A new approach to LIBOR modeling

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Based on joint work with Antonis Papapantoleon and Josef Teichmann

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Interest rates – Notation

- $T_1 \leq T_2 \leq \cdots \leq T_N = T$: fixed *tenor structure* with uniform spacing $\delta = T_{k+1} - T_k$.
- $B(t, T_k)$: time-t price of a zero coupon bond maturing at T_k;
- $L(t, T_k)$: time-t forward LIBOR for $[T_k, T_{k+1}]$;
- $F(t, T_k, T_l)$: time-t forward price for T_k and T_l;

Fundamental Relationship

$$F(t, T_k, T_{k+1}) = \frac{B(t, T_k)}{B(t, T_{k+1})} = 1 + \delta L(t, T_k)$$

- Forward measure P_{T_k}: Corresponds to using $B(t, T_k)$ as numeraire.
- Terminal measure P_T: equals P_{T_N}; $B(t, T)$ is the numeraire.

LIBOR model: Axioms I

Economic thought (absence of arbitrage) dictates:

Axiom 1

The LIBOR rate should be non-negative, i.e. $L(t, T_k) \geq 0$ *for all* t.

Axiom 2

The LIBOR rate process should be a martingale under the corresponding forward measure, i.e. $L(\cdot, T_k) \in \mathcal{M}(P_{T_{k+1}})$.

Practical applications require:

Axiom 3

Models should be analytically tractable ($\rightsquigarrow$ *fast calibration*).

Models should have rich structural properties ($\rightsquigarrow$ *good calibration*).

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LIBOR model: Axioms II

What axioms do the existing models satisfy?

- LIBOR models
- Forward Price models

LIBOR models I (Sandmann et al, Brace et al, . . . , Eberlein & Özkan)

Ansatz: model the LIBOR rate as the exponential of a semimartingale H (with tractable structure under P_T).

The change of measure from the terminal to the forward measures is of a complicated functional form; the model structure for simple models (e.g. H a Levy process) is not preserved.

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Consequences for continuous semimartingales:

1. Caplets can be priced in closed form or by Fourier methods;
2. swaptions and multi-LIBOR products cannot be priced with Fourier methods;
3. Monte-Carlo pricing is very time consuming $\rightsquigarrow$ coupled high dimensional SDEs!

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Consequences for general semimartingales:

1. Even caplets cannot be priced with Fourier methods!
2. ditto for Monte-Carlo pricing.

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The change of measure from the terminal measure to the forward measure is of a simple functional form (Esscher transform).

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So, what is wrong?

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Negative LIBOR rates can occur!

Towards the Affine LIBOR Model

Aim: *design a model where the model structure is preserved and LIBOR rates are positive.*

Tool: *Affine processes on* $\mathbb{R}^d_{\geqslant 0}$.

Affine processes I

Model Setup:

- $X = (X_t)_{0 \leq t \leq T}$: A time-homogeneous Markov process
- X takes values in $D = \mathbb{R}^d_{\geqslant 0}$
- X is called affine, if its moment generating function satisfies

$$E_x\big[\exp\langle u, X_t\rangle\big] = \exp\big(\phi_t(u) + \langle\psi_t(u), x\rangle\big)$$

for some functions $\phi_t(u)$, $\psi_t(u)$ taking values in $\mathbb{R}$ and $\mathbb{R}^d$ respectively.

Affine processes II

- $\phi_t(u)$ and $\psi_t(u)$ are defined on $[0, T] \times \mathcal{I}_T$, where

$$\mathcal{I}_T := \left\{ u \in \mathbb{R}^d : E_x\left[e^{\langle u, X_T \rangle}\right] < \infty, \text{ for all } x \in D \right\}$$

  the 'domain of finite exponential moments'.
- Additional technical assumption: $0 \in \mathcal{I}_T^\circ$;

The above definition includes the CIR-process, many Levy subordinators, Non-Gaussian Ornstein-Uhlenbeck processes, and multivariate extensions of these.

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Affine processes III

The process X is also a regular affine process in the sense of Duffie, Filipovic and Schachermayer [2003], and in particular a semi-martingale.

