Portfolio Selection

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Definition

A utility function is a strictly increasing, strictly concave map $U \in C^1(\mathbb{R}_+;\mathbb{R})$ satisfying the *Inada conditions* (named after after the economist Ken-Ichi Inada):

1. $U'(0) := \lim_{x \searrow 0} U'(x) = +\infty$,
2. $U'(\infty) := \lim_{x \to \infty} U'(x) = 0$.

For $y > 0$ we introduce the conjugate or Legendre transform of $-U(-\cdot)$ in the sense of convex analysis,

$$J(y) := \sup_{x>0}(U(x) - xy),$$

and denote by $I := (U')^{-1}$ the inverse of the derivative of U.

Basics on utility functions

$J \in C^1(\mathbb{R}_+;\mathbb{R})$ is strictly decreasing, strictly convex, $J'(0) = -\infty$, $J'(\infty) = 0$, $J(0) = U(\infty)$ and $J(\infty) = U(0)$. Moreover for any $x > 0$ we have the conjugacy relation

$$U(x) = \inf_{y>0}(J(y) + xy),$$

in addition $J' = -I$ and for any $y > 0$ we have

$$J(y) = U(I(y)) - yI(y).$$

J is clearly decreasing and convex, as it is a supremum of convex (even affine) functions. To show that $J \in C^1(\mathbb{R}_+;\mathbb{R})$ we assume that $U \in C^2(\mathbb{R}_+;\mathbb{R})$. Then $I \in C^1(\mathbb{R}_+;\mathbb{R})$ and for a fixed $y > 0$, $\sup_{x>0}(U(x) - xy)$ is attained in $x = I(y)$, so that

$$J(y) = U(I(y)) - I(y)y.$$

This last expression shows that $J \in C^1(\mathbb{R}_+;\mathbb{R})$ and

$$J'(y) = \underbrace{U'(I(y))}_{=y} I'(y) - I'(y)y - I(y) = -I(y).$$

Basics on utility functions

Classical utility functions on $\mathbb{R}_+$ are

$$U(x) := \log(x),$$

with corresponding conjugate

$$J(y) = \sup_{x>0}(U(x) - xy) \underbrace{=}_{x=\frac{1}{y}} -\log(y) - 1;$$

and for $\gamma \in (-\infty, 1) \setminus \{0\}$,

$$U(x) := \frac{1}{\gamma}(x^\gamma - 1),$$

with Legendre transform

$$J(y) = \sup_{x>0}(U(x) - xy) \underbrace{=}_{x=y^{\frac{1}{\gamma-1}}} \frac{1-\gamma}{\gamma} y^{\frac{\gamma}{\gamma-1}} - \frac{1}{\gamma}.$$

Note that for $\gamma < 0$, U is bounded from above by zero, while for $\gamma > 0$, U is unbounded. Moreover, for $\gamma \to 0$ we obtain the first case.

Utility Optimization Problem

For $x > 0$ we introduce

$$v(x) := \{V(x, \vartheta) = x + (\vartheta \bullet S) \mid (x, \vartheta) \ x\text{-admissble strategy}\}$$

Suppose U is a utility function and define

$$u(x) := \sup_{V \in v(x)} \mathbb{E}[U(V_T(x, \vartheta))],$$

for which we will assume that $u(x_0) < \infty$ for some $x_0 > 0$.

Utility Optimization Problem

U quantifies the subjective preferences by assigning to a monetary amount z a subjective utility of $U(z)$. The fact that U is increasing means that more is better and the concavity of U captures the idea of risk aversion or the effect that an extra dollar means more to a beggar than to a millionaire.

For a given $x > 0$, $u(x)$ can be interpreted as the maximal expected utility one can obtain via investment from an initial wealth x and the standing assumption implies that the optimisation problem is well-posed for at least one x_0.

Note that U is defined on $\mathbb{R}_+$ and $V_T \geq 0$, but $U(0) \in [-\infty, \infty)$ exists (as a limit $x \to 0$), so that $U(V_T(x, \vartheta))$ is well-defined in $[-\infty, \infty)$.

Utility Optimization Problem

Moreover we set $\mathbb{E}[U(V_T)] := -\infty$ if $(U(V_T))^- \notin L^1(P)$, since $u(x) \geq U(x) > -\infty$ for any $x > 0$, i.e. we do not lose any information if we exclude such strategies.

