& x\star(z\star y) + (y\star z)\star x - z\star(x\star y) - (y\star x)\star z \\ &+2z\star(y\star x) + x\star(y\star z) - (x\star y)\star z - 2(z\star y)\star x \\ =& x\star(z\star y) + (y\star z)\star x - z\star(x\star y) - (y\star x)\star z \\ &-(z\star y)\star x + z\star(y\star x). \end{aligned}$$

The multiplication for a para-Hurwitz composition is $(x,y)\mapsto x\star y = \bar{x}\,\bar{y}$ where $x\mapsto\bar{x}$ is conjugation and $(x,y)\mapsto xy$ is the multiplication in the Hurwitz algebra. We recall that

$$b(x,y) = \frac{1}{2}(x\bar{y} + y\bar{x}) = \frac{1}{2}(\bar{x}y + \bar{y}x)$$

for $x,y\in V$. Let d_3 be the difference of the right and the left hand side of (21.4). Using that the multiplication in the Hurwitz algebra is assumed to be associative, we have:

$$d_3 = \bar{y}(zx - xz) + (zy - yz)\bar{x} + \bar{z}\,xy - xy\,\bar{z}. \tag{21.6}$$

One readily checks that $d_3 = 0$ if x, y or z is the identity element. Thus we may assume that x, y and z are pure, i.e., $\bar{x} = -x$, $\bar{y} = -y$ and $\bar{z} = -z$. Then (21.6) reduces to

$$d_3 = (xy + yx)z - z(xy + yx)$$

and the right hand side is zero since $\frac{1}{2}(xy + yx) = b(x,y)$ for pure elements. □

Let SCA^0 be the set of tensor identities obtained by adding the third relation of Lemma 21.1 to the relations defining symmetric compositions. Let Rib_{SCA^0} be the corresponding category of ribbon graphs and $E_{SCA^0} = \mathrm{Mor}_{\mathrm{Rib}_{SCA^0}}(\emptyset, \emptyset)$ be the ring of numerical invariants of the category Rib_{SCA^0}.

Theorem 21.7. *Let J be the ideal of the polynomial ring $\mathbb{Q}[\bar{d}, \bar{e}]$, generated by the elements*

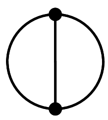
$$p_1 = \bar{e}\big(\bar{e} - (2-\bar{d})^2\big),$$
$$p_2 = (\bar{d}-2)(\bar{d}-8)(\bar{d}-\bar{e}),$$
$$p_3 = \bar{d}(\bar{d}-2)(\bar{d}-8) + \bar{e}(5\bar{d}-12).$$

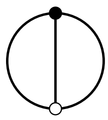
Then the homomorphism $\Phi\colon \mathbb{Q}[\bar{d}, \bar{e}] \to E_{SCA^0}$ given by $\Phi(\bar{d}) = d$, $\Phi(\bar{e}) = e$, induces an isomorphism of rings

$$\bar{\Phi}\colon \mathbb{Q}[\bar{d}, \bar{e}]/J \to E_{SCA^0}.$$

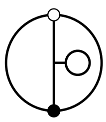
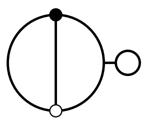
In particular, only the values $(d, e) = (0,0)$, $(1,1)$, $(2,0)$, $(4,4)$ and $(8,0)$ are admissible and all are simple.

Proof. Clearly, all relations obtained in the proof of Theorem 17.6 are still valid. In particular we have $\Phi(p_1) = \Phi(p_2) = 0$. We check that $\Phi(p_3) = 0$. The relations (17.14) and (17.15) imply

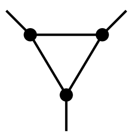
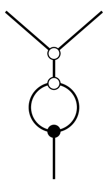
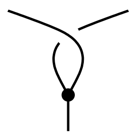
(21.8)  $= 2d - e,$

(21.9)  $= d^2.$

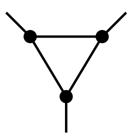
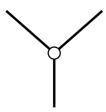
Moreover,

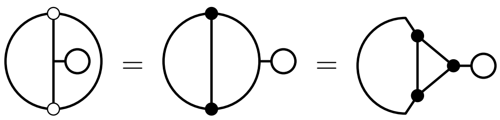
 $=$  $\overset{(17.15)}{=} ed$

(the second and the third bands joining the black and the white vertices have been permuted in the first equality; this implies a change of orientation). We compose (17.5) with the graph μ:

 $+$  $= 2 \cdot$ 

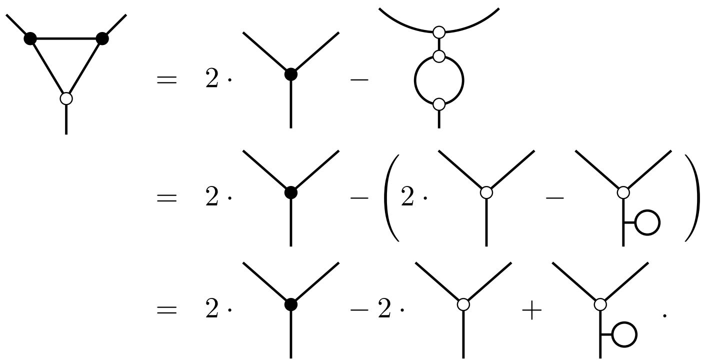
and hence

(21.10)  $= (2-d) \cdot$  , which implies

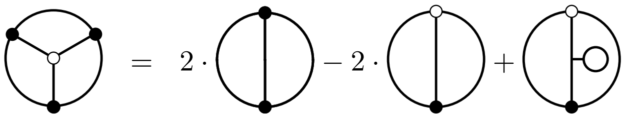


$$= (2-d)e.$$

A consequence of (17.5) is:

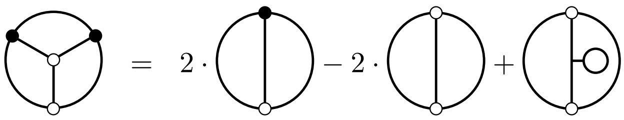


Close the three ends together with a clockwise orientation:



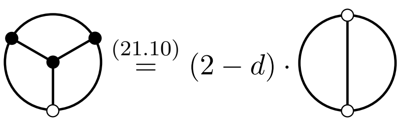
$$= 2(2d-e) - 2d^2 + ed = (2-d)(2d-e)$$

and with a counterclockwise orientation:



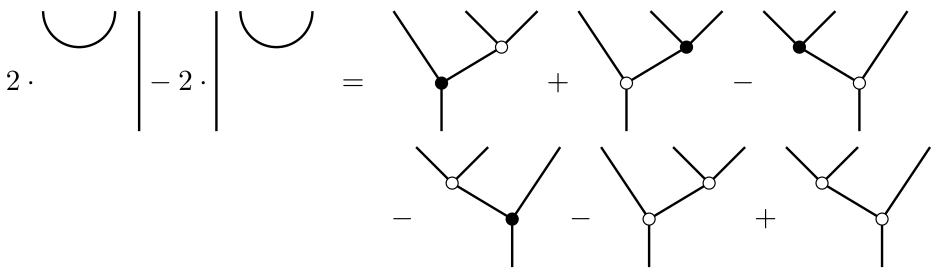
$$= 2d^2 - 2(2d-e) + (2-d)e = 2d^2 - 4d + 4e - de.$$

A similar computation yields

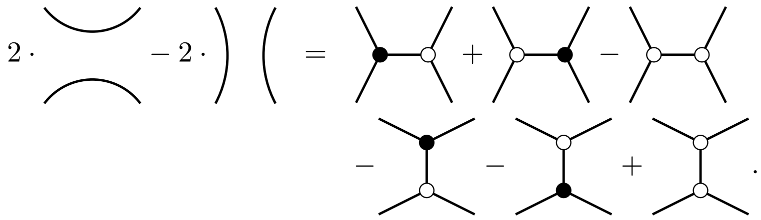


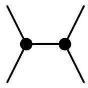
$$= (2-d)(2d-e).$$

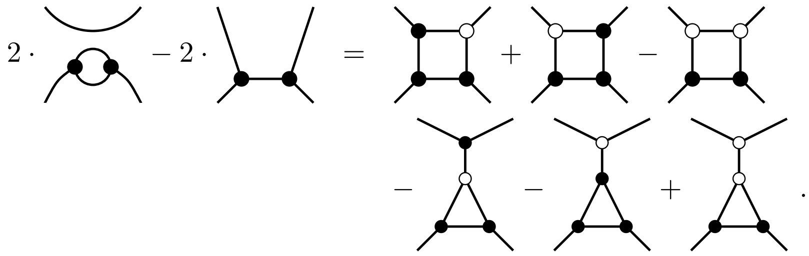
Now the third relation in Lemma 21.1 translates into a graph identity of ribbon $(3,1)$-graphs:



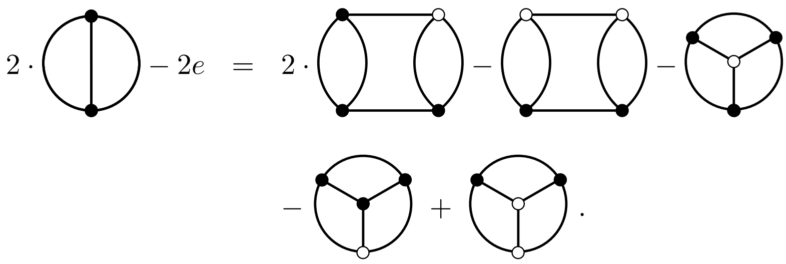
or, equivalently, of ribbon $(2,2)$-graphs by applying updown transformations:



We compose it with the graph  (glued at the bottom):



The next step is to collate the left bands and also the right ones together. We find:



Hence,

$$2(2d-e)-2e \quad = \quad 2d(2d-e)-d^3-(2-d)(2d-e)-(2-d)d^2+2d^2,$$

which implies the wanted relation $d(d-2)(d-8)+e(5d-12)=0$. The fact that $\bar{\Phi}$ is an isomorphism is a consequence of the fact that all values are simple and can be realized by compositions over $\mathbb{Q}$. The last claim follows from Corollary 11.4. $\square$

Chapter V

J-algebras

22 Jordan algebras

Definition 22.1. A *Jordan algebra* J is a commutative finite-dimensional unital algebra over F such that the multiplication $m(a,b) = a \cdot b$ satisfies

$$\big((a \cdot a) \cdot b\big) \cdot a = (a \cdot a) \cdot (b \cdot a)$$

for all $a, b \in J$. For any associative algebra A, the product $a \cdot b = \frac{1}{2}(ab + ba)$ gives A the structure of a Jordan algebra, denoted by A^+. A Jordan algebra A is *special* if there exists an injective homomorphism $A \to D^+$ for some associative algebra D and is *exceptional* otherwise.

Jordan algebras admit generic minimal polynomials (see [KMRT98, Chapter IX]). The degree of the generic minimal polynomial is called the *degree of the Jordan algebra*.

Example 22.2. (Jordan algebras of quadratic forms, see [Jac68, p. 13]). Let V be a vector space over a field F which is equipped with a symmetric bilinear form f. The vector space $J(V, f) = F \cdot 1 \oplus V$ with the product

$$(\alpha 1 + x)(\beta 1 + y) = \big(\alpha\beta + f(x,y)\big)1 + (\beta x + \alpha y)$$

for $\alpha, \beta \in F$, $x, y \in V$, is a Jordan algebra, called the *Jordan algebra of the quadratic form* f, where $f(x) = f(x,x)$. Such a Jordan algebra is of degree 2 and is special.

Example 22.3. (Freudenthal algebras, see [KMRT98, p. 516]). Let C be a Hurwitz algebra over a field F of characteristic not 2 and let $M_3(C) = M_3(F) \otimes C$. For $X = (c_{ij}) \in M_3(C)$, let $\overline{X} = (\bar{c}_{ij})$ where $c \mapsto \bar{c}$, $c \in C$, denotes conjugation. Let $\alpha = \operatorname{diag}(\alpha_1, \alpha_2, \alpha_3) \in GL_3(F)$ and

$$\mathcal{H}_3(C, \alpha) = \{X \in M_3(C) \mid \alpha^{-1}\overline{X}^t \alpha = X\}.$$

The space $\mathcal{H}_3(C,\alpha)$ is a cubic Jordan algebra for the product $X\cdot Y = \frac{1}{2}(XY+YX)$, where XY is the usual matrix product. We call Jordan algebras which are isomorphic to algebras $\mathcal{H}_3(C,\alpha)$, for some Hurwitz algebra C, *reduced Freudenthal algebras*, and we call twisted forms of $\mathcal{H}_3(C,\alpha)$ *Freudenthal algebras*. Freudenthal algebras are of degree 3. The possible dimensions are 1, 3, 6, 9, 15, or 27. The 27-dimensional Freudenthal algebras are exceptional, the others are special.

Let $P_{J,x}(X) = X^3 - T_J(x)X^2 + S_J(x)X - N_J(x)1$ be the generic minimal polynomial of the cubic Jordan algebra J. The form T_J is the generic trace and N_J is the generic norm. The bilinear trace form $T(x,y) = T_J(x\cdot y)$ is symmetric and associative. According to Dieudonné's theorem, its non-singularity implies that the Jordan algebra is separable (cfr. [KMRT98, Corollary (32.3) and Theorem (32.4)]). A separable Jordan algebra of degree 3 is either a Freudenthal algebra or is of the form $F^+ \times J(V,q)$, where $J(V,q)$ is the Jordan algebra of a quadratic space of dimension ≥ 2 (see [KMRT98, Theorem (37.2)]).