We can show that the derivatives

$$F(u) := \frac{\partial}{\partial t}\Big|_{t=0+}\phi_t(u) \qquad \text{and} \qquad R(u) := \frac{\partial}{\partial t}\Big|_{t=0+}\psi_t(u)$$

exist for all $u \in \mathcal{I}_T$ and are continuous in u. Moreover, F and R satisfy the Lévy–Khintchine-type equations:

$$F(u) = \langle b, u\rangle + \int_D \left(e^{\langle \xi, u\rangle} - 1\rangle\right) m(d\xi);$$

$$R_i(u) = \langle \beta_i, u\rangle + \left\langle \frac{\alpha_i}{2} u, u\right\rangle + \int_D \left(e^{\langle \xi, u\rangle} - 1 - \langle u, h^i(\xi)\rangle\right)\mu_i(d\xi),$$

where $(b, m, \alpha_i, \beta_i, \mu_i)_{1\le i\le d}$ are admissible parameters.

Affine processes IV

Lemma (generalized Riccati equations)

The functions ϕ *and* ψ *satisfy the* **generalized Riccati equations***:*

$$\begin{aligned} \frac{\partial}{\partial t}\phi_t(u) &= F(\psi_t(u)), \qquad & \phi_0(u) &= 0\ , \\ \frac{\partial}{\partial t}\psi_t(u) &= R(\psi_t(u)), \qquad & \psi_0(u) &= u\ , \end{aligned}$$

for all $t \in [0, T]$ *and* $u \in \mathcal{I}_T$*.*

Affine LIBOR model: martingales $\geqslant 1$

Construction of the martingale M_t^u

Let $u \in \mathbb{R}^d_{\geqslant 0} \cap \mathcal{I}_T$, and define

$$M_t^u = E\left[e^{\langle X_T, u\rangle} \middle| \mathcal{F}_t \right] , \qquad (t \in [0,T]) \ :$$

1. M_t^u is a martingale:

$$\mathbb{E}\left[M_t^u \middle| \mathcal{F}_r \right] = \mathbb{E}\left[\mathbb{E}\left[e^{X_T \cdot u} \middle| \mathcal{F}_t \right] \middle| \mathcal{F}_r \right] = \mathbb{E}\left[e^{X_T \cdot u} \middle| \mathcal{F}_r \right] = M_r^u \, .$$

2. M_t^u is greater than 1.
3. Due to the affine property of X, M_t^u is of the tractable form

$$M_t^u = \exp\left(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle \right) .$$

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Affine LIBOR model: Ansatz

Discounted bond prices must satisfy:

$$\frac{B(\cdot, T_k)}{B(\cdot, T_N)} \in \mathcal{M}(P_{T_N}), \qquad \text{for all } k \in \{1, \dots, N-1\}.$$

The Affine LIBOR Model

We model quotients of bond prices using the martingales M:

$$\frac{B(t, T_1)}{B(t, T_N)} = M_t^{u_1}$$

$$\vdots$$

$$\frac{B(t, T_{N-1})}{B(t, T_N)} = M_t^{u_{N-1}},$$

with initial conditions: $\frac{B(0,T_k)}{B(0,T_N)} = M_0^{u_k}$, for all $k \in \{1, \dots, N-1\}$.

Affine LIBOR model: initial values

Proposition

Let $L(0,T_1),\dots,L(0,T_N)$ be a tenor structure of non-negative initial LIBOR rates; let X be an affine process on $\mathbb{R}^d_{\geqslant 0}$.