If U is unbounded and S allows arbitrage, then $u \equiv +\infty$, so the problem just makes sense in an arbitrage-free model.

The standing assumption, i.e. $u(x_0) < \infty$ for some $x_0 > 0$, implies that $u(x) < \infty$ for any $x > 0$.

Setting

Recall the basic problem: maximise $\mathbb{E}[u(V_T(x, \vartheta))]$ over all $\vartheta \in \Theta^x_{adm}$.
Here, we have that
$\Theta^x_{adm} = \{\text{predictable } S\text{-integrable } \mathbb{R}^d\text{-valued } \vartheta \text{ with } \int \vartheta dS \geq -x\}$.
With no loss of generality we can also impose that
$(u(V_T(x, \vartheta)))^- \in L^1(P)$.

Setting

We now fix $t \in [0, T]$, $\vartheta \in \Theta^x_{adm}$ and define

$$\Theta(t, \vartheta) := \{\psi \in \Theta^x_{adm} | \psi = \vartheta \text{ on } [0, t]\} \,.$$

The key idea now is to look at all the conditional problems to maximize $\mathbb{E}[u(V_T(x, \psi))|\mathcal{F}_t]$ over all $\psi \in \Theta(t, \vartheta)$ (for every $\vartheta \in \Theta^x_{adm}$). So we define the maximal conditional expected utility, given the initial wealth and an initial strategy ϑ, i.e.

$$J_t(\vartheta) := \text{ess-sup}_{\psi \in \Theta(t,\vartheta)} \underbrace{\mathbb{E}[u(V_T(x, \psi))|\mathcal{F}_t]}_{=:\Gamma_t(\psi)} \,.$$

If $\mathcal{F}_0$ is trivial, then for all $\vartheta \in \Theta^x_{adm}$ we have

$$J_0(\vartheta) = J_0 = \sup_{\psi \in \vartheta^x_{adm}} \mathbb{E}[u(V_T(x, \psi))] = U(x) \,.$$

Remark

One should be careful with the conditions on u and ϑ to ensure in the sequel that there are no integrability problems, e.g. $u \geq 0$ or u bounded above might be useful assumptions. We do not take care of the exact details here.

Martingale Optimality Principle

The main result is then the following version of the martingale optimality principle from stochastic calculus (dynamic programming principle):

Theorem (Martingale Optimality Principle (MOP) - with suitable integrability assumed)

The following hold:

1. *For every $\vartheta \in \Theta^x_{adm}$, the process*
$$(J_t(\vartheta))_{0 \le t \le T}$$
is a P-supermartingale.
2. *A strategy $\vartheta^* \in \Theta^x_{adm}$ is optimal, i.e.*
$$\mathbb{E}[u(V_T(x, \vartheta^*))] = \sup_{\vartheta \in \Theta^x_{adm}} \mathbb{E}[u(V_T(x, \vartheta))]$$
if and only if $(J_t(\vartheta^*))_{0 \le t \le T}$ is a P-martingale.

Proof

First we check that $\{\Gamma_t(\psi) | \psi \in \Theta(t, \vartheta)\}$ is upward directed: for $t \in [0, T]$, $A \in \mathcal{F}_t$, $\psi^1, \psi^2 \in \Theta(t, \vartheta)$, we have $\psi^1 I_A + \psi^2 I_{A^c} \in \Theta(t, \vartheta)$ so with $A := \{\Gamma_t(\psi^1) \geq \Gamma_t(\psi^2)\} \in \mathcal{F}_t$, we get $\max\{\Gamma_t(\psi^1), \Gamma_t(\psi^2)\} = \Gamma_t(\psi^1 I_A + \psi^2 I_{A^c})$.

So there exists an sequence $(\psi^n)_{n \in \mathbb{N}}$ in $\Theta(t, \vartheta)$ with $J_t(\vartheta) = \nearrow - \lim_{n \to \infty} \Gamma_t(\psi^n)$ and so monotone convergence holds:

$$
\begin{aligned}
\mathbb{E}[J_t(\vartheta) | \mathcal{F}_s] = \lim_{n \to \infty} \mathbb{E}[\Gamma_t(\psi^n) | \mathcal{F}_s] = \lim_{n \to \infty} \overbrace{\mathbb{E}[u(V_T(x, \psi^n)) | \mathcal{F}_s]}^{= \Gamma_s(\psi^n) \text{ and } \psi^n \in \Theta(t,\vartheta) \subseteq \Theta(s,\vartheta)} \\
\leq \text{ess-sup}_{\psi \in \Theta(s, \vartheta)} \Gamma_s(\psi) = J_s(\vartheta) .
\end{aligned}
$$

Proof

Integrability of $J(\vartheta)$ goes analogously; one needs control on J_0, e.g. $U \geq 0$ or $J_0 = U(x) < \infty$ work.

