A new perspective on the fundamental theorem of asset pricing for large financial markets

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(based on joint work with Christa Cuchiero and Irene Klein)

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  - call options (with even a continuum of strikes), etc.
- Usual assumption to preclude arbitrage is to suppose the existence of an equivalent (local/ σ-) martingale measure for the (uncountably many) discounted assets.
- In contrast to classical small financial markets, this property has not been characterized in an economically satisfying way, (only the Kreps-Yan theorem involving weak-$*$-closures is available in this setting).

Goal and outline of today's talk

Version of the fundamental theorem of asset pricing (FTAP) for large financial markets:

Certain economically meaningful "No asymptotic arbitrage" condition

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3. On (σ)-martingale measures in large financial markets

Setting and notation

- Finite time horizon $[0, 1]$
- One fixed filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,1]}, P)$
- Financial market with countably many (discounted) assets modeled by a sequence of $\mathbb{R}$-valued semimartingales $(S_t^n)_{t\in[0,1],n\in\mathbb{N}}$
- Vector of first n assets: $\mathbf{S}^n = (S^1, \ldots, S^n)^\top$
- 1-admissible portfolio wealth processes in the small financial market n:

$$\mathcal{X}_1^n = \{(\mathbf{H} \bullet \mathbf{S}^n) \,|\, \mathbf{H} \in \mathcal{H}_1^n\},$$

with

$\mathcal{H}_\lambda^n = \{\mathbf{H} \,|\, \mathbb{R}^n\text{-valued, predictable, } \mathbf{S}^n\text{-integrable, } \lambda\text{-admissible process}\}.$

As usual λ-admissibility means $(\mathbf{H} \bullet \mathbf{S}^n)_t \geq -\lambda$ for all $t \in [0, 1]$.

Generalized strategies in the large financial market

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  - generalized stochastic integration with respect to a sequence of semimartingales using so-called generalized strategies as integrands (introduced by M.De Donno et M.Pratelli (2006))

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- The precise definitions are introduced via the Emery topology on the space of $\mathbb{R}$-valued semimartingales $\mathbb{S}$:

$$d_{\mathbb{S}}(X_1, X_2) := \sup_{K \in b\mathcal{E}, \, \|K\|_\infty \leq 1} E\left[|(K \bullet (X_1 - X_2))|_1^* \wedge 1 \right],$$

where $|X|_1^* = \sup_{t \leq 1} |X_t|$ and $b\mathcal{E}$ denotes the set of simple predictable strategies. The Emery topology makes the space of semimartingales a complete metric space.

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where $|X|_1^* = \sup_{t \leq 1} |X_t|$ and $b\mathcal{E}$ denotes the set of simple predictable strategies. The Emery topology makes the space of semimartingales a complete metric space.

- Two semimartingales (portfolios) are close in the Emery topology if all their increments are close or in more financial terms if all investments in the difference of two portfolios are small.

Definition of (admissible) generalized strategies

Definition

1. For each $n \in \mathbb{N}$, let $\mathbf{H}^n$ be an $\mathbb{R}^n$-valued predictable $\mathbf{S}^n$-integrable process. A sequence $(\mathbf{H}^n)_{n\in\mathbb{N}}$ is called generalized strategy if $(\mathbf{H}^n \bullet \mathbf{S}^n)$ converges in the Emery topology to a semimartingale

$$Z := \mathbb{S}\text{-}\lim(\mathbf{H}^n \bullet \mathbf{S}^n),$$

which is called generalized stochastic integral and where $\mathbb{S}$-lim denotes the limit in the Emery topology.

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- Considering generalized strategies, means including portfolios Z of which the difference between the increments of Z and those of a small market portfolio can be made arbitrarily small in the Emery metric.
- $\Rightarrow$ Economically meaningful to include these limits in no arbitrage requirements

Admissible generalized portfolio processes in the LFM

Definition

1. Consider the set

$$\mathcal{X}_1 = \overline{\bigcup_{n\geq 1} \mathcal{X}_1^n}^{\mathbb{S}}$$
$$= \{\mathbb{S}\text{-}\lim(\mathbf{H}^n \bullet \mathbf{S}^n) \,|\, (\mathbf{H}^n) \text{ 1-admissible generalized strategy}\},$$

where $\overline{(\cdot)}^{\mathbb{S}}$ denotes the closure in the Emery-topology. The elements of $\mathcal{X}_1$ are called 1-admissible generalized portfolio wealth processes in the large financial market.