1. *If*[1] $\gamma_X > \frac{B(0,T_1)}{B(0,T_N)}$, *then there exists a decreasing sequence* $u_1 \geq u_2 \geq \cdots \geq u_N = 0$ *in* $\mathcal{I}_T \cap \mathbb{R}^d_{\geqslant 0}$, *such that*

$$M_0^{u_k} = \frac{B(0,T_k)}{B(0,T_N)}, \qquad \textit{for all } k \in \{1,\dots,N\}.$$

   In particular, if $\gamma_X = \infty$, *the affine LIBOR model can fit any term structure of non-negative initial LIBOR rates.*

2. *If X is one-dimensional, the sequence* $(u_k)_{k\in\{1,\dots,N\}}$ *is unique.*
3. *If all initial LIBOR rates are positive, the sequence* $(u_k)_{k\in\{1,\dots,N\}}$ *is strictly decreasing.*

---

[1] $\gamma_X := \sup_{u\in\mathcal{I}_T\cap\mathbb{R}^d_{>0}} E_x\left[e^{\langle u,X_T\rangle}\right]$

Affine LIBOR model: forward prices

Forward prices have the following exponential-affine form

$$\frac{B(t,T_k)}{B(t,T_{k+1})} = \frac{B(t,T_k)}{B(t,T_N)}\frac{B(t,T_N)}{B(t,T_{k+1})} = \frac{M_t^{u_k}}{M_t^{u_{k+1}}}$$
$$= \exp\Big(\phi_{T_N-t}(u_k) - \phi_{T_N-t}(u_{k+1}) + \big\langle \psi_{T_N-t}(u_k) - \psi_{T_N-t}(u_{k+1}), X_t\big\rangle\Big).$$

Now, $\phi_t(\cdot)$ and $\psi_t(\cdot)$ are order-preserving, i.e.

$$u \geq v \;\Rightarrow\; \phi_t(u) \geq \phi_t(v) \;\text{ and }\; \psi_t(u) \geq \psi_t(v).$$

Consequently: positive initial LIBOR rates lead to positive LIBOR rates for all times.

Affine LIBOR model: forward measures

Forward measures are related via:

$$\frac{dP_{T_k}}{dP_{T_{k+1}}}\bigg|_{\mathcal{F}_t} = \frac{F(t, T_k, T_{k+1})}{F(0, T_k, T_{k+1})} = \frac{B(0, T_{k+1})}{B(0, T_k)} \times \frac{M_t^{u_k}}{M_t^{u_{k+1}}}$$

or equivalently:

$$\frac{dP_{T_{k+1}}}{dP_{T_N}}\bigg|_{\mathcal{F}_t} = \frac{B(0, T_N)}{B(0, T_{k+1})} \times \frac{B(t, T_{k+1})}{B(t, T_N)} = \frac{B(0, T_N)}{B(0, T_{k+1})} \times M_t^{u_{k+1}}.$$

Hence, we can easily see that

$$\frac{B(\cdot, T_k)}{B(\cdot, T_{k+1})} = \frac{M^{u_k}}{M^{u_{k+1}}} \in \mathcal{M}(P_{T_{k+1}}) \quad \text{since} \quad M^{u_k} \in \mathcal{M}(P_{T_N}).$$

Affine LIBOR model: dynamics under forward measures

The moment generating function of X_t under any forward measure is again of exponential-affine form…

$$
\begin{aligned}
E_{P_{T_{k+1}}}\left[e^{vX_t}\right] &= M_0^{u_{k+1}} E_{P_{T_N}}[M_t^{u_{k+1}} e^{vX_t}] \\
&= \exp\Big(\phi_t\big(\psi_{T_N-t}(u_{k+1})+v\big) - \phi_t(\psi_{T_N-t}(u_{k+1})) \\
&\qquad + \big\langle \psi_t\big(\psi_{T_N-t}(u_{k+1})+v\big) - \psi_t(\psi_{T_N-t}(u_{k+1})), x\big\rangle\Big),
\end{aligned}
$$

hence X is a time-inhomogeneous affine process under any $P_{T_{k+1}}$.

Note also the "Esscher structure" of the measure change $\frac{dP_{T_{k+1}}}{dP_{T_N}}$.

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hence X is a time-inhomogeneous affine process under any $P_{T_{k+1}}$. Note also the "Esscher structure" of the measure change $\frac{dP_{T_{k+1}}}{dP_{T_N}}$.