23 J-algebras

In his book [SV00, Chapter 5], Springer introduced a new kind of Jordan algebras, which are a generalization of Freudenthal algebras. We recall some of the results of [SV00, Chapter 5] (because of our definition of polar form, some factors are modified):

Definition 23.1. Let F be a field of characteristic not $2,3$. A *J-algebra* over F is a finite-dimensional commutative, not necessarily associative, F-algebra A with identity element ε together with a nondegenerate quadratic form Q on A such that the conditions

$$Q(x^2) = Q(x)^2 \quad \text{if } \langle x,\varepsilon\rangle = 0, \tag{23.2}$$

$$\langle xy,z\rangle = \langle x,yz\rangle, \tag{23.3}$$

$$Q(\varepsilon) = \frac{3}{2}, \tag{23.4}$$

are satisfied. The form Q is called the *norm* of A, and the polar form of Q is denoted by $\langle,\rangle$.

A linearized version of (23.2) will be used for graphical computations:

Lemma 23.5. ([SV00, Lemma 5.1.3]). *If x, y, z and u are elements of $\varepsilon^\perp$, then*

$$\langle xy,zu\rangle + \langle xz,yu\rangle + \langle xu,yz\rangle = \langle x,y\rangle\langle z,u\rangle + \langle x,z\rangle\langle y,u\rangle + \langle x,u\rangle\langle y,z\rangle.$$

Proof. By substituting $\lambda x + \mu y + \nu z + \rho u$ for x in (23.2), writing both sides out as polynomials in λ, μ, ν and ρ, and equating the coefficients of $\lambda\mu\nu\rho$ on either side, we immediately get the formula. Here we use that the degree of the polynomials is 4 and that $|F| > 4$. □

Lemma 23.6. (Hamilton-Cayley equation, [SV00, Proposition 5.1.5]). *Any J-algebra A is a Jordan algebra and an element $x \in A$ satisfies a cubic equation*

$$x^3 - 2\langle x, \varepsilon\rangle x^2 - (Q(x) - 2\langle x, \varepsilon\rangle^2)x - \det(x)\,\varepsilon = 0,$$

where det *is a cubic form on A. Note that* $\det(\varepsilon) = 1$.

A J-algebra is called *reduced* if it contains an idempotent $\neq 0, \varepsilon$. By ([SV00], Lemma 5.2.2) it also contains an idempotent u such that $Q(u) = \frac{1}{2}$. Such an idempotent is called *primitive idempotent*. J-algebras are reduced over algebraically closed fields. It follows from Lemma 23.6 that $\det(u) = 0$, $\langle u, \varepsilon\rangle = \frac{1}{2}$, $u(\varepsilon - u) = 0$, $\langle u, \varepsilon - u\rangle = 0$ and $Q(\varepsilon - u) = 1$ hold for a primitive idempotent.

Let A be a reduced J-algebra with a primitive idempotent u. Define $E = (F\varepsilon \oplus Fu)^{\perp} = \{x \in A \mid \langle x, \varepsilon\rangle = \langle x, u\rangle = 0\}$. The restriction of Q to $F\varepsilon \oplus Fu$ is nondegenerate, hence the same holds for E. Let $t\colon E \to E$ be the linear transformation defined by $x \mapsto ux$. Then

$$E = E_0 \oplus E_1 \qquad \text{where} \qquad E_i = \{x \in E \mid t(x) = ux = \frac{1}{2}ix\}$$

and we have a decomposition $A = Fu \oplus F(\varepsilon - u) \oplus E_0 \oplus E_1$.

Proposition 23.7. (J-algebra of quadratic type, [SV00, Proposition 5.3.5]). *Let A be a reduced J-algebra with a primitive idempotent u, and let E, E_0 and E_1 be as before, with respect to u.*
(i) $E_0 = 0$ if and only if A is 2-dimensional; then $A = Fu \oplus F(\varepsilon - u)$.
(ii) If $E_1 = 0$, then $A = Fu \oplus F(\varepsilon - u) \oplus E_0$. For λ, λ', μ, $\mu' \in F$, x, $x' \in E_0$ product and norm are given by

$$\big(\lambda u + \mu(\varepsilon - u) + x\big)\big(\lambda' u + \mu'(\varepsilon - u) + x'\big) = \lambda\lambda' u + \big(\mu\mu' + q(x, x')\big)(\varepsilon - u) + \mu x' + \mu' x$$

and

$$Q\big(\lambda u + \mu(\varepsilon - u) + x\big) = \tfrac{1}{2}\lambda^2 + \mu^2 + q(x),$$

where q is a nondegenerate quadratic form on E_0 with polar form $q(\ ,\ )$. Conversely, for any vector space E_0 (possibly 0) with a nondegenerate quadratic form q, the above formulas define a J-algebra A.

Such a J-algebra is closely related to the Jordan algebra of a quadratic form q as defined in Example 22.2: in fact A is the direct sum of a one-dimensional algebra Fu and of the Jordan algebra $F(\varepsilon - u) \oplus E_0$ of q with $\varepsilon - u$ as identity element.

One can show that if $\dim_F E_0 = 1$, then A is of quadratic type. The same holds for $E_1 = 0$. We assume now that A is a reduced J-algebra with a primitive idempotent u such that $\dim_F E_0 > 1$ and $E_1 \neq 0$. One can construct an element $x_1 \in E_0$ such that $Q(x_1) = \frac{1}{4}$ by taking any $y \in E_1$ with $Q(y) \neq 0$. Then $x_1 y = \frac{1}{4} y$. Let

$$C = x_1^{\perp} \cap E_0 = \{x \in E_0 \mid \langle x, x_1 \rangle = 0\}.$$

It is possible to turn C into a composition algebra with an appropriate product and a multiplicative norm. Then one can construct an isomorphism $A \xrightarrow{\sim} \mathcal{H}_3(C, \alpha)$. The main theorem states that there are only two different types of J-algebras.

Theorem 23.8. ([SV00, Theorem 5.4.5]). *A reduced J-algebra A over a field F of characteristic not $2, 3$ with identity element ε and a quadratic form Q is of one of the following two types:*

(I) $A = Fu \oplus F(\varepsilon - u) \oplus E_0$, where u is a primitive idempotent, $ux = 0$ for $x \in E_0$, and $xx' = \langle x, x' \rangle (\varepsilon - u)$ for $x, x' \in E_0$.
Here E_0 can be any vector space (possibly 0), and the restriction of Q to E_0 can be any nondegenerate quadratic form on it. These algebras are called J-algebras of quadratic type.

(II) $A \simeq \mathcal{H}_3(C, \alpha)$ for C a composition algebra over F. They are called proper J-algebras*; in this case* $\dim_F A = 6, 9, 15$ *or* 27.

Moreover, a J-algebra of type (I) cannot be isomorphic to one of type (II).

Proof. For a proof we refer to [SV00], Section 5.4, pages 137-140. □

24 Graphical calculus for reduced J-algebras

For graphical computations we utilize the identity of Lemma 23.5, which is valid only in $\varepsilon^{\perp}$.

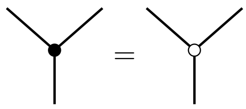
In $\varepsilon^{\perp}$ we have $T_J(x) = 2\langle x, \varepsilon \rangle = 0$, and the Hamilton-Cayley equation becomes $x^3 - Q(x)x - \det(x)\,\varepsilon = 0$. Hence, $S_J(x) = -Q(x)$. As for symmetric composition algebras, we define a multiplication in $\varepsilon^{\perp}$ by

$$x \star y = xy - \frac{1}{3} T_J(yx)\,\varepsilon, \qquad x, y \in \varepsilon^{\perp}.$$

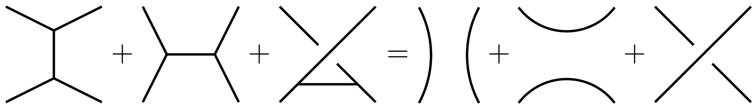
Since $T_J(xy) = -2S_J(x, y) = 2\langle x, y \rangle$ we have $x \star y = xy - \frac{2}{3}\langle x, y \rangle\,\varepsilon$.

The bilinear form $\langle , \rangle$ is associative, hence we can again depict the multiplication graph μ as a trivalent vertex. Commutativity of the multiplication and Lemma 23.5 are the tensor identities defining the algebra $(\varepsilon^{\perp}, \star)$. We choose $R = \mathbb{Q}$ as coefficients and let $\mathrm{Rib}^{+}_{\mathcal{V}_1}$ be the associated ribbon category. The set Γ defining the corresponding category of ribbon graph Rib_{Γ}

contains the following relations:

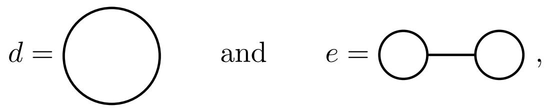


(24.1)



(in view of the first relation, it is not necessary to mark trivalent vertices).

As in the previous chapters, set



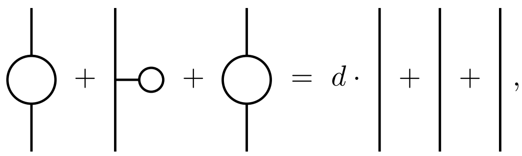
so $d = \dim_F(\varepsilon^\perp)$.

Theorem 24.2. *The relation*

$$e(e - (2 - d)^2) = 0$$

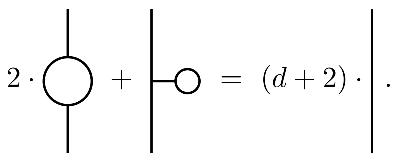
holds in the ring E_Γ of numerical invariants.

Proof. We start by gluing together the two bands on the right of (24.1)

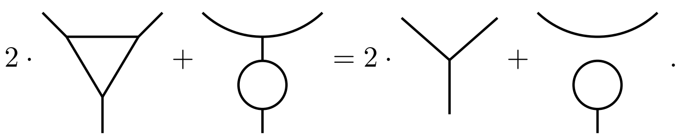


hence

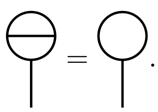
(24.3)



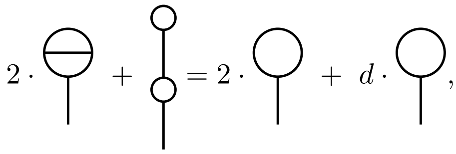
Moreover, $\mu \circ (24.1)$ gives



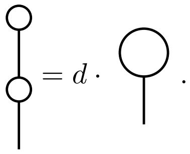
The last identity can be modified by closing the left upper band with the bottom one. Then



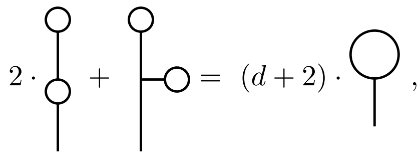
One can also modify it by closing the two upper bands together:



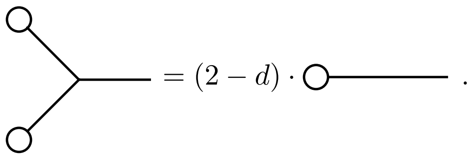
which yields



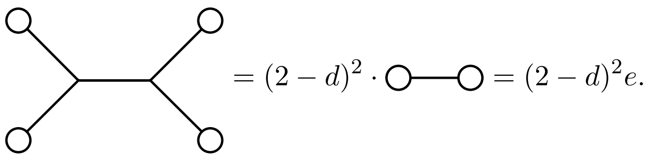
Relation (24.3) implies similarly



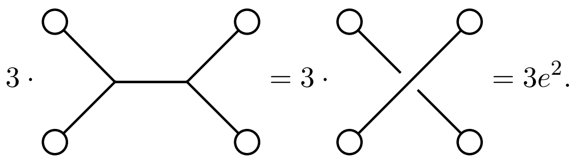
and consequently



Composing both sides with each other yields



The next computation is a direct consequence of (24.1)



We conclude that $(2-d)^2 e = e^2$ or $e(e-(2-d)^2) = 0$. □

Proposition 24.4. *The invariant e distinguishes the two types of J-algebras: its value is equal to zero for Freudenthal algebras $\mathcal{H}_3(C, \alpha)$ and to $(d-2)^2$ for J-algebras of quadratic type (d denotes the dimension of $\varepsilon^\perp$).*

Proof. We start with a J-algebra A of type (I). Since A is a direct sum of a one-dimensional algebra and a Jordan algebra of a quadratic form, A is alternative (a direct proof can be done by testing the identity $x(xy) = (xx)y$ with the product given in Proposition 23.7). It follows from [KMRT98, Proposition (34.19)] that the algebra $(A^0, \star)$ is a symmetric composition algebra with norm $n(x) = -\frac{1}{3}S_J(x) = \frac{1}{3}Q(x)$. For a generic element $x = \lambda u + \mu(\varepsilon - u) + v$ the Cayley-Hamilton identity has the form

$$x^3 - (\lambda + 2\mu)x^2 - (q(v) - \mu^2 - 2\lambda\mu)x - \lambda\big(\mu^2 - q(v)\big)\,\varepsilon = 0.$$