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2. We denote by $\mathcal{X}$ the set $\mathcal{X} := \cup_{\lambda>0}\lambda\mathcal{X}_1$ and call its elements admissible generalized portfolio wealth processes in the large financial market.

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3. We denote by K_0, respectively K_0^1 the evaluations of elements of $\mathcal{X}$, respectively $\mathcal{X}_1$, at terminal time $T = 1$.

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- In perfect analogy to the notion of (NFLVR) in the classical setting of small financial markets, we define (NAFLVR) as follows:

Definition (NAFLVR)

The set $\mathcal{X}$ is said to satisfy no asymptotic free lunch with vanishing risk if

$$\overline{C} \cap L^\infty_{\geq 0} = \{0\},$$

where $C = (K_0 - L^0_{\geq 0}) \cap L^\infty$ and $\overline{C}$ denotes the norm closure in L^∞.

No free lunch and equivalent separating measures

Definition

The set $\mathcal{X}$ satisfies the (ESM) (equivalent separating measure) property if there exists an equivalent measure $Q \sim P$ such that $E_Q[X_1] \leq 0$ for all $X \in \mathcal{X}$.

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- Under the condition

$$\overline{C}^* \cap L^\infty_{\geq 0} = \{0\}, \tag{NFL}$$

where $\overline{C}^*$ denotes the weak-$*$-closure in L^∞, the (ESM) property is a consequence of the Kreps-Yan Theorem (80, 81), which in turn follows from Hahn-Banach's Theorem. Condition (NFL) is the classical no free lunch condition for the abstract set C.

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- It is clear that (NFL) $\Rightarrow$ (NAFLVR). The goal is to show the reverse implication, that is...

A fundamental theorem of asset pricing for large financial markets

Theorem (C. Cuchiero, I. Klein, JT (2014))

Under (NAFLVR), $C = \overline{C}^*$, i.e., the cone C is already weak-$*$-closed and (NFL) holds.

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Theorem (Fundamental Theorem of Asset Pricing (C. Cuchiero, I. Klein, JT (2014)))

(NAFLVR) $\Leftrightarrow$ (ESM), i.e., $\exists Q \sim P$ such that $E_Q[X_1] \leq 0$ for all $X \in \mathcal{X}$.

Remarks on the proof

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- The so-called concatenation property of $\mathcal{X}_1$ does *not* hold, namely on the dense set (w.r.t. the Emery topology) $\bigcup_{n \geq 1} \mathcal{X}_1^n$:
  - For all $X, Y \in \bigcup_{n \geq 1} \mathcal{X}_1^n$ and all bounded predictable strategies $H, G \geq 0$, with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \bigcup_{n \geq 1} \mathcal{X}_1^n$, but not necessarily on the closure.

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- The other crucial properties of $\mathcal{X}_1$ (axiomatized by Y. Kabanov (1997)), namely convexity, boundedness from below by -1 and closedness in the Emery topology, hold true.

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- The other crucial properties of $\mathcal{X}_1$ (axiomatized by Y. Kabanov (1997)), namely convexity, boundedness from below by -1 and closedness in the Emery topology, hold true.
- Adding admissible generalized portfolios to $\bigcup_{n\geq 1}\mathcal{X}_1^n$ such that $\mathcal{X}_1$ is closed in the Emery topology is the crucial insight.

Connection to no asymptotic arbitrage of the first kind

- The notion of arbitrage of the first kind (Ingersoll (1987)) was introduced in the context of LFMs by Y. Kabanov and D. Kramkov (1994).
- It describes the possibility of getting arbitrarily rich with positive probability by taking an arbitrarily small risk.

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Definition (NAA1)

There exists an asymptotic arbitrage of the first kind (AA1) if there exist some $\alpha > 0$ and sequences $\varepsilon_n \to 0$, $c_n \to \infty$ and a sequence of strategies $(\mathbf{H}^n)$ such that for each $n \in \mathbb{N}$

1. $(\mathbf{H}^n \bullet \mathbf{S}^n)_t \geq -\varepsilon_n$ for all $t \in [0, 1]$
2. $P[(\mathbf{H}^n \bullet \mathbf{S}^n)_1 \geq c_n] \geq \alpha$,

No asymptotic arbitrage of the first kind (NAA1) holds if there exists no (AA1).