Moreover, denote by $\frac{M_t^{u_k}}{M_t^{u_{k+1}}} = e^{A_k + B_k \cdot X_t}$; then

$$E_{P_{T_{k+1}}}\left[e^{v(A_k+B_k\cdot X_t)}\right] = \frac{B(0,T_N)}{B(0,T_{k+1})} \exp\big(A'_k + \langle B'_k, x\rangle\big), \qquad (1)$$

where A'_k and B'_k are explicitly known in terms of ϕ and ψ.

Affine LIBOR model: caplet pricing (Fourier-methods)

We can re-write the payoff of a caplet as follows (here $\mathcal{K} := 1 + \delta K$):

$$\begin{aligned}\delta(L(T_k, T_k) - K)^+ &= (1 + \delta L(T_k, T_k) - 1 + \delta K)^+ \\ &= \left(\frac{M^{u_k}_{T_k}}{M^{u_{k+1}}_{T_k}} - \mathcal{K} \right)^+ = \left(e^{A_k + B_k \cdot X_{T_k}} - \mathcal{K} \right)^+ .\end{aligned} \tag{2}$$

Then we can price caplets by Fourier-transform methods:

$$\begin{aligned}\mathbb{C}(T_k, K) &= B(0, T_{k+1}) E_{P_{T_{k+1}}} \left[\delta(L(T_k, T_k) - K)^+ \right] \\ &= \frac{\mathcal{K} B(0, T_{k+1})}{2\pi} \int_{\mathbb{R}} \mathcal{K}^{iv-R} \frac{\Lambda_{A_k + B_k \cdot X_{T_k}}(R - iv)}{(R - iv)(R - 1 - iv)} dv\end{aligned} \tag{3}$$

where $\Lambda_{A_k + B_k \cdot X_{T_k}}$ is given by (1).

Similar formula for swaptions (1D affine process).

Affine LIBOR model: caplet pricing (closed form)

In some cases even closed-form valuation of caplets is possible!

Suppose that...

- The distribution function of X_t is known, and
- belongs to an exponential family of distributions,

then caplets can be priced by closed form.

Here is an example...

CIR martingales

The Cox-Ingersoll-Ross (CIR) process is given by

$$dX_t = -\lambda\left(X_t - \theta\right)dt + 2\eta\sqrt{X_t}dW_t, \quad X_0 = x \in \mathbb{R}_{\geqslant 0},$$

where $\lambda, \theta, \eta \in \mathbb{R}_{\geqslant 0}$. This is an affine process on $\mathbb{R}_{\geqslant 0}$, with

$$E_x\left[e^{uX_t}\right] = \exp\Big(\phi_t(u) + x \cdot \psi_t(u)\Big),$$

where

$$\phi_t(u) = -\frac{\lambda\theta}{2\eta}\log\left(1 - 2\eta b(t)u\right) \quad \text{and} \quad \psi_t(u) = \frac{a(t)u}{1 - 2\eta b(t)u},$$

with

$$b(t) = \begin{cases} t, & \text{if } \lambda = 0 \\ \frac{1-e^{-\lambda t}}{\lambda}, & \text{if } \lambda \neq 0 \end{cases}, \qquad \text{and} \qquad a(t) = e^{-\lambda t}.$$

CIR martingales: closed-form formula I

Definition

A random variable Y has location-scale extended non-central chi-square distribution, $Y \sim \mathrm{LSNC}{-}\chi^2(\mu, \sigma, \nu, \alpha)$, if $\frac{Y-\mu}{\sigma} \sim \mathrm{NC}{-}\chi^2(\nu, \alpha)$

We have that

$$X_t \overset{P_{T_N}}{\sim} \mathrm{LSNC}{-}\chi^2\left(0, \eta b(t), \frac{\lambda\theta}{\eta}, \frac{xa(t)}{\eta b(t)}\right) ,$$

and

$$X_t \overset{P_{T_{k+1}}}{\sim} \mathrm{LSNC}{-}\chi^2\left(0, \frac{\eta b(t)}{\zeta(t, T_N)}, \frac{\lambda\theta}{\eta}, \frac{xa(t)}{\eta b(t)\zeta(t, T_N)}\right),$$

hence

$$\log\left(\frac{B(t, T_k)}{B(t, T_{k+1})}\right) \overset{P_{T_{k+1}}}{\sim} \mathrm{LSNC}{-}\chi^2\left(A_k, \frac{B_k \eta b(t)}{\zeta(t, T_N)}, \frac{\lambda\theta}{\eta}, \frac{xa(t)}{\eta b(t)\zeta(t, T_N)}\right).$$