Hence for an element in $\varepsilon^\perp$ we have that $\lambda = -2\mu$, $x = \mu(\varepsilon - 3u) + v$ and $n(x) = \mu^2 + \frac{1}{3}q(v)$. The goal is to compute the invariant e given by $n(c) = b_n(c,c)$, where $c = \sum x_i \star x_i$ is the Casimir element. We have to fix an orthonormal base $\{x_i\}$ of $\varepsilon^\perp$. First we remark that the element $x_0 = \varepsilon - 3u$ lies in $\varepsilon^\perp$ and that $n(x_0) = \frac{1}{3}Q(\varepsilon - 3u) = 1$, so we have the first basis element. The other $d-1$ elements may be found in E_0 by choosing an orthogonal basis $\{x_1, \ldots, x_{d-1}\}$ with $q(x_i) = 3$, $1 \le i \le d-1$. Finally:

$$\begin{aligned}
n\Big(\sum_{i=0}^{d-1} x_i \star x_i\Big) &= n\Big(x_0 \star x_0 + \sum_{i=1}^{d-1} x_i \star x_i\Big)\\
&= n\Big(x_0^2 - \frac{2}{3}Q(x_0)\,\varepsilon + \sum_{i=1}^{d-1} x_i^2 - \frac{2}{3}q(x_i)\,\varepsilon\Big)\\
&= n\Big(\varepsilon + 3u - 2\,\varepsilon + \sum_{i=1}^{d-1} q(x_i)(\varepsilon - u) - \frac{2}{3}q(x_i)\,\varepsilon\Big)\\
&= n\Big(3u - \varepsilon + (\varepsilon - 3u)\sum_{i=1}^{d-1} 1\Big)\\
&= n\Big(3u - \varepsilon + (\varepsilon - 3u)(d-1)\Big)\\
&= n\Big((d-2)(\varepsilon - 3u)\Big) = (d-2)^2 n(x_0) = (d-2)^2.
\end{aligned}$$

We next consider J-algebras of type (II). The elements of a Freudenthal algebra $\mathcal{H}_3(C,\alpha)$ can be represented as matrices

$$a = \begin{pmatrix} \xi_1 & c_3 & \alpha_1^{-1}\alpha_3\bar{c}_2 \\ \alpha_2^{-1}\alpha_1\bar{c}_3 & \xi_2 & c_1 \\ c_2 & \alpha_3^{-1}\alpha_2\bar{c}_1 & \xi_3 \end{pmatrix}, \quad c_i \in C,\ \xi_i \in F,$$

and the generic minimal polynomial is

$$P_{J,x}(X) = X^3 - T_J(x)X^2 + S_J(x)X - N_J(x)1,$$

where

$$
\begin{aligned}
T_J(x) &= \xi_1 + \xi_2 + \xi_3, \\
S_J(x) &= \xi_1\xi_2 + \xi_2\xi_3 + \xi_1\xi_3 - \alpha_3^{-1}\alpha_2 N_C(c_1) - \alpha_1^{-1}\alpha_3 N_C(c_2) \\
&\quad -\alpha_2^{-1}\alpha_1 N_C(c_3), \\
N_J(x) &= \xi_1\xi_2\xi_3 - \alpha_3^{-1}\alpha_2\xi_1 N_C(c_1) - \alpha_1^{-1}\alpha_3\xi_2 N_C(c_2) - \alpha_2^{-1}\alpha_1\xi_3 N_C(c_3) \\
&\quad +T_C(c_3c_1c_2).
\end{aligned}
$$

The forms $N_C(x) = x\bar{x}$ and $T_C(x) = b_C(x, 1_C)$ are the norm and the trace, respectively, of the Hurwitz algebra C with identity 1_C. For an element

$$
b = \begin{pmatrix} \xi_1' & c_3' & \alpha_1^{-1}\alpha_3\bar{c}_2' \\ \alpha_2^{-1}\alpha_1\bar{c}_3' & \xi_2' & c_1' \\ c_2' & \alpha_3^{-1}\alpha_2\bar{c}_1' & \xi_3' \end{pmatrix}, \quad c_i' \in C,\ \xi_i' \in F,
$$

it turns out that

$$
\begin{aligned}
S_J(a,b) &= \frac{1}{2}\big(S_J(x+y) - S_J(x) - S_J(y)\big) \\
&= \frac{1}{2}(\xi_1\xi_2' + \xi_1'\xi_2 + \xi_2\xi_3' + \xi_2'\xi_3 + \xi_1\xi_3' + \xi_1'\xi_3) \\
&\quad -\alpha_3^{-1}\alpha_2 b_C(c_1, c_1') - \alpha_1^{-1}\alpha_3 b_C(c_2, c_2') - \alpha_2^{-1}\alpha_1 b_C(c_3, c_3').
\end{aligned}
$$

The identity element ε is represented by the matrix $\mathrm{diag}(1,1,1)$; the multiplication in $\mathcal{H}_3(C,\alpha)^0 = \{x \in \mathcal{H}_3(C,\alpha) \mid T_J(x) = 0\}$ is given by

$$
x \star y = \frac{1}{2}(xy + yx) - \frac{1}{3}T_J(xy)\,\varepsilon
$$

and the norm of the symmetric composition algebra $\big(\mathcal{H}_3(C,\alpha)^0, \star\big)$ is given by $n(x) = -\frac{1}{3}S_J(x)$. Without loss of generality we set $\alpha = (1,1,1)$. We assume (by extending the base field if necessary) that the standard basis of the Hurwitz algebra C is orthonormal with respect to N_C and denote it by $(e_i)_{i \le \dim_F C}$. An orthogonal basis for $\mathcal{H}_3(C,\alpha)^0$ is the set of matrices:

$$
x_{0,1} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad x_{0,2} = \sqrt{3} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},
$$

$$
x_{1,i} = \sqrt{3} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & e_i \\ 0 & \bar{e}_i & 0 \end{pmatrix}, \quad x_{2,i} = \sqrt{3} \cdot \begin{pmatrix} 0 & 0 & \bar{e}_i \\ 0 & 0 & 0 \\ e_i & 0 & 0 \end{pmatrix},
$$

$$
x_{3,i} = \sqrt{3} \cdot \begin{pmatrix} 0 & e_i & 0 \\ \bar{e}_i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
$$

Then,

$$\begin{aligned} x_{0,1} \star x_{0,1} &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 - \frac{1}{3} T_J(x_{0,1}^2) \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = x_{0,1} \end{aligned}$$

and analogously

$$x_{0,2} \star x_{0,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -x_{0,1}.$$

For $j = 1, 2, 3$ and $i = 1, \ldots, 8$ we find

$$x_{j,i} \star x_{j,i} = \begin{pmatrix} 1 - 3\delta_{1j} & 0 & 0 \\ 0 & 1 - 3\delta_{2j} & 0 \\ 0 & 0 & 1 - 3\delta_{3j} \end{pmatrix}$$

and from the formula for S_J we conclude that all elements are orthonormal with respect to n, in fact $S_J(x_{j,i}, x_{k,l}) = -3\delta_{jk}\delta_{il}$ for $0 \leq j, k \leq 3$ and $1 \leq i, l \leq 8$. It readily follows that the Casimir element is zero:

$$\begin{aligned} c &= \sum_{i=1}^{2} x_{0,i} \star x_{0,i} + \sum_{j=1}^{3} \sum_{i=1}^{8} x_{j,i} \star x_{j,i} \\ &= x_{0,1} - x_{0,1} + \sum_{i=1}^{8} x_{1,i} \star x_{1,i} + \sum_{i=1}^{8} x_{2,i} \star x_{2,i} + \sum_{i=1}^{8} x_{3,i} \star x_{3,i} \\ &= 8 \begin{pmatrix} -2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 8 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + 8 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = 0. \end{aligned}$$

Hence $e = n(c) = 0$. □

Unfortunately, the axioms used to determine the graphical relations are valid for a generic J-algebra, i.e., also for a J-algebra of quadratic type $A = Fu \oplus F(e - u) \oplus E_0$. The space E_0 can have arbitrary dimension. It follows that all integers are possible dimensions. To reduce to the four cases $6, 9, 15, 27$ one would need one or more relations which only hold for Freudenthal algebras.

Chapter VI

3-Vector products

It follows from the list of possible dimensions of r-vector products given in Section 13 that non trivial 3-vector products can only exist in dimension 4 and 8. In this chapter we associate a ribbon category to 3-vector products and give a diagrammatic proof of this result.

25 Tensor identities for 3-vector products

Let V be a d-dimensional vector space over a field F of characteristic zero and let b be a nondegenerate, symmetric bilinear form on V. By setting $r = 3$ in Definition 13.1 we get the following:

Definition 25.1. A 3-*vector product* in V is a trilinear map

$$P_3 \colon V \times V \times V \longrightarrow V,$$

satisfying the following properties:

$$b\big(P_3(v_1, v_2, v_3), v_i\big) = 0 \qquad \text{for } v_i \in V,\ i = 1, 2, 3 \tag{25.2}$$

$$b\big(P_r(v_1, v_2, v_3), P_r(v_1, v_2, v_3)\big) = \det\big(b(v_i, v_j)\big)_{i,j} \tag{25.3}$$

It follows from (25.2) that the 3-vector product P_3 and the multilinear form

$$f(v_1, v_2, v_3,\, v_4) = b\big(P_3(v_1, v_2, v_3), v_4\big)$$

are alternating. Denoting the 3-vector product by $m \colon V \otimes V \otimes V \to V$ and putting $m(v_1, v_2, v_3) = v_1 v_2 v_3$ we have

$$v_{\pi(1)} v_{\pi(2)} v_{\pi(3)} = \operatorname{sgn}(\pi) v_1 v_2 v_3, \quad v_i \in V \tag{25.4}$$

and

$$b(xyz, t) = -b(x, yzt) \quad \text{for } x,\ y,\ z \text{ and } t \in V. \tag{25.5}$$

Relation (25.3) becomes

$$b(xyz, xyz) = \det\left|\begin{pmatrix} b(x,x) & b(x,y) & b(x,z) \\ b(y,x) & b(y,y) & b(y,z) \\ b(z,x) & b(z,y) & b(z,z) \end{pmatrix}\right|. \tag{25.6}$$

Linearizing (25.6) we get:

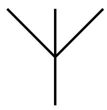
$$\begin{aligned} (25.7)\ & b(uxt, vys) + b(uxs, vyt) + b(uys, vxt) + b(uyt, vxs) = \\ = \ & 4b(x,y)b(u,v)b(s,t) - 2b(x,y)b(u,s)b(v,t) - 2b(x,y)b(u,t)b(v,s) \\ & -2b(x,u)b(y,v)b(s,t) - 2b(x,v)b(y,u)b(s,t) - 2b(x,s)b(y,t)b(u,v) \\ & -2b(x,t)b(y,s)b(u,v) + b(x,u)b(y,t)b(v,s) + b(x,v)b(u,t)b(y,s) \\ & +b(x,u)b(y,s)b(v,t) + b(x,v)b(u,s)b(y,t) + b(x,s)b(y,u)b(t,v) \\ & +b(x,s)b(y,v)b(t,u) + b(x,t)b(y,u)b(s,v) + b(x,t)b(y,v)b(s,u) \end{aligned}$$

for all u, v, x, y, s and $t \in V$.

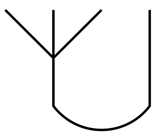
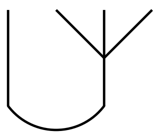
Thus 3-vector products are characterized by the tensor identities (25.7) and

$$f(x_{\pi(1)}, x_{\pi(2),} x_{\pi(3)}, x_{\pi(4)}) = \operatorname{sgn}(\pi) f(x_1, x_2, x_3, x_4). \tag{25.8}$$

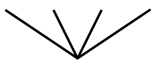
By the general formalism of Section 11 we may associate a ribbon category to 3-vector products. The multiplication $m\colon (x,y,z) \mapsto xyz$ is represented by a ribbon $(3,1)$-graph μ. For drawings it will be convenient to replace the coupon in μ by a quadrivalent vertex, as we did for 2-vector products. A difficulty is that the form $f(x,y,z,u) = b(xyz,u)$ is alternating. If we represent μ as



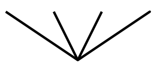
then relation (25.5) translates to

(25.9)  $= -$  .

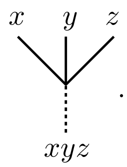
Both graphs are isotopic to the graph

 ,

hence one concludes that

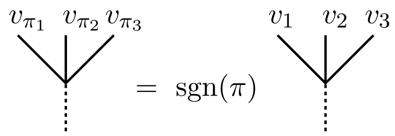
 $= 0$,

which is absurd. To take into account the fact that $f(x,y,z,u) = b(xyz,u)$ is alternating, we represent the outgoing edge as a dotted edge:

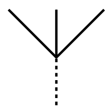
(25.10) $\mu =$ 

.

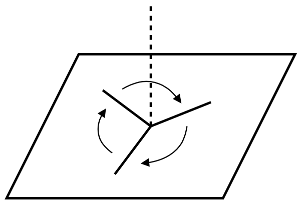
Since the product is alternating we have relations

(25.11) 

for any permutation $\pi \in S_3$. In view of (25.11) we can fix an ordering on the three undotted edges. We shall always assume that the three undotted edges are positive (clockwise) oriented. This yields a well defined representation of the ribbon $(3,1)$-graph μ as a graph with four edges and one quadrivalent vertex:

(25.12) 

One can also view the three normal (undotted) edges in a two-dimensional plane and the dotted edge as a vector perpendicular to the plane:

(25.13) 

26 Bicolored ribbon graphs

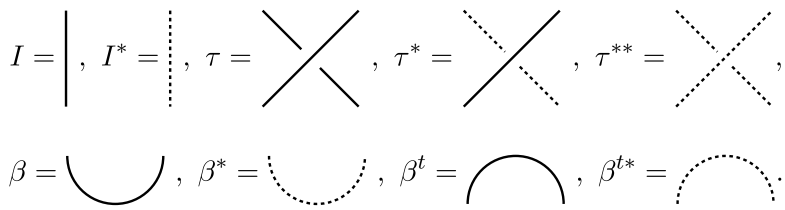
In order to use the formalism of ribbon graphs for 3-vector products as represented in (25.10) we need to have two types of bands, corresponding to the dotted and undotted edges. For this we introduce the notion of a *bicolored category of ribbon graphs*. For simplicity reasons we start with the category $\mathrm{Rib}_{\mathcal{V}}$ instead of a general ribbon graph category. We give two colors $\chi = 1$ or $\chi = -1$ to the space V generating $\mathrm{Rib}_{\mathcal{V}}$. Objects of the bicolored ribbon category of graphs $\mathrm{BiRib}_{\mathcal{V}}$ associated to $\mathrm{Rib}_{\mathcal{V}}$ are finite sequences $\big((V_1,\chi_1),\dots,(V_n,\chi_n)\big)$, $\chi_k \in \{\pm 1\}$. A ribbon graph is *bicolored* over $\mathcal{V}$ if each band and each annulus of the graph are equipped with a bicolored object (V,χ); we represent bands with $\chi = +1$ with a normal line, and bands with $\chi = -1$ with a dotted line. Morphisms of $\mathrm{BiRib}_{\mathcal{V}}$ are bicolored ribbon graphs. Composition is by gluing bands of the same color.