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Remark

(NAA1) $\Leftrightarrow$ K_0^1 *is a bounded subset of* L^0 *(NUPBR)*

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- Similarly, to small markets we have...

Proposition

(NA) + (NAA1) $\Leftrightarrow$ *(NAFLVR)*

"(NAFLVR)" without Emery closure

- Consider $\bigcup_{\lambda>0} \lambda \bigcup_{n\geq 1} \mathcal{X}_1^n$ the set of admissible portfolios in all small markets n, but without the closure in the Emery-topology. We use calligraphic red letters for quantities referring to these portfolios, e.g., $\mathcal{K}_0 = \{X_1 \,|\, X \in \bigcup_{\lambda>0} \lambda \bigcup_{n\geq 1} \mathcal{X}_1^n\}$.

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- Consider the analogous notion of (NAFLVR) for this set i.e., $\overline{\mathcal{C}} \cap L^\infty_{\geq 0} = \{0\}$, where $\mathcal{C} = (\mathcal{K}_0 - L^0_{\geq 0}) \cap L^\infty$.

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Proposition

Suppose every small market satisfies (NFLVR). Then

$$\overline{\mathcal{C}} \cap L^\infty_{\geq 0} = \{0\} \Leftrightarrow \text{(NAA1)}$$

Remarks

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- The crucial issue to obtain an FTAP without using weak-$*$-closures is to strengthen the no arbitrage condition , i.e., not only requiring "No arbitrage" for each small market, but also for the portfolios obtained via generalized strategies in the large market.

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- The crucial issue to obtain an FTAP without using weak-$*$-closures is to strengthen the no arbitrage condition , i.e., not only requiring "No arbitrage" for each small market, but also for the portfolios obtained via generalized strategies in the large market.
- This is precisely achieved by taking the Emery closure $\mathcal{X}_1$ of $\bigcup_{n\geq 1}\mathcal{X}_1^n$ and considering $\mathcal{X}=\bigcup_{\lambda>0}\lambda\mathcal{X}_1$, which is equivalent to weak-$*$-closing the set $\mathcal{C}$.

Abstract portfolio wealth process setting including uncountably many assets

- In order to allow for a unified treatment of different financial markets, involving e.g. a continuum of assets such as in the case of bond markets, we adapt the abstract portfolio wealth process setting introduced Y. Kabanov (1997) to large financial markets.
- Notation:
  - $I \subseteq [0, \infty)$: parameter space which can be any subset, countable or uncountable of $[0, \infty)$.
  - For each $n \in \mathbb{N}$, define

$$\mathcal{A}^n = \{\text{some/all subsets } A \subseteq I, \text{ such that } |A| = n\},$$

such that if $A^1, A^2 \in \bigcup_{n \geq 1} \mathcal{A}^n$, then $A^1 \cup A^2 \in \bigcup_{n \geq 1} \mathcal{A}^n$.

Example

- Markets consisting of on a continuum of tradeable assets, such as bonds, modeled by families of semimartingales $(S_t^\alpha)_{0\leq t\leq 1, \alpha\in[0,T^*]}$ correspond to
  - $I = [0, T^*]$,
  - $\mathcal{A}^n = \{\text{all subsets } A \subseteq [0, T^*] \,|\, |A| = n\}$, where $\alpha \in [0, T^*]$ can e.g., be thought of as the maturity of a bond.
  - For $A := \{\alpha_1, \ldots, \alpha_n\} \in \mathcal{A}^n$ and $\alpha_1, \ldots, \alpha_n \in [0, T^*]$, define

    $\mathcal{X}_1^A = \{(\mathbf{H}^A \bullet \mathbf{S}^A) \,|\, \mathbf{H}^A \text{ is } \mathbb{R}^n\text{-valued, predictable, } \mathbf{S}^A\text{-integrable, 1-adm.}\}$,

    where $\mathbf{S}^A = (S^{\alpha_1}, \ldots, S^{\alpha_n})$.
  - $\mathcal{X}_1^n = \bigcup_{A\in\mathcal{A}^n} \mathcal{X}_1^A$ are then the 1-admissible portfolio processes built by strategies that include at most n assets.