CIR martingales: closed-form formula II

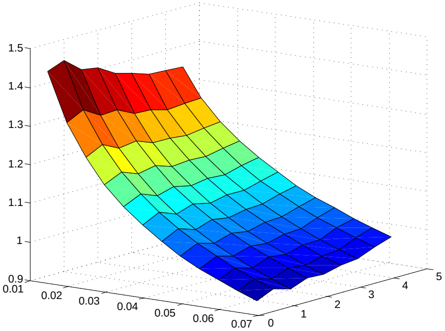
Then, denoting by $M = \log\left(\frac{B(T_k,T_k)}{B(T_k,T_{k+1})}\right)$ the log-forward rate, we arrive at:

$$\begin{aligned}
\mathbb{C}(T_k, K) &= B(0, T_{k+1})\, E_{P_{T_{k+1}}}\left[\left(e^M - \mathcal{K}\right)^+\right] \\
&= B(0, T_{k+1})\, \left\{E_{P_{T_{k+1}}}\left[e^M 1_{\{M \geq \log \mathcal{K}\}}\right] - \mathcal{K}\, P_{T_{k+1}}\left[M \geq \log \mathcal{K}\right]\right\} \\
&= B(0, T_k) \cdot \overline{\chi}^2_{\nu,\alpha_1}\left(\frac{\log \mathcal{K} - A_k}{\sigma_1}\right) - \mathcal{K}^\star \cdot \overline{\chi}^2_{\nu,\alpha_2}\left(\frac{\log \mathcal{K} - A_k}{\sigma_2}\right),
\end{aligned}$$

where $\mathcal{K}^\star = \mathcal{K} \cdot B(0, T_{k+1})$ and $\overline{\chi}^2_{\nu,\alpha}(x) = 1 - \chi^2_{\nu,\alpha}(x)$, with $\chi^2_{\nu,\alpha}(x)$ the non-central chi-square distribution function, and all the parameters are known explicitly.

Similar closed-form solution for swaptions!

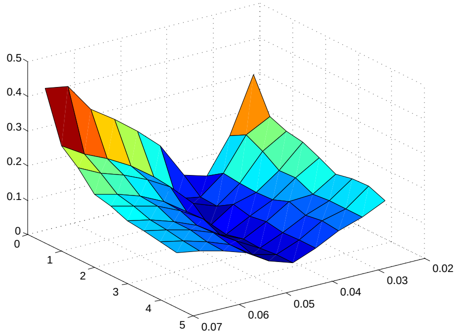
CIR martingales: volatility surface



Example of an implied volatility surface for the CIR martingales.

Γ-OU martingales: volatility surface

$$dX_t = -\lambda(X_t - \theta)dt + dH_t, \quad X_0 = x \in \mathbb{R}_{\geqslant 0}$$



Example of an implied volatility surface for the Γ-OU martingales.

Summary and Outlook

1. We have presented a LIBOR model that
    - is very simple, and yet . . .
    - captures all the important features . . .
    - especially positivity and analytical tractability
2. Future work:
    - thorough empirical analysis
    - extensions: multiple currencies, default risk
    - connections to HJM framework and short rate models
3. M. Keller-Ressel, A. Papapantoleon, J. Teichmann (2009)
   A new approach to LIBOR modeling
   Preprint, arXiv/0904.0555

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1. We have presented a LIBOR model that
    - is very simple, and yet . . .
    - captures all the important features . . .
    - especially positivity and analytical tractability
2. Future work:
    - thorough empirical analysis
    - extensions: multiple currencies, default risk
    - connections to HJM framework and short rate models
3. M. Keller-Ressel, A. Papapantoleon, J. Teichmann (2009)
   A new approach to LIBOR modeling
   Preprint, arXiv/0904.0555

Thank you for your attention!