Let $Alph^* = \{I, \tau, \beta, \beta^t\}$ of $\mathrm{Rib}_{\mathcal{V}}$ be the basic alphabet. Then we have the following coloring rules:

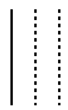
1. The ribbon graphs $I\colon (V,\chi_1) \to (V,\chi_2)$, $\beta\colon \big((V,\chi_1),(V,\chi_2)\big) \to \emptyset$ and $\beta^t\colon \emptyset \to \big((V,\chi_1),(V,\chi_2)\big)$ are bicolored if $\chi_1\chi_2 = 1$;

2. The ribbon graph $\tau\colon \big((V,\chi_1),(V,\chi_2)\big) \to \big((V,\chi_3),(V,\chi_4)\big)$ is bicolored if $\chi_1\chi_4 = \chi_2\chi_3 = 1$.

Thus we get the following bicolored basic alphabet $BiAlph^*$ for $\mathrm{BiRib}_{\mathcal{V}}$:



For example the identity morphism of an object is represented by a series of vertical undotted and dotted lines. Thus the graph



represents the identity morphism for $\big((V,+1),(V,-1),(V,-1)\big)$.

We have accordingly bicolored updown transformations extending the updown transformations defined in Section 9. Let $Alph_1^*$ be an alphabet containing $Alph^*$ and let $BiAlph_1^*$ be a bicoloration of $Alph_1^*$ containing $BiAlph^*$. For example we can adjoin a graph μ representing a multiplication to $Alph^*$ and a corresponding bicolored multiplication (also denoted μ) to $BiAlph^*$. The bicoloring of the multiplication is such that the incoming bands have color $+1$ and the outcoming band has color -1. Let $\mathrm{Rib}_{\mathcal{V}1}$ be the subcategory of $\mathrm{Rib}_{\mathcal{V}}$ whose morphisms are generated by the alphabet $Alph_1^*$ and let $\mathrm{BiRib}_{\mathcal{V}_1}$ be the associated bicolored ribbon graph category. As in Section 11 we can R-linearize $\mathrm{BiRib}_{\mathcal{V}_1}$ and we get a well defined functor $\mathcal{R}^+\colon \mathrm{BiRib}^+_{\mathcal{V}_1} \to \mathcal{V}$ sending elements of $BiAlph_1^*$ to corresponding elements of $Alph_1^*$. Given a set Γ of morphisms in $\mathrm{BiRib}^+_{\mathcal{V}_1}$, we define the factor category BiRib_Γ by going "modulo" Γ, again as in Section 11. Typically Γ is a set of relations arising from an algebra of tensor type. Let $\mathcal{P}\colon \mathrm{BiRib}_{\mathcal{V}_1} \to \mathrm{BiRib}_\Gamma$ be the projection functor. Theorem 11.2 generalizes to

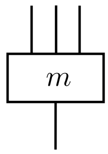
Theorem 26.1. *Let $\mathfrak{C} = \{c_i\}$ be a set of tensors in $\mathcal{V}_1$ and let $\Gamma = \{\gamma_i\}$ be a set of bicolored graphs in* $\mathrm{BiRib}^+_{\mathcal{V}_1}$ *such that $\mathcal{R}^+(\gamma_i) = c_i$ for all $c_i \in \mathfrak{C}$.*

Let BiRib_Γ *be the corresponding bicolored ribbon category of type* Γ. *Then there is a unique functor* $\mathcal{R}_\Gamma\colon \mathrm{BiRib}_\Gamma \to \mathcal{V}$ *of ribbon categories such that* $\mathcal{R}^+ = \mathcal{R}_\Gamma \circ \mathcal{P}$. *In particular we have* $\mathcal{R}_\Gamma(\gamma_i) = 0$ *for all* $\gamma_i \in \Gamma$. *Moreover* $\mathcal{R}_\Gamma$ *is compatible with updown transformations.*

27 Possible dimensions for 3-vector products

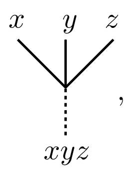
We now apply Theorem 26.1 to a 3-vector product (V, b, m). The bicolored ribbon $(3, 1)$-graph

(27.1)



representing the multiplication m has to be added to the basic alphabet *BiAlph**. The relations in $\mathcal{V}$ are (25.7) and (25.8). Let $\Gamma = 3\mathrm{vec}$ be a set of bicolored ribbon graphs representing (25.7) and (25.8) and let $\mathrm{Rib}_{3\mathrm{vec}}$ be the corresponding factor category. As already observed, the relations (25.8) allow a representation of μ in $\mathrm{Rib}_{3\mathrm{vec}}$ as a quadrivalent vertex with colored edges:

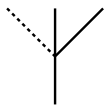
(27.2) $\mu =$



,

with a fixed (clockwise) ordering of the three undotted edges. We can move the dotted edge to any position with the help of updown transformations. For example

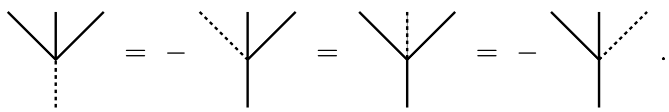
(27.3)



$= {}^{[-1]}\mu^{[1]}$

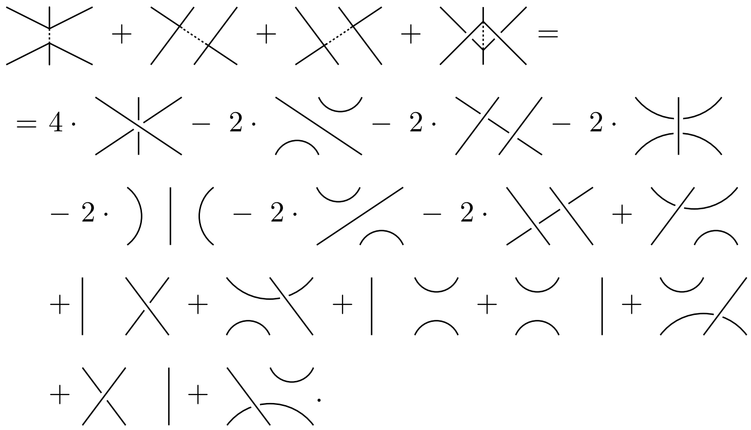
and (25.8) leads to

(27.4)



Remark 27.5. Relations (27.4) can be viewed as rules for changing colors. It is important to have such a rule since, by the definition of the composition of bicolored graphs, one is only allowed to glue bands of the same color.

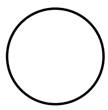
Relation (25.7) has now the following diagrammatic translation:

(27.6) 

The functor $\mathcal{R}_\Gamma\colon \mathrm{Rib}_{3\mathrm{vec}} \to \mathcal{V}$ induces a ring homomorphism

$$E_{3\mathrm{vec}} = \mathrm{Mor}_{\mathrm{Rib}_{3\mathrm{vec}}}(\emptyset, \emptyset) \to \ F,$$

and the aim is to compute $E_{3\mathrm{vec}}$ (or at least part of it). Here again a simple element of $E_{3\mathrm{vec}}$ is the circle

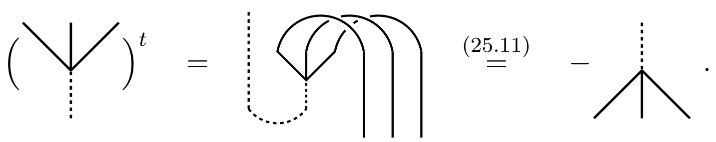
$\beta \circ \beta^t =$  .

Since $\mathcal{R}_\Gamma(\beta \circ \beta^t) = \dim_F V$, we denote the circle also by d.

Lemma 27.7. *Let $(e_1, \ldots, e_d)$ be an orthonormal basis of V. The image of μ^t under $\mathcal{R}_\Gamma$ is $m^t\colon V \to V \otimes V \otimes V$ and m^t is given by*

$$m^t(v) = -\sum_{j,k} e_j e_k v \otimes e_j \otimes e_k$$

and is depicted as



Proof. The transpose is given by $\mu^t = (\beta^* \Box I_3) \circ (I^* \Box \mu \Box I_3) \circ (I \Box \beta_3^t)$. Thus

$$\begin{aligned} m^t(v) &= \mathcal{R}_\Gamma\big((\beta^*\Box I_3)\circ(I^*\Box\mu\Box I_3)\circ(I\Box\beta_3^t)\big)(v) \\ &= \mathcal{R}_\Gamma\big((\beta^*\Box I_3)\circ(I^*\Box\mu\Box I_3)\big)\big(\sum_{i,j,k} v\otimes e_i\otimes e_j\otimes e_k\otimes e_i\otimes e_j\otimes e_k\big) \\ &= \mathcal{R}_\Gamma(\beta^*\Box I_3)\sum_{i,j,k} v\otimes e_ie_je_k\otimes e_i\otimes e_j\otimes e_k \\ &= \sum_{i,j,k} b(v, e_ie_je_k)e_i\otimes e_j\otimes e_k \overset{(25.5)}{=} -\sum_{i,j,k} b(e_i, e_je_kv)e_i\otimes e_j\otimes e_k \\ &= -\sum_{j,k} e_je_kv\otimes e_j\otimes e_k. \end{aligned}$$

□

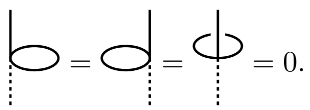
Theorem 27.8. *Let*

$$p(\bar{d}) = \bar{d}(\bar{d}-1)(\bar{d}-2)(\bar{d}+2)(\bar{d}-4)(\bar{d}+4)(\bar{d}-8) \in \mathbb{Q}[\bar{d}].$$

The map $\mathbb{Q}[\bar{d}] \to E_{3vec}$, induced by $\bar{d} \mapsto d$, induces a $\mathbb{Q}$-algebra homomorphism

$$\mathbb{Q}[\bar{d}]/p(\bar{d}) \to E_{3vec}.$$

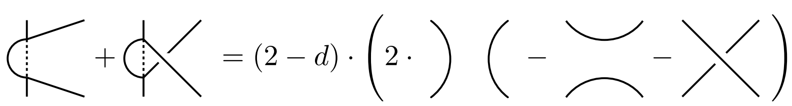
Proof. We verify that $p(d) = 0$ in E_{3vec}. Relation (25.11) implies that

(27.9) 

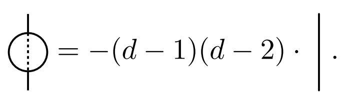
In fact:

$$\mathcal{R}_\Gamma\big(\mu\circ(I\Box\beta^t)\big)(v) = \sum_i ve_ie_i \overset{(25.4)}{=} 0.$$

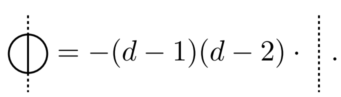
Closing the leftmost bands on the left of (27.6) gives

(27.10) 

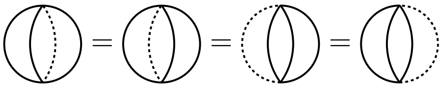
and closing the last relation on the right:

(27.11) 

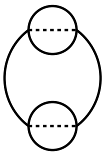
As a consequence of (27.4) we have

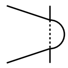


Relation (27.11) yields immediately

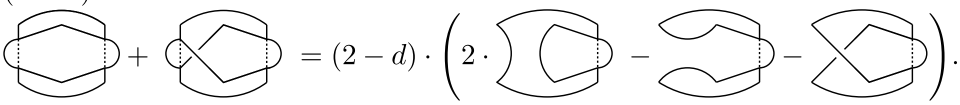
(27.12)  $= -d(d-1)(d-2).$

Composing both sides of (27.11) with each other, one finds

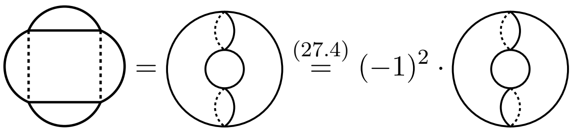
(27.13)  $= d(d-1)^2(d-2)^2.$

We next close (27.10) with the graph  to get:

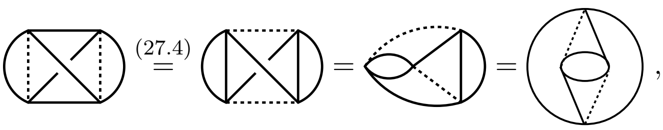
(27.14)



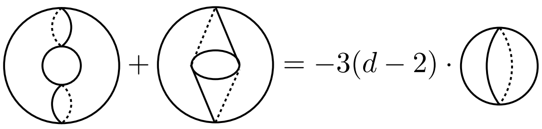
The fourth graph is zero by (27.9). The first graph can be redrawn as



and the second one as

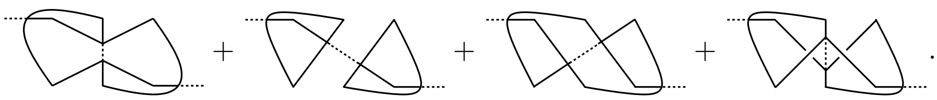


resulting in the useful relation:

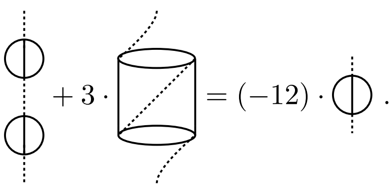
(27.15) 

$= 3d(d-1)(d-2)^2\,.$

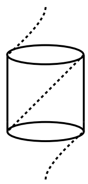
The next step is to look for an equation which permits the computation of the second graph on the left side of (27.15). The strategy is to attach two copies of μ at the six bands of the graphs in (27.6). For example, the graphs on the left become



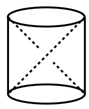
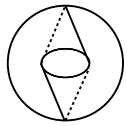
This yields



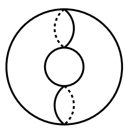
Applying equation (27.11) we get

$3 \cdot$  $= -(d-1)(d-2)(d+2)(d-5) \cdot$  .

Connecting the dotted lines we obtain the second graph of (27.15):

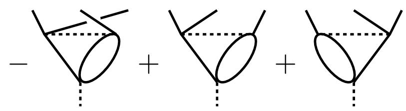
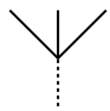
(27.16)  $=$  $= -\frac{1}{3} d(d-1)(d-2)(d+2)(d-5)\,,$

so that

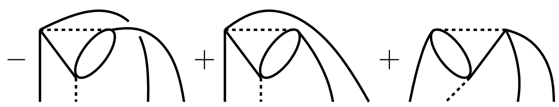
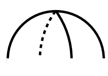
(27.17)  $= \frac{1}{3} d(d-1)(d-2)(d^2+6d-28).$

The graphs in (27.13), (27.16) and (27.17) play a central role in the next computations, since all other computed graphs will be linear combinations of these three.

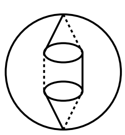
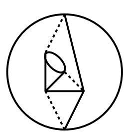
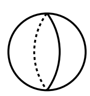
As further application, $\mu \circ (27.6)$, gives

(27.18)  $= (d-5)(d+2) \cdot$  .

We transform the last relation into a relation of ribbon $(0,4)$-graphs by updown transformations. We obtain

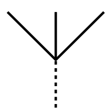
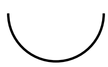
 $= (d-5)(d+2) \cdot$  .

Then, composing this relation with itself:

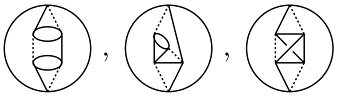
$3 \cdot$  $+ 6 \cdot$  $= (d-5)^2(d+2)^2$ 

$= -d(d-1)(d-2)(d+2)^2(d-5)^2.$

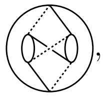
The graph relation (27.6) can be collated in many different ways with other graphs of type

 and 

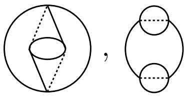
to get on the left side of (27.6) linear combinations of the diagrams



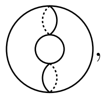
and



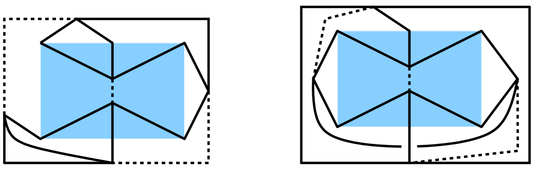
and on the right side linear combinations of the diagrams



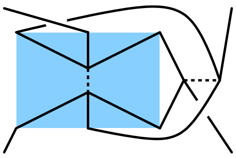
and



which are already computed. In the sequel, we show three of these ways. As an example, we only draw the composition for the first graph on the left side of (27.6); to make the lecture easier, we shadow it with a grey box. Here are the first two ways:



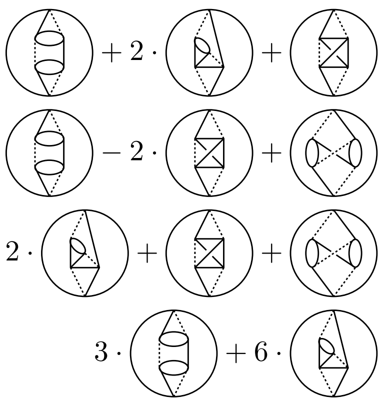
The third way is obtained by applying



and then successively by composing $\varphi \circ (I \square G \square I^*) \circ \rho$, where G is the resulting graph,

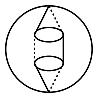
$\varphi =$  and $\rho =$  .

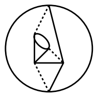
We get a system of four equations

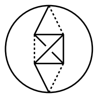


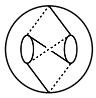
$$= -d(d-1)(d-2)(d+2)(7d-26),$$
$$= -6d(d-1)(d-2)^2(d-4),$$
$$= d(d-1)(d-2)(d^2+24d-100),$$
$$= -d(d-1)(d-2)(d+2)^2(d-5)^2,$$

which has the solutions

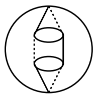
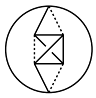
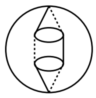
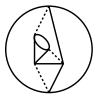
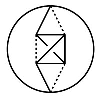
 $= \frac{1}{3}d(d-1)(d-2)(d+2)(d^3-8d^2-37d+206),$

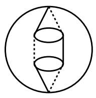
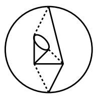
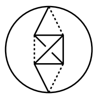
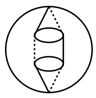
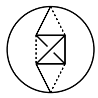
 $= -\frac{1}{3}d(d-1)(d-2)(d-4)(d+4)(d-8)(d+2),$

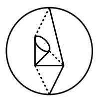
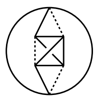
 $= \frac{1}{3}d(d-1)(d-2)(d-4)(d+4)(d-8)(d+2),$

 $= \frac{1}{3}d(d-1)(d-2)(d^4-6d^3-29d^2+168d-44).$

Two other ways of collating (27.6) yield the two equations:

 $-$  $=$  $+\,2\cdot$  $+$  ,

 $+\,2\cdot$  $-$  $=$  $-$  ,

which imply  $= -$  $= 0$, hence

$$-\frac{1}{3}d(d-1)(d-2)(d+2)(d-4)(d+4)(d-8) = 0$$

in $E_{3\text{vec}}$. □

Corollary 27.19. *The possible values for the dimension of a vector space admitting a* 3*-vector product are* 0*,* 1*,* 2*,* 4 *and* 8*. In dimension* 0*,* 1 *and* 2 *the product is degenerate.*

Proof. The existence of a 3-vector product in dimension $d^* > 0$ implies the existence of an evaluation map $E_{3\text{vec}} \to F$, sending d to d^*. Such a map induces a homomorphism $\mathbb{Q}[\bar{d}]/p(\bar{d}) \to F$ (where $\mathbb{Q}[\bar{d}]/p(\bar{d})$ is as in Theorem 27.8), sending again $\bar{d}$ to d^*. Thus d^* has to be equal to 0, 1, 2, 4 or 8. In dimension 0, 1 and 2 the product is trivial. □

Remark 27.20. The polynomial $p(\bar{d})$ also admits the negative zeroes -2 and -4. We do not know if there is another relation in $E_{3\text{vec}}$ which would exclude these values.

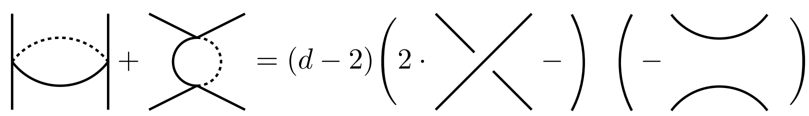
Remark 27.21. From all the computations of the proof, it seems plausible that there are no other invariants except for the dimension d. However, we are not able to show that d is the only generator of $E_{3\text{vec}}$, for example there is not a similar argument to the one for symmetric composition algebras (see Proposition 17.27).

Remark 27.22. Let

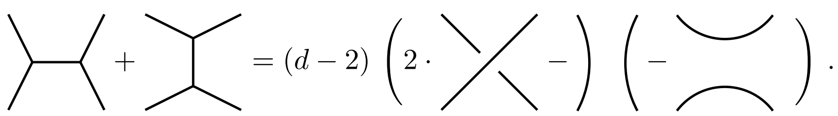
$$X = \big((V_1, \chi_1), \ldots, (V_k, \chi_k)\big) \text{ and } Y = \big((V_{k+1}, \chi_{k+1}), \ldots, (V_m, \chi_m)\big),$$

with $\chi_i = 1$ for $1 \leq i, k \leq m$. One can ask if a bicolored ribbon graph $G\colon X \to Y$ with a given number of inner 4-valented vertices always exists. Without loss of generality, we investigate a graph $G'\colon X \Box Y \to \emptyset$ (apply updown transformations), where $X \Box Y = \big((V_1, 1), \ldots, (V_m, 1)\big)$. Let n be the number of inner trivalent vertices. Since each vertex corresponds to four edges, we have $3n$ normal and n dotted edges which have to be glued together to compose a graph. The number of the couples of dotted edges is $\frac{n}{2}$, hence n is even. Moreover, only $3n - m$ normal lines can be used for the "inner" of the graph, the other being part of the boundary. So, $\frac{1}{2}(3n - m)$ is the number of couples, hence m is even and $3n \geq m$. Finally, a ribbon graph only exists if n, m are even and $3n \geq m$.

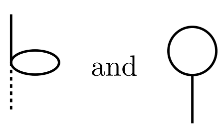
There is another interesting remark about 3-vector products and 2-vector products investigated in Chapter III. By carefully examining some of the identities found in both chapters, one observes that some relations are very similar. Imagine drawing all graphs in such a way that all undotted lines lie in a two-dimensional plane and all dotted edges are perpendicular to the plane, as in (25.13). Look at the relation drawn in the plane (ignore the dotted edges, as if you eliminate the third dimension), then the composition $(27.10) \circ \tau$



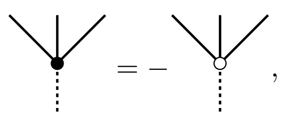
becomes the identity



This is the axiom (16.3) defining 2-vector product algebras, but modified with a $(d-2)$ term on the right. The same happens by ignoring the dotted edges from (27.11): we get relation (16.6) multiplied on the right with the same term $(d-2)$. Moreover, both graphs



are zero. If we draw black and white points to indicate the orientation of the normal lines in a 4-valented vertex, i.e.



then we obtain the skew-symmetry axiom (16.2) for 2-vector products. We do not know how to formalize this similarity, since it does not apply for all the identities of this chapter: it is not always possible to “cancel” the dotted edge and find a similar graph for 2-vector product algebras. First of all, it is necessary that only undotted lines form the boundary of the graph. Possible ribbon $(m, 0)$-graphs by 2-vector products with n inner vertices exist if $3n - m \geq 0$, and n, m are both even or both odd. Remark 27.22 states that 3-vector products always exist for $3n - m \geq 0$ and n, m are both even. At last, it is possible that after eliminating the dotted edge the resulting graph in the plane is no longer connected.

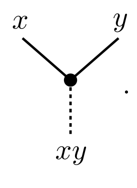
Chapter VII

Unital algebras

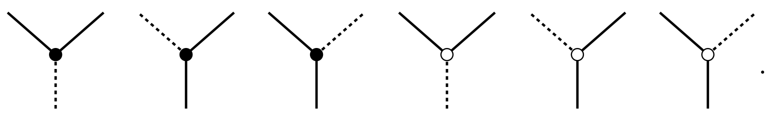
In this chapter we apply diagrammatic calculus to algebras with identity. The identity is viewed as a morphism. We first consider some cases where the bilinear form is a trace form and end with composition algebras with identity, in which case the bilinear form is not a trace form and the formalism of bicolored graphs developed in Section 26 has to be applied.

28 Unital algebras

Let F be a field of characteristic not 2 and let $A = (V, m)$ be an algebra, where V is the underlying vector space over F and $m\colon V \otimes V \to V$ is the bilinear multiplication. As in Section 10 we assume that V carries a nonsingular symmetric bilinear form and we view the multiplication m as a morphism in the ribbon category $\mathcal{V}$ attached to V. The morphism μ is represented by a ribbon $(2,1)$-graph in the category of ribbon graphs $\mathrm{Rib}_{\mathcal{V}}$. Combining the notations of Sections 12 and 26 and working in the bicolored category $\mathrm{BiRib}_{\mathcal{V}}$, we are allowed to represent μ by a bicolored oriented graph



By applying updown transformations we get the six bicolored ribbon $(2,1)$-graphs (see (12.1)), represented by:

(28.1) 

Remark 28.2. This representation is not completely satisfactory, since by definition of a category of colored ribbon graphs, only bands of the same color can be glued together. Thus, for example, the composition $\mu \circ (\mu \Box I)$ does not make sense even if it makes sense in $\mathcal{V}$. In this situation rules for changing colors help. Such rules occur for example in Remark 27.5. If the bilinear form is a trace form, then no bicoloring is needed and we only have two types of ribbon $(2,1)$-graphs (see Section 12).

Definition 28.3. A *unit* in A is an F-linear map $1\colon F \to V$ satisfying the condition:

$$m \circ (1 \otimes 1_V) = 1_V = m \circ (1_V \otimes 1) \tag{28.4}$$

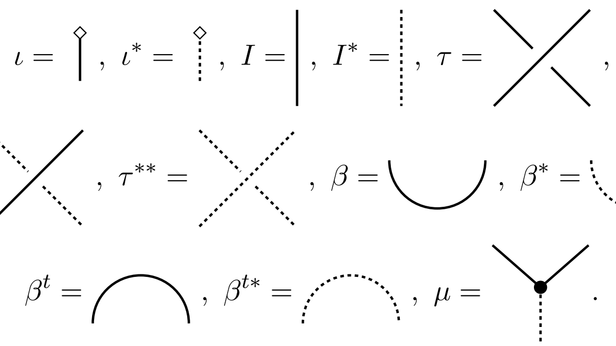
and the algebra $A = (V, m)$ is called *unital* if it admits a unit.

We introduce the *unit ribbon graph* ι representing the morphism $1\colon F \to V$. From Example 5.9 follows that it is graphically depicted as the ribbon $(0,1)$-graph

$\iota =$  .

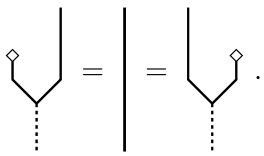
Let *Alph* be the alphabet $\{1_V, \tau, b, b^t, m, 1\}$ in $\mathcal{V}$. The corresponding alphabet $BiAlph^*$ in $\mathrm{BiRib}_{\mathcal{V}}$ is:

(28.5)



The unitality relation (28.4) is represented by

(28.6)



Note that here, even if the multiplication in A is not (necessarily) commutative, the orientation of the trivalent vertex is irrelevant.

By replacing the undotted line by one of the graphs containing the unit, one obtains two bands with different colored ends; hence identity (28.6) allows to glue undotted with dotted lines. Moreover, we can always replace dotted with normal lines (and conversely) in graphs without trivalent vertices: it suffices to exchange the color χ with $-\chi$ (the color χ has the unique function to differentiate the result of the multiplication in the graph μ). This allows to overcome the difficulty mentioned in Remark 28.2 if A admits a unit.

29 Unital Jordan algebras of degree 2

We consider unital Jordan algebras $J = (V, m)$ of degree 2, i.e., Jordan algebras with a generic minimal polynomial

$$P_{J,x}(X) = X^2 - T_J(x)X + N_J(x)1.$$

T_J is the generic trace and N_J the generic norm. We denote the product by $m(x \otimes y) = x \cdot y$ and choose as a bilinear form the trace form $T_J(x, y) = T_J(x \cdot y)$. Linearizing the generic polynomial and taking traces shows that

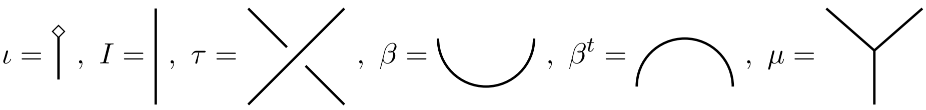
$$2b_{N_J}(x, y) = T_J(x)T_J(y) - T_J(x \cdot y), \tag{29.1}$$

where b_{N_J} is the polar of N_J, and

$$x \cdot y = \frac{1}{2}\big(T_J(y)x + T_J(x)y + T_J(x, y)1 - T_J(x)T_J(y)1\big). \tag{29.2}$$

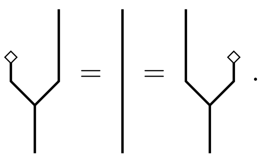
It follows from (29.1) that the trace form is symmetric and it is easy to deduce from (29.2) that the trace form is associative with respect to the product $(x, y) \mapsto x \cdot y$.

Jordan algebras are commutative by definition and T_J is associative with respect to the multiplication. Graphically, this means that we can avoid using bicolored ribbon graphs and vertex orientations. The alphabet *Alph** is given by the basic graphs



and the existence of a unit is represented by

(29.3)

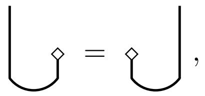


Let $(V, T_J, m, 1)$ be a commutative unital algebra satisfying the relation (29.2), i.e., a Jordan algebra of degree 2. Let $R = \mathbb{Q}$ and and let $\mathrm{Rib}^+_{\mathcal{V}_1}$ the associated category of ribbon graphs. Lifting the relation (29.2) to a relation in $\mathrm{Rib}^+_{\mathcal{V}_1}$ and going modulo this relation, we get the category Rib_{2Jord} associated to Jordan algebras of degree 2. Let $E_{2Jord} = \mathrm{Mor}_{\mathrm{Rib}_{2Jord}}(\emptyset, \emptyset)$ be its ring of numerical invariants.

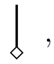
Theorem 29.4. *There is a unique functor* $\mathcal{R}\colon \mathrm{Rib}_{2Jord} \to \mathcal{V}$ *sending tensor product to tensor product and such that*

$$\mathcal{R}(\iota) = 1, \qquad \mathcal{R}(I) = 1_V, \qquad \mathcal{R}(\tau) = \tau,$$
$$\mathcal{R}(\beta) = T_J, \qquad \mathcal{R}(\beta^t) = {T_J}^t, \qquad \mathcal{R}(\mu) = m.$$

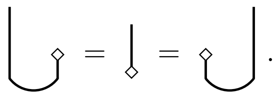
As before, we identify graphs whose images under $\mathcal{R}$ are equal. Since $T_J(x,1) = T_J(1,x)$ we have



and we shall denote the graph corresponding to the morphism $T_J\colon V \to F$, $x \mapsto T_J(x,1)$, as

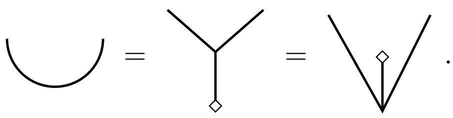
(29.5) 

so that:



Observe that the graph (29.5) is the transpose of the unity ι.

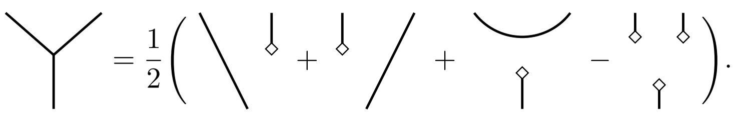
Lemma 29.6. *The relations*



hold in the category Rib_{2Jord}.

Proof. The first equality is a direct consequence of $T_J(x,y) = T_J(x \cdot y) = T_J\big((x \cdot y) \cdot 1\big) = T_J(x \cdot y, 1)$. The second equality follows from the unitality condition $m \circ (1 \otimes 1_V) = m \circ (1_V \otimes 1)$, graphically depicted in (29.3). Applying an updown transformation to (29.3) shows that the first graph is equal to the third one. □

Equation (29.2) is then drawn as

(29.7) 

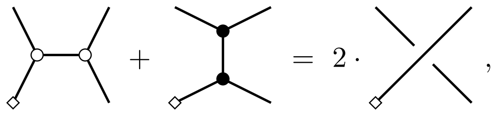
Theorem 29.8. *The ring of numerical invariants E_{2Jord} is isomorphic to the polynomial ring $\mathbb{Q}[\bar{d}]$.*

Proof. In view of the relation (29.7) any graph containing a trivalent vertex can be replaced by graphs without trivalent vertices. Thus E_{2Jord} is generated by d and we have a surjective map $\mathbb{Q}[\bar{d}] \to E_{2Jord}$, $\bar{d} \mapsto d$. Since there are Jordan algebras of degree 2 in any dimension ≥ 2, namely Jordan algebras of quadratic forms, see Example 22.2, the map is also injective. □

30 Unital symmetric composition algebras

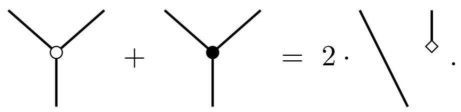
In this section we show graphically that the field F is the only unital symmetric composition algebra. Axiom (17.5) and all relations established in the proof of Theorem 17.6 are still valid. We recompute some relations, using the existence of a unit $1 \colon F \to V$.

We start by composing the graph $(\beta \square I) \circ \big(\iota \ \square (17.5)\big)$. We obtain

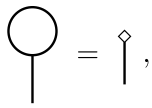


and the unitality condition for graphs yields

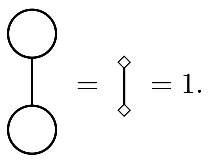
(30.1)



It readily follows from $(30.1) \circ \beta^t$ that



and this yields



We set $e = 1$ in the polynomials

$$\begin{aligned} 0 &= e\big(e - (2-d)^2\big), \\ 0 &= (d-2)(d-8)(d-e), \end{aligned}$$

(see Theorem 17.6) and find that $(d, e) = (1, 1)$ is the only solution. Thus, symmetric composition algebras of dimension ≥ 2 never admit an identity.

31 Unital composition algebras

By the Hurwitz Theorem unital composition algebras C have dimension 1, 2, 4 or 8. An indirect diagrammatic proof was given in Chapter III by determining the possible dimensions of vector product algebras and using the relation $\dim_F C = 1 + \dim_F VPA$. In this section we give a direct proof, which illustrates the diagrammatic calculus for unital algebras.

We recall that a unital composition algebra C is a triple (V, m, q) where m is a bilinear multiplication $m\colon V\otimes V \to V$ on V, $m(x\otimes y) = xy$, with a unit 1, and $q\colon V \to F$ is a nonsingular multiplicative quadratic form, i.e., satisfying $q(xy) = q(x)q(y)$ for all $x, y \in V$. Let b be the polar of q. Linearizing $q(xy) = q(x)q(y)$ gives

$$b(xz, uy) + b(xy, uz) = 2b(x, u)b(y, z). \tag{31.1}$$

The bilinear form b is not associative. Thus for diagrammatic calculus we have to work with bicolored graphs and oriented trivalent vertices. We use the alphabet $BiAlph^*$ given in (28.5).

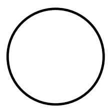
Let $R = \mathbb{Q}$ and let $\mathrm{Bi\,Rib}^+_{\mathcal{V}_1}$ be the corresponding category of ribbon graphs. We define BiRib_{CA} to be the category of bicolored ribbon graphs associated to the relations induced by (28.4) and (31.1). Denote by $E_{CA} = \mathrm{Mor}_{\mathrm{BiRib}_{CA}}(\emptyset, \emptyset)$ the ring of numerical invariants of the category BiRib_{CA}.

Theorem 31.2. *Let $C = (V, m, b)$ be a unital composition algebra. There is a unique functor $\mathcal{R}\colon \mathrm{BiRib}_{CA} \to \mathcal{V}$ sending a bicolored object (V, χ) to V, tensor product to tensor product and μ to m. For the other basic graphs of $BiAlph^*$ we have:*

$$\mathcal{R}(I) = \mathcal{R}(I^*) = 1_V, \qquad \mathcal{R}(\tau) = \mathcal{R}(\tau^*) = \mathcal{R}(\tau^{**}) = \tau,$$
$$\mathcal{R}(\beta) = \mathcal{R}(\beta^*) = b, \qquad \mathcal{R}(\beta^t) = \mathcal{R}(\beta^{t*}) = b^t, \qquad \mathcal{R}(\iota) = \mathcal{R}(\iota^*) = 1.$$

Moreover, the functor $\mathcal{R}$ is compatible with updown transformations.

Again, the functor $\mathcal{R}$ induces a ring homomorphism $E_{CA} \to F$, and we set

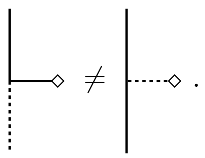
$d =$  .

Theorem 31.3. *Let J be the ideal of the polynomial ring $\mathbb{Q}[\bar{d}]$, generated by the element $p(\bar{d}) = (\bar{d} - 1)(\bar{d} - 2)(\bar{d} - 4)(\bar{d} - 8)$. Then, the homomorphism $\Phi\colon \mathbb{Q}[\bar{d}] \to E_{CA} = \mathrm{Mor}_{\mathrm{BiRib}_{CA}}(\emptyset, \emptyset)$, given by $\Phi(\bar{d}) = d$, induces an isomorphism*

$$\bar{\Phi}\colon \mathbb{Q}[\bar{d}]/J \to E_{CA}.$$

In particular the only possible values of the invariant d are 1, 2, 4 and 8.

Remark 31.4. Note that



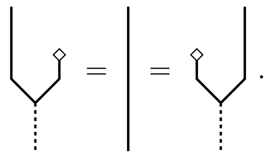
In fact, the image of the graph on the right under the functor $\mathcal{R}$ is the morphism $\vartheta = (1_V \otimes b)\circ(1_V \otimes m \otimes 1)\circ(\beta^t \otimes 1_V)$. Then $\vartheta(x) = \sum_i b(e_i x, 1)e_i \neq x$, for b not associative. Relation (32.10) on page 115 describes explicitely the difference between the two graphs.

32 Graphical proof of Hurwitz Theorem

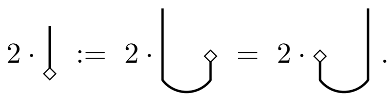
This section is dedicated to the proof of Theorem 31.3. The proof is similar to the proof of the corresponding result for vector products or symmetric compositions with the added difficulty of the bicoloring of the graphs. We only prove injectivity of the map $\bar{\Phi}$. Surjectivity follows from the fact that d generates E_{CA} and this can also be proved as for vector products or symmetric compositions.

We recall the unitality condition:

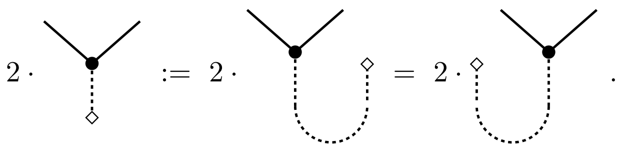
(32.1)



As in Section 29, the trace $T_C(x) = 2b(x,1) = 2b(1,x)$ of an element $x \in C$ is represented by the ribbon $(1,0)$-graph

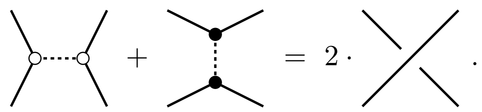


Analogously, the trace $T_C(xy)$ is represented by the $(2,0)$-graph

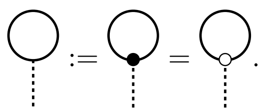


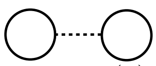
The axioms defining composition algebras with identity are given by (32.1) and

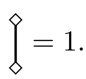
(32.2)



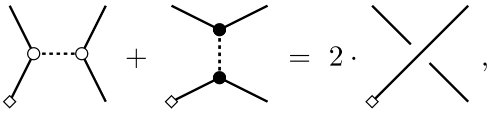
Trivially the following rules are also valid:



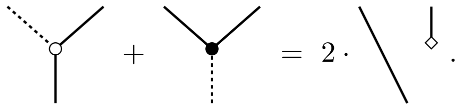
Thus the invariant $e =$  does not have orientations. Moreover, $q(x) = q(1x) = q(1)q(x)$ yields $q(1) = b(1,1) = 1$. Hence



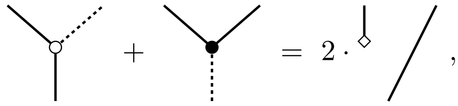
It follows from (32.2) that



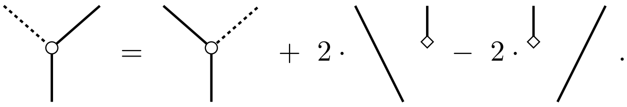
and (32.1) yields

(32.3) 

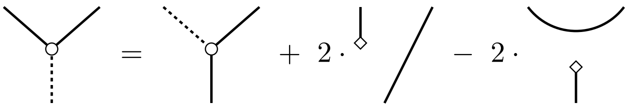
Similarly, by composing $(I\Box\beta)\circ\big((32.2)\Box\iota\big)$ we find

(32.4) 

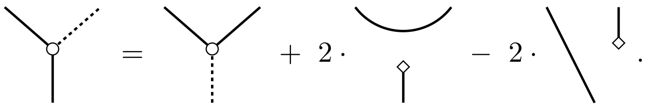
and subtracting the two relations gives the useful rule



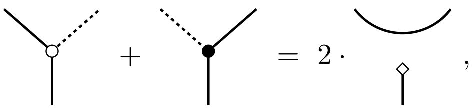
Then

(32.5) 

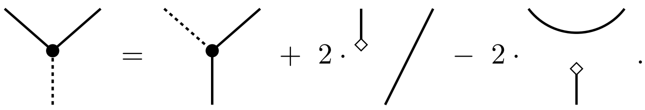
and

(32.6) 

We next rotate the three vertices of (32.3) by a 120° degree clockwise orientation:

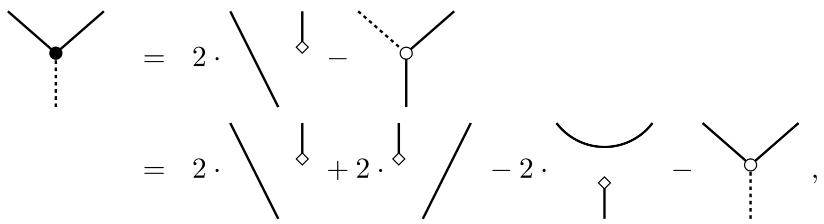


and (32.4) implies

(32.7) 

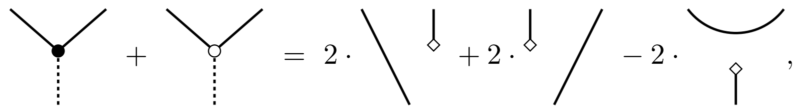
This means that (32.5) holds for positive and negative orientations.

Remark 32.8. Using (32.3) and (32.5) we obtain



that is

(32.9)

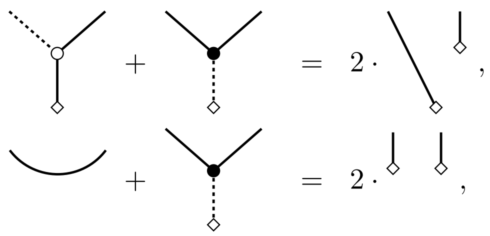


which is the graphical incarnation of the algebraic relation

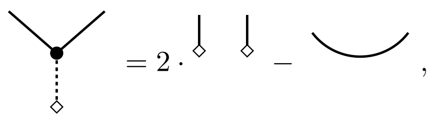
$$xy + yx = T_C(y)x + T_C(x)y - b(x, y)1.$$

This gives a diagrammatic proof of the fact that the elements x of a composition algebra satisfy the quadratic equation $x^2 - 2T_C(x)x + q(x) = 0$.

Equations (32.1) and (32.3) imply

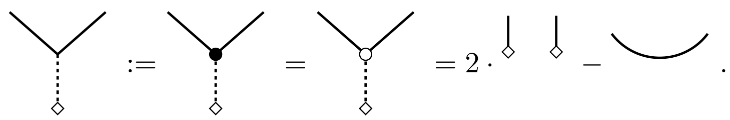


so that

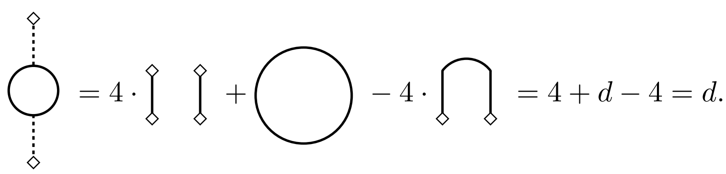


which corresponds to the formula $b(xy, 1) = b(x, \bar{y})$. The last equation composed with τ yields the same relation but with a white trivalent vertex on the left. We get:

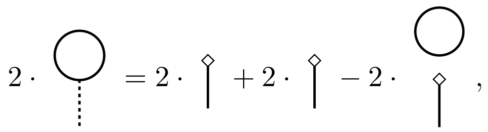
(32.10)



Composing (32.10) with itself leads to

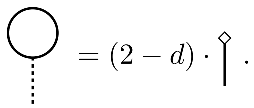


Equation (32.9) $\circ\, \beta^t$ becomes



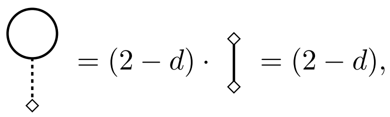
hence

(32.11)

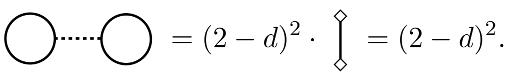


This implies

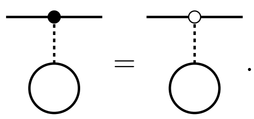
(32.12)



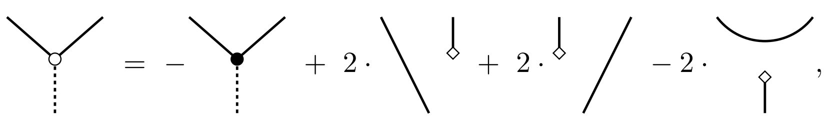
and



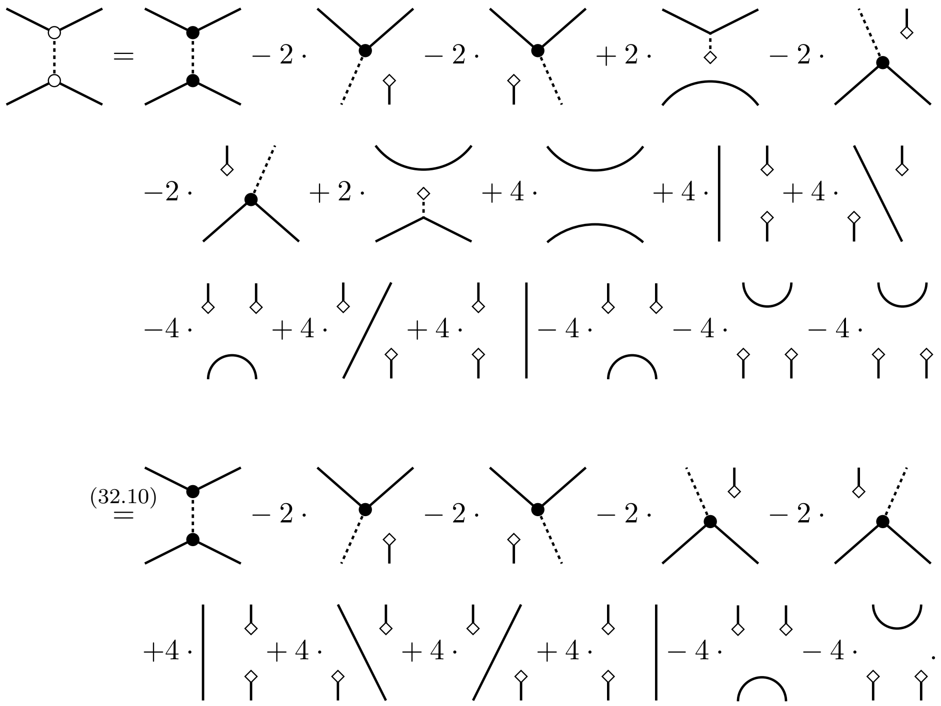
The next step is to show that



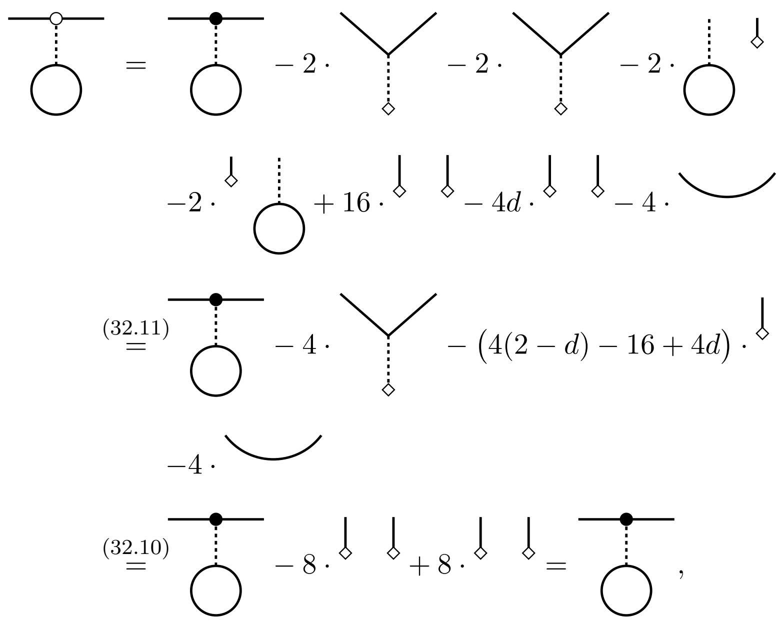
We begin with relation (32.9)



which composed with itself gives

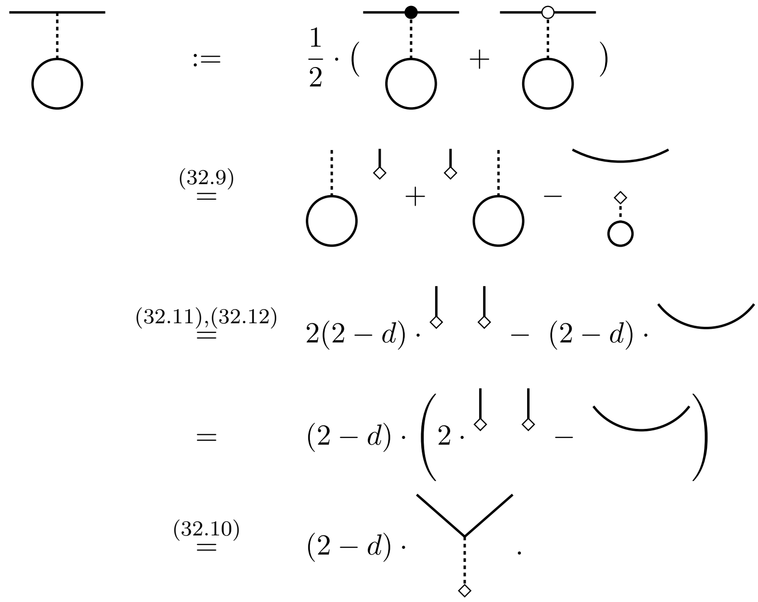


This yields



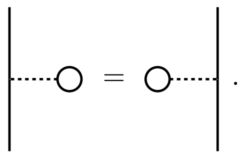
as wanted.

This permits us to find another relation involving the last graph, since

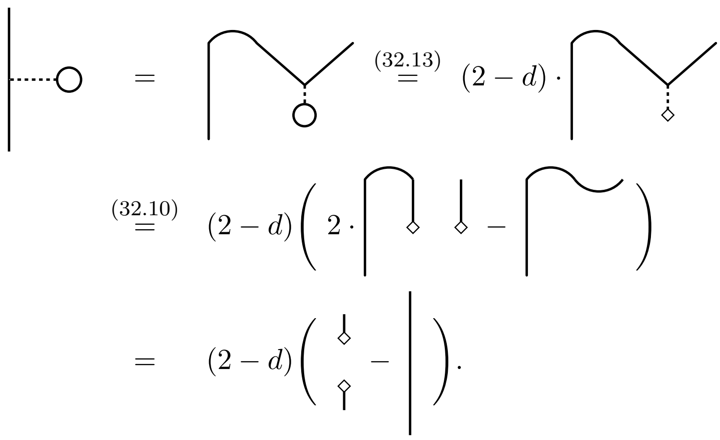


(32.13)

Lemma 32.14. *We have*

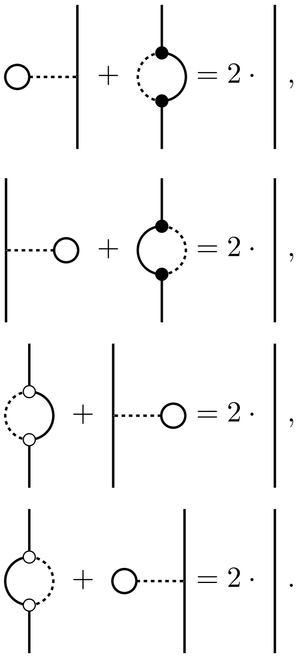


Proof. We compute the first graph:

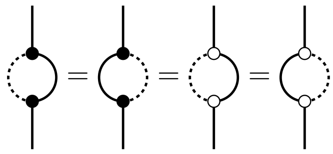


Similar steps applied to the second graph of the lemma lead to the same formula. □

We come to the last steps of the proof. We start with the defining axiom of unital composition algebras (32.2) and collate its bands in four different ways (the two on the left, on the right, on the top and on the bottom). We obtain the four relations

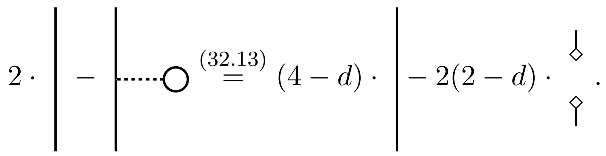


The last lemma implies that

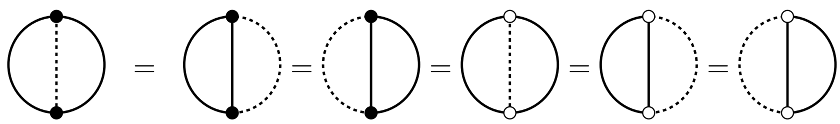


and these diagrams are all equal to

(32.15)

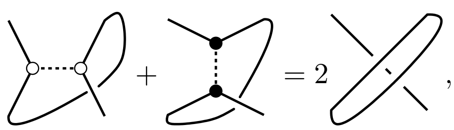


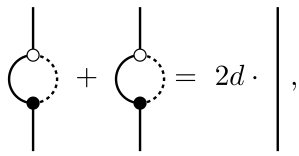
Hence

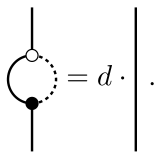


$$= (4-d)d - 2(2-d) = 6d - 4 - d^2.$$

We now glue two bands of axiom (32.2) diagonally. A first example shows the left band on the bottom glued with the right band on the top:

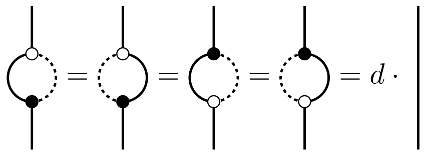




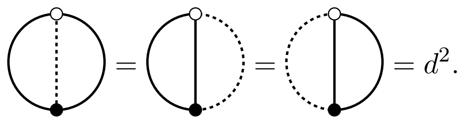


Analogously,

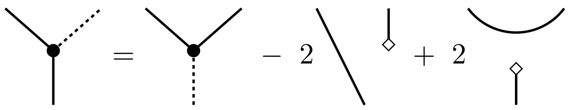
(32.16)



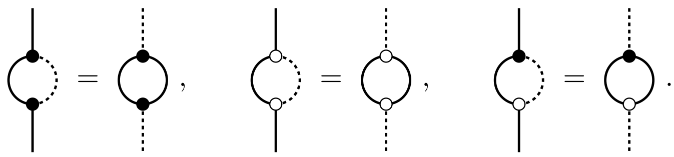
and



Remark 32.17. We compose relation (32.6) (with black vertices)

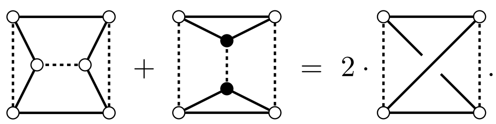


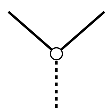
with its equivalent ribbon $(1, 2)$-graph. We obtain

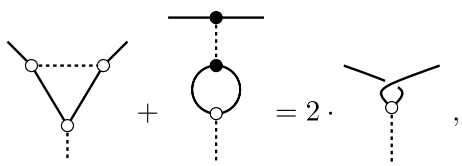


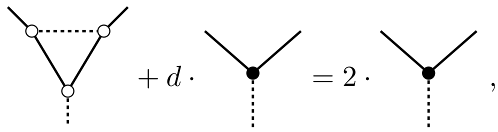
The final step of the proof is computing the three invariants obtained from axiom (32.2):

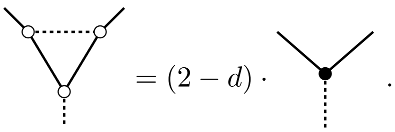
(32.18)



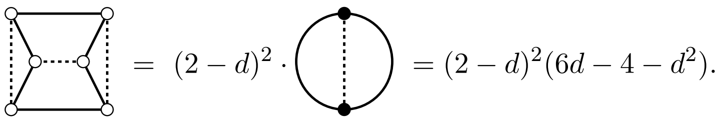
We start with the left one. Compose axiom (32.2) with  :

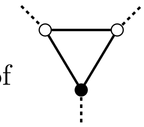
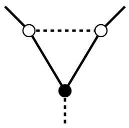




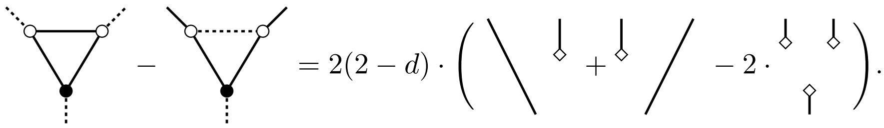
(32.19) 

With the help of updown transformations, we modify equation (32.19) into two relations of ribbon $(0,3)$-graphs and ribbon $(3,0)$-graphs, then collate them:

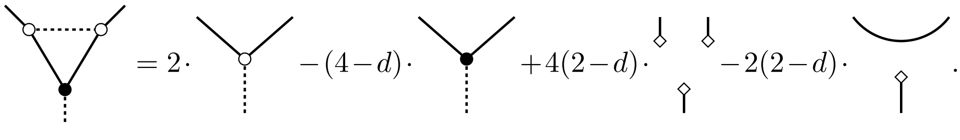


The second invariant of (32.18) is computed as the composition of  with itself. One can erroneously think that this graph is equal to the graph . They are slightly different as shown in next lemma.

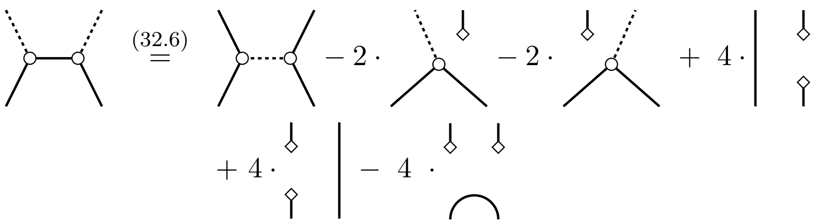
Lemma 32.20.

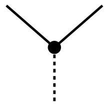


Proof. As for (32.19), we have

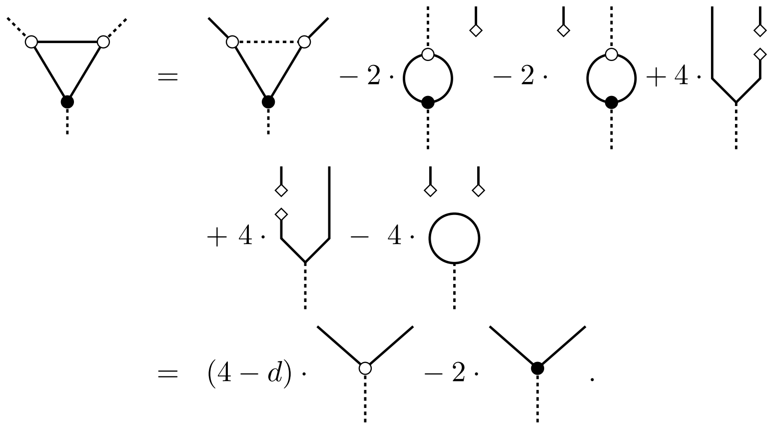


The dotted edges of the first graph must be on the boundary and not in the middle of the graph. We must compute



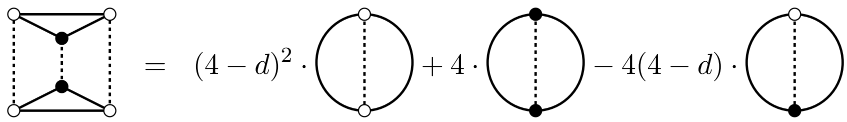
and collate it with  to obtain, with the help of (32.9), (32.11), (32.15), (32.16), Remark (32.17) and axiom (32.1):

(32.21)



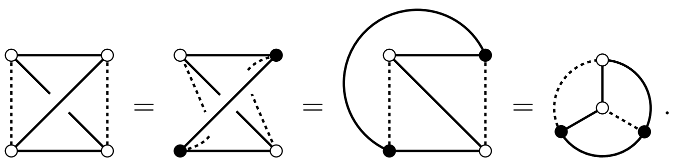
The lemma follows immediately. □

From (32.21) we obtain the computation for the second invariant

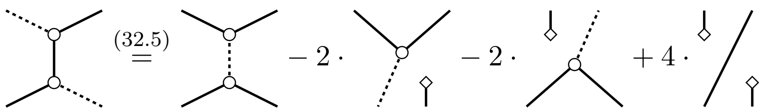


$$= \big((4-d)^2+4\big)(6d-4-d^2) - 4(4-d)d^2$$
$$= (2-d)(d^3-16d^2+56d-40).$$

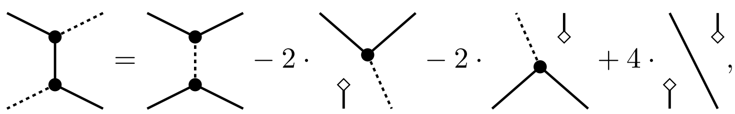
It only remains to compute the third invariant on the right of (32.18). We start by remarking that



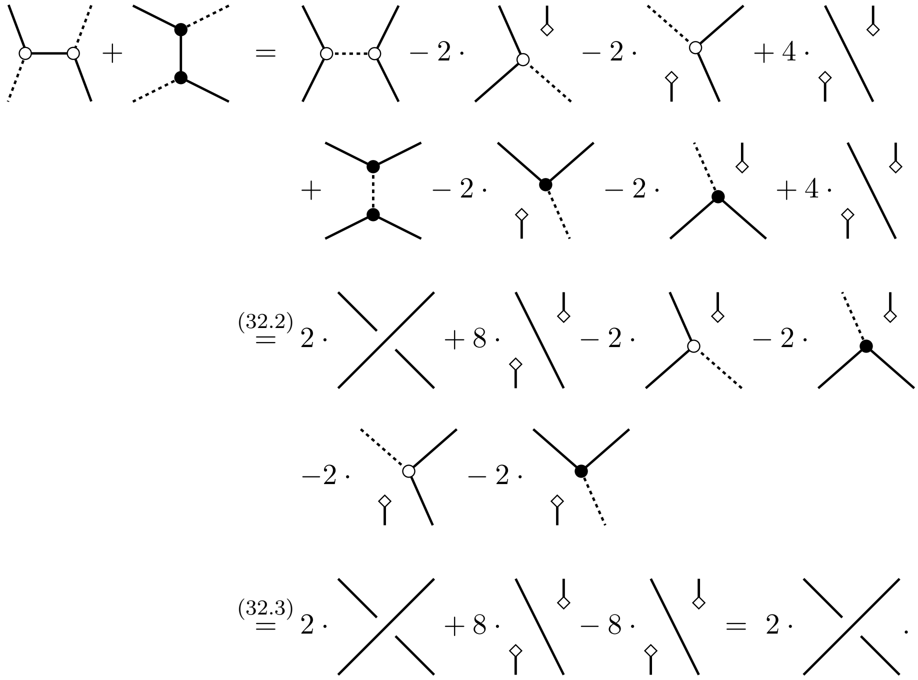
Since



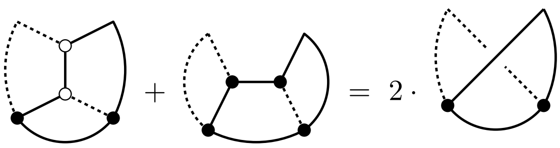
and analogously



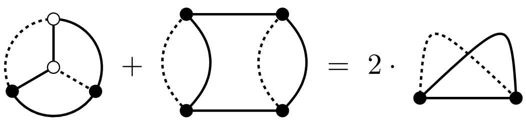
we find



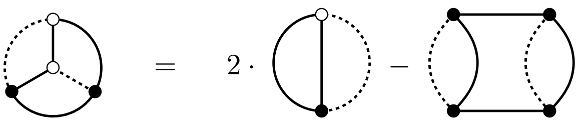
Closing the last relation with two trivalent vertices as shown



gives

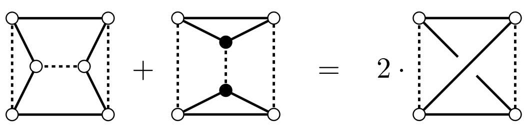


and hence



$$\overset{(32.15)}{=} 2d^2 - \big((4-d)(6d-4-d^2) - 2(2-d)d\big)$$
$$= (2-d)(d^2-8d+8).$$

So far we have computed the three invariant graphs of the equation



$$\begin{aligned} 2(2-d)(d^3-12d^2+36d-24) &= 2(2-d)(d^2-8d+8) \\ (d-1)(d-2)(d-4)(d-8) &= 0, \end{aligned}$$

which has solutions

$$d = 1, 2, 4 \text{ and } 8.$$

Since all possible dimensions can be realized over $\mathbb{Q}$ we have injectivity for $\bar{\Phi}$. As already mentioned we renounce to a proof of the surjectivity. The last claim of the theorem follows from Corollary 11.4.

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Curriculum Vitae

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