Now we take $\vartheta^* \in \Theta^x_{adm}$; then $J(\vartheta^*)$ is a P-supermartingale by 1). So $J(\vartheta^*)$ is a P-martingale if and only if it has constant expectation; and on $[0, T]$ this is equivalent to:

$$\mathbb{E}[u(V_T(x, \vartheta^*))] = \mathbb{E}[J_T(\vartheta^*)] = J_0 = \sup_{\psi \in \Theta^x_{adm}} \mathbb{E}\left[u(V_T(x, \psi))\right].$$

This means that ϑ^* is optimal.

Remark

Note that 2) includes the condition $\vartheta^* \in \Theta^x_{adm}$. So if we just exhibit some predictible S-integrable $\overline{\vartheta}$ s.t. $J(\overline{\vartheta})$ is a P-martingale, we can only conclude optimality of $\overline{\vartheta}$ after we check that $\overline{\vartheta} \in \Theta^x_{adm}$. [This is quite often not handled properly in applications.]

Remark

Now we want to exploit martingale optimality to get more information on ϑ^*. First, we can prove that $J(\vartheta)$ has a càdlàg version; we use that and decompose uniquely (by Doob-Meyer) as $J(\vartheta) = J_0 + M(\vartheta) - B(\vartheta)$ with $M(\vartheta) \in \mathcal{M}_{0,loc}$, $B(\vartheta)$ predictable, increasing, null at $t = 0$. Can we say even more?

We look at

$$\begin{aligned} J_t(\vartheta) &= \text{ess-sup}_{\psi \in \Theta(t,\vartheta)} \mathbb{E}\left[u(V_T(x,\psi))|\mathcal{F}_t\right] \\ &= \text{ess-sup}_{\psi \in \Theta(t,\vartheta)} \mathbb{E}\left[u(V_t(x,\vartheta) + \int_t^T \psi_u dS_u)|\mathcal{F}_t\right]. \end{aligned}$$

We expect that each of the conditional expectations, and hence also $J_t(\vartheta)$ is an $\mathcal{F}_t$-measurable functional of $V_t(x,\vartheta)$. So we also expect that $B_t(\vartheta)$ depends on ϑ, $V_t(x,\vartheta)$ in a "nice" way.

Remark

From martingale optimality we see that $B(\vartheta)$ is always increasing for each ϑ and it is constant (null) for optimal ϑ^*. In other words, the "drift" $b(\vartheta)$" is always ≥ 0, and $\equiv 0$ for ϑ^*. This can be exploited to obtain (non-linear) PDEs for the solution of the optimization problem.

We have a bank account $\tilde{B}$ and a stock $\tilde{S}$ with:

$$d\tilde{B}_t = \tilde{B}_t r dt,\ \tilde{B}_0 = 1$$
$$d\tilde{S}_t = \tilde{S}_t \left(\mu dt + \sigma dW_t\right),\ \tilde{S}_0 > 0$$

for $\mu, r \in \mathbb{R}, \sigma \in \mathbb{R}^+$.

For finite time horizon T, we want to maximize the expected utility for terminal wealth! We do this by re-parametrizing: u is defined on $(0, \infty)$, so $V(x, \vartheta)$ must be > 0, so we can describe a strategy not via number of shares (ϑ) but by fractions of wealth (π).

The Merton Problem

Call $V(x,\vartheta)$, $\tilde{V}(x,\vartheta)$ the discounted and undiscounted wealth in terms of ϑ, and define $\pi_t := \frac{\vartheta_t \tilde{S}_t}{\tilde{V}_t(x,\vartheta)} = \frac{\vartheta_t S_t}{V(x,\vartheta)}$. π_t is the fraction at time t of total wealth that is invested in stock; the fraction $1-\pi_t$ is in the bank account.

Call $X^\pi := \tilde{V}(x,\vartheta)$ the undicounted wealth expressed with π, with x fixed. The self-financing condition for X^π is then: $dV(x,\vartheta) = \vartheta dS$, so

$$d\left(\frac{X^\pi}{\tilde{B}}\right) = \frac{\pi X^\pi}{\tilde{B} S} dS = \frac{X^\pi}{\tilde{B}} \pi \frac{dS}{S}$$

and so

$$\begin{aligned} dX_t^\pi &= d\left(\tilde{B}_t \frac{X_t^\pi}{\tilde{B}_t}\right) = \tilde{B}_t d\left(\frac{X_t^\pi}{\tilde{B}_t}\right) + \frac{X_t^\pi}{\tilde{B}_t} d\tilde{B}_t = \pi_t X_t^\pi \overbrace{\frac{dS_t}{S_t}}^{=(\mu-r)dt+\sigma dW} + X_t^\pi r dt \\ &= r X_t^\pi dt + \pi_t X_t^\pi ((\mu - r)dt + \sigma dW_t). \end{aligned}$$

It is our goal to maximize $\mathbb{E}[U(X_T^\pi)]$ over all allowed $\pi = (\pi)_{0 \le t \le T}$ in the sequel. For this purpose fix $t \in [0, T]$, strategy π and another strategy ψ with $\psi = \pi$ on $[0, t]$. Consider

$$\begin{aligned}\Gamma_t(\psi) &= \mathbb{E}[U(X_T^\psi)|\mathcal{F}_t] = \mathbb{E}\left[U(X_t^\pi + \int_t^T dX_u^\psi)|\mathcal{F}_t\right] \\ &= \mathbb{E}\left[U(X_t^\pi + \int_t^T (rX_u^\psi + \psi_u X_u^\psi(\mu - r))du + \psi_u X_u^\psi \sigma dW_u)|\mathcal{F}_t\right].\end{aligned}$$

The Merton Problem

Our filtration $\mathbb{F}$ is generated by $\tilde{S}$, $\tilde{B}$ or equivalently by W. Recall that W has the Markov property, so "the situation is Markovian": it seems plausible that

- $\Gamma_t(\psi)$ should only depend on the current wealth X_t^π and
- it is sufficient to consider strategies ψ which only depend on current wealth, $\psi_t = g(t, X_t^\psi)$, since the optimal strategy has to be of this type. Notice that this defines a stochastic differential equation for X.

So it is natural to guess that also after optimisation, this persists; we guess that

$$J_t(\pi) = \text{ess-sup}_{\psi \in \Theta(t,\vartheta)} \mathbb{E}[U(X_T^\psi)|\mathcal{F}_t] = k(t, X_t^\pi)$$

for some function $k(t, x)$. What do we get then?

The Merton Problem

Assume k is nice and use Itô's formula. This gives:

$$dJ(\pi) = k_t dt + k_x \underbrace{dX^\pi}_{=\cdots} + \frac{1}{2} k_{xx} \underbrace{d\langle X^\pi \rangle}_{=\pi^2 (X^\pi)^2 \sigma^2 dt} .$$

So we get:

$$dJ_t(\pi) = \underbrace{k_x(t, X_t^\pi) \pi_t X_t^\pi \sigma dW_t}_{=dM_t(\pi)}$$
$$+ \underbrace{\left(\frac{\partial k}{\partial t} + \frac{\partial k}{\partial x} rx + \frac{\partial k}{\partial x} px(\mu - r) + \frac{1}{2} \frac{\partial^2 k}{\partial x^2} p^2 x^2 \sigma^2 \right)_{(t, x = X_t^\pi, p = \pi_t)} dt}_{=-dB_t(\pi) = -b(t, \pi_t, X_t^\pi) dt} .$$

By the martingale optimality principle, $B(\pi)$ is always increasing and constant at optimal π^*; so $b(\pi)$ (respectively $-b(\pi)$) is always ≥ 0 (≤ 0), and $= 0$ at optimal π^*.

The Merton Problem

Treating $p \widehat{=} \pi_t$ and $x \widehat{=} X_t^\pi$ as independent variables leads us to guess that $k(t, x)$ should satisfy

$$\sup_{p>0} \left(k_t(t,x) + rxk_x(t,x) + (\mu - r)pxk_x(t,x) + \frac{1}{2}\sigma^2 p^2 x^2 k_{xx}(t,x) \right) = 0.$$

This is the so called *Hamilton-Jacobi-Bellman* (HJB) equation for our control problem. It is a nonlinear PDE. Since $k(T, X_T^\pi) = J_T(\pi) = u(X_T^\pi)$ we impose $k(T, x) = u(x)$ for $x > 0$ as our boundary condition.

The Merton Problem

The idea now is to try and solve the HJB equation to come up with a candidate for the optimal strategy, π^*.

If we formally maximise over p we get the optimiser $p^*(t,x) = -\frac{\mu - r}{\sigma^2}\frac{k_x(t,x)}{x k_{xx}(t,x)}$. Plugging this in yields the HJB equation in the form:

$$0 = k_t(t,x) + rxk_x(t,x) - \frac{1}{2}\frac{(\mu - r)^2}{\sigma^2}\frac{(k_x(t,x))^2}{k_{xx}(t,x)},\ k(T,x) = U(x).$$

The Merton Problem

This is a nonlinear second order PDE for k. Conceptually, we should now to the following:

1. Find a sufficiently smooth solution $k(t, x)$ to the HJB equation.
2. Define function $p^*(t, x)$ from k as above.
3. Consider the SDE: $dX_t = rX_t dt + p^*(t, X_t)X_t((\mu - r)dt + \sigma dW_t)$ obtained by using the "candidate strategy" $p^*(t, X_t)$ for π^* (and writing the self-financing equation), and prove that this has a solution X^*.
4. Define $\pi_t^* := p^*(t, X_t^*)$ and show that π^* is an allowed strategy. (Then, by 3), $X^{\pi^*} = X^*$.)
5. Prove that π^* is optimal, either by direct argument (by comparing it to all other allowed π), or by showing that $X^* = X^{\pi^*}$ is such that $(J(\pi^*_\bullet) = k(\bullet, X^{\pi^*}_\bullet) = k(\bullet, X^*_\bullet)$ is a martingale.

The most difficult step is usually the first one.

The Merton Problem: an example

For power utility $u(x) = \frac{1}{\gamma}x^\gamma$ with $\gamma < 1$, $\gamma \neq 0$, we can solve the PDE explicitly. This goes as follows. the wealth dynamics

$$\frac{dX_t^\pi}{X_t^\pi} = rdt + \pi_t((\mu - r)dt + \sigma dW_t),\ X_0^\pi = x$$

give

$$X_t^\pi = x\mathcal{E}\left(rs + \int \pi_s((\mu - r)ds + \sigma dW_s)\right)_t$$

and so, for $\psi \in \Theta(t, \pi)$,

$$X_T^\psi = X_t^\pi \mathcal{E}\left(rs + \int \psi_s((\mu - r)ds + \sigma dW_s)\right)_{t,T}$$

The Merton Problem: an example

So,

$$\Gamma_t(\psi) = \mathbb{E}[U(X_T^\psi)|\mathcal{F}_t] = \left[\begin{array}{c} X_T^\psi = X_t^\pi \frac{X_T^\psi}{X_t^\pi} \\ U(x) = \frac{1}{\gamma} x^\gamma \end{array}\right] = \frac{1}{\gamma}(X_t^\pi)^\gamma \overbrace{\mathbb{E}[U(\mathcal{E}(\cdots\psi)_{t,T})|\mathcal{F}_t]}^{=:\overline{\Gamma}_t(\psi)}.$$

The Merton Problem: an example

So of course we set

$$J_t(\pi) = \frac{1}{\gamma}(X_t^\pi)^\gamma \operatorname{ess-sup}_{\psi\in\Theta(t,\vartheta)} \overline{\Gamma}_t(\psi)$$

and we guess that $k(t,x) = \frac{1}{\gamma}x^\gamma f(t)$.

Then $k_t = \frac{1}{\gamma}x^\gamma \dot{f}(t)$, $k_x = x^{\gamma-1}f(t)$, $k_{xx} = (\gamma-1)x^{\gamma-2}f(t)$ and plugging this into the HJB equation yields

$$0 = \frac{1}{\gamma}x^\gamma\left(\dot{f}(t) + \gamma r f(t) - \frac{1}{2}\frac{(\mu-r)}{\sigma^2}\frac{\gamma}{\gamma-1}f(t)\right),\ \frac{1}{\gamma}x^\gamma = \frac{1}{\gamma}x^\gamma f(T),$$

or $f(T) = 1$.

The Merton Problem: an example

This ODE for f can be solved explicitly. The explicit candidate for the optimal strategy is then $\pi_t^* = p^*(t, X_t^*) = -\frac{\mu - r}{\sigma^2}\frac{1}{\gamma - 1} = \frac{\mu - r}{\sigma^2(1-\gamma)}$ which prescribes to always hold a fixed proportion of total wealth (the so called *Merton proportion*) in the stock (and the rest in the bank account). One can check that this strategy is allowed and optimal.

The strategy π^* being constant still involves trading, because the corresponding ϑ^* (optimal number of shares) is not constant. In case of the Merton problem one could also argue directly that the strategy can neither depend on time nor on current wealth, hence it has to be constant. Given this fact, it is easy to calculate the value of π directly. The solution of the HJB-equation is just making precise this type of reasoning.

Abstract Utility optimization

In this section we study the basic problem of an optimal portfolio choice with preferences given by expected utility. We take the standard model with finite time $T < \infty$, $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ a filtered probability space satisfying the usual conditions, $B \equiv 1$ the bank account and the discounted asset prices $S = (S_t)_{0 \le t \le T}$, where S is an $\mathbb{R}^d$-valued semimartingale. We impose absence of arbitrage via $\mathbb{P} \neq \emptyset$.

Abstract Utility optimization

We fix an initial capital $x > 0$ and consider a self-financing strategy (x, ϑ), where ϑ is an $\mathbb{R}^d$-valued predictable S-integrable process. We impose that the strategy ϑ is x-admissible so that the wealth process

$$V(x, \vartheta) = x + (\vartheta \bullet S) \geq 0.$$

Our goal is to find a x-admissible strategy ϑ, so that this strategy maximizes the expected utility from terminal wealth over ϑ, i.e. maximize $\mathbb{E}[U(V_T(x, \vartheta))]$, where U is a utility function on $\mathbb{R}_+$.

Abstract Utility optimization

Note that imposing (x, ϑ) to be a x-admissible strategy ties up with $\mathrm{dom}(U) = \mathbb{R}_+$ and we could have just imposed that $V_T(x, \vartheta) \geq 0$. Moreover, if $\mathrm{dom}(U) = (-a, \infty)$ with $0 < a < \infty$, then we can just translate by a, but if $\mathrm{dom}(U) = \mathbb{R}$, finding a good class of allowed strategies becomes tricky.

Abstract Utility optimization (primal problem)

Let U be a utility function as above and $x > 0$, the primal problem is

$$u(x) = \sup_{V \in v(x)} \mathbb{E}[U(V_T)].$$

Abstract Utility optimization (primal problem)

Consider the set of positions that can be superreplicated from initial wealth $x > 0$, with x-admissible self-financing strategies, i.e.

$$\mathcal{C}(x) := \left\{ f \in L^0_+(\mathcal{F}_T) \mid \exists V \in v(x) : f \leq V_T \right\} = (x + G_T(\Theta^x_{adm}) - L^0_+) \cap L^0_+,$$

where

$$\Theta^x_{adm} := \{ \vartheta = (\vartheta_t)_{0 \leq t \leq T} \mid \vartheta \in \Theta_{adm} : (\vartheta \bullet S) \geq -x \} .$$

Abstract Utility optimization (primal problem)

Note that $v(x)_T \subseteq \mathcal{C}(x)$ and if $f \in \mathcal{C}(x)$ then $\mathbb{E}[U(f)] \leq u(x)$, for the latter take some $V \in v(x)$ so that $V_T \geq f$; since U is increasing we have $U(f) \leq U(V_T)$ and hence $\mathbb{E}[U(f)] \leq \mathbb{E}[U(V_T)] \leq u(x)$.

Abstract Utility optimization (primal problem)

So the primal problem can be written as

$$u(x) = \sup_{f \in \mathcal{C}(x)} \mathbb{E}[U(f)].$$

As we know already $\mathcal{C}(x)$ is easier to describe than $v(x)$. Note also that if $f^* \in \mathcal{C}(x)$ is optimal, then there is some $\vartheta^* \in \Theta^x_{adm}$ so that

$$f^* \leq x + G_T(\vartheta^*)$$

and $V(x, \vartheta^*) \in v(x)$ is a solution to the primal problem, because

$$u(x) = \mathbb{E}[U(f^*)] \leq \mathbb{E}[U(V_T(x, \vartheta^*))] \leq u(x).$$

Abstract Utility optimization (dual problem)

In order to gain more information about the primal problem we want to introduce a suitable dual problem using the conjugacy relation of U and J, and exploiting the absence of arbitrage condition. Take $Q \in \mathbb{P}(\neq \emptyset)$ and denote by Z the density process of Q with respect to P, then $S \in \mathcal{M}_{loc}(Q)$ is a local martingale with respect to Q.

Let $V = V(x, \vartheta) \in v(x)$, then $(\vartheta \bullet S)$ is well-defined and bounded below by $-x$, hence by Ansel-Stricker $(\vartheta \bullet S) \in \mathcal{M}_{loc}(Q)$ is a local martingale with respect to Q, so it is also a Q-super-martingale.

Moreover, since Z is a density process of an equivalent probability measure we have $Z > 0$ and $\mathbb{E}[Z_0] = 1$. So, if $\mathcal{F}_0$ is trivial or if we insist on $Q = P$ on $\mathcal{F}_0$, then $Z_0 \equiv 1$.

This motivates the following set: for $z > 0$ we introduce the family of all $(\mathcal{F}_t)_{t\in[0,T]}$-adapted, positive, RCLL processes Z starting at z such that for any $V \in v(1)$, ZV is a P-supermartingale, i.e.

$$\mathcal{Z}(z) := \{Z \mid Z \geq 0 \ : Z_0 = z, \ \forall V \in v(1) : ZV \ \text{ cadlag super-mart.}\}.$$

Note that for any $x > 0$, $v(x) = xv(1)$; so the last condition is equivalent to saying that for any $V \in v(x)$, ZV is a P-super-martingale.

Abstract Utility optimization (dual problem)

Any $Z \in \mathcal{Z}(z)$ is itself a super-martingale, to see this take $(x, \vartheta) = (1, 0)$, then $V(1, 0) \equiv 1 \in v(1)$ so that $ZV = Z$ is a super-martingale. Moreover, $\mathcal{Z}(z)$ contains all density processes $Q \in \mathbb{P}$ with $Q = P$ on $\mathcal{F}_0$. Finally, $\mathcal{Z}(z) = z\mathcal{Z}(1)$.

Abstract Utility optimization (dual problem)

This set allows us to derive the dual problem in the following way: let $x, z > 0$, $V \in v(x)$ and $Z \in \mathcal{Z}(z)$, then ZV is a P-super-martingale starting at $Z_0 V_0 = zx$, so

$$\mathbb{E}[Z_T V_T] \leq zx.$$

Recall the Legendre transform of U, i.e. for any $y > 0$,

$$J(y) = \sup_{x>0}(U(x) - xy) \geq U(x) - xy,$$

to obtain, using the super-martingale property that

$$\mathbb{E}[U(V_T)] \leq \mathbb{E}[J(Z_T) + V_T Z_T] \leq \mathbb{E}[J(Z_T)] + zx.$$

Taking the supremum over $V \in v(x)$ and the infimum over $Z \in \mathcal{Z}(z)$ yields the following expression

$$u(x) \leq \inf_{Z \in \mathcal{Z}(z)} \mathbb{E}[J(Z_T)] + zx.$$

So, for $z > 0$ it is a natural dual problem to look for

$$j(z) := \inf_{Z \in \mathcal{Z}(z)} \mathbb{E}[J(Z_T)].$$

Abstract Utility optimization (duality)

The primal problem maximizes a concave functional, while the dual problem minimizes a convex functional.

In analogy to $\mathcal{C}(x)$, we introduce the set

$$\mathcal{D}(z) := \left\{ h \in L^0_+ \mid \exists Z \in \mathcal{Z}(z) : h \leq Z_T \right\}$$

to get the abstract equivalent version of the dual problem

$$j(z) = \inf_{h \in \mathcal{D}(z)} \mathbb{E}[J(h)],$$

this follows from the following two observations: $\mathcal{Z}(z)_T \subseteq \mathcal{D}(z)$ and if $h \in \mathcal{D}(z)$, then $\mathbb{E}[J(h)] \geq j(z)$.

Abstract Utility optimization (duality)

Moreover, note that if we fix $z > 0$ we obtain that

$$j(z) \geq \sup_{x>0}(u(x) - xz),$$

and if we fix $x > 0$ we get

$$u(x) \leq \inf_{z>0}(j(z) + zx).$$

This is very reminiscent of the