Definition of 1-admissible portfolio wealth processes

Definition (1-admissible portfolio processes in small markets)

1. For each $A \in \bigcup_{n\geq 1} \mathcal{A}^n$ we consider a convex set $\mathcal{X}_1^A \subset \mathbb{S}$ of semimartingales
   - starting at 0,
   - bounded from below by -1,
   - satisfying the following concatenation property: for all bounded, predictable strategies $H, G \geq 0$, $X, Y \in \mathcal{X}_1^A$ with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \mathcal{X}_1^A$.
   - For $A^1, A^2 \in \bigcup_{n\geq 1} \mathcal{A}^n$ with $A^1 \subseteq A^2$ we have that $\mathcal{X}_1^{A^1} \subseteq \mathcal{X}_1^{A^2}$,

   and call its elements 1-admissible portfolio wealth processes in the small financial market A.

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   - satisfying the following concatenation property: for all bounded, predictable strategies $H, G \geq 0$, $X, Y \in \mathcal{X}_1^A$ with $HG = 0$ and $Z = (H \bullet X) + (G \bullet Y) \geq -1$, it holds that $Z \in \mathcal{X}_1^A$.
   - For $A^1, A^2 \in \bigcup_{n\geq 1} \mathcal{A}^n$ with $A^1 \subseteq A^2$ we have that $\mathcal{X}_1^{A^1} \subseteq \mathcal{X}_1^{A^2}$,

   and call its elements 1-admissible portfolio wealth processes in the small financial market A.
2. For each $n \in \mathbb{N}$, we denote by $\mathcal{X}_1^n$ the set $\mathcal{X}_1^n := \bigcup_{A\in\mathcal{A}^n} \mathcal{X}_1^A$ corresponding to 1-admissible portfolio wealth processes with respect to strategies that include at most n assets (but all possible different choices of n assets).

FTAP in the abstract portfolio wealth process setting

As in the exemplary large financial market with countably many assets, we define completely analogously

- $\mathcal{X}_1 = \overline{\bigcup_{n \geq 1} \mathcal{X}_1^n}^{\mathbb{S}}$, $\mathcal{X} := \cup_{\lambda > 0} \lambda \mathcal{X}_1$
- $K_0 = \{X_1 | X \in \mathcal{X}\}$, $C = K_0 - L_{\geq 0} \cap L^\infty$
- (NAFLVR): $\overline{C} \cap L^\infty_{\geq 0} = \{0\}$

and get in this abstract setting

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and get in this abstract setting

Theorem (FTAP (C. Cuchiero, I. Klein, JT))

(NAFLVR) $\Leftrightarrow$ (ESM), i.e., $\exists Q \sim P$ such that $E[X_1] \leq 0$ for all $X \in \mathcal{X}$.

On (σ-)martingale measures in large financial markets

- In the case of (possibly uncountably many) locally bounded assets, equivalent separating measures correspond to equivalent local martingale measures. Hence in this case (NAFLVR) is equivalent to the existence of a local martingale measures.
- In contrast to classical small financial markets (NAFLVR) however does not necessarily imply the existence of a σ-martingale measure in the non-locally bounded case (counterexample in a one period market).

Conclusion

- Formulation of the notion of no asymptotic free lunch with vanishing risk (NAFLVR) by considering the Emery-closure of 1-admissible portfolio wealth processes in small markets (corresponding to generalized integrals in the setting of M. De Donno et al. with 1-admissible generalized strategies.)
- Proof of a version of the fundamental theorem of asset pricing (FTAP) in markets with an (even uncountably) infinite number of assets, i.e., (NAFLVR) $\Leftrightarrow$ (ESM), in particular in the case of locally bounded assets (NAFLVR) $\Leftrightarrow$ (ELMM).
- In the non locally bounded case, (NAFLVR) does not yield the existence of an equivalent (σ)-martingale measure in general.

Thanks for your inspiring works as researcher and teacher, Steve!

Ad multos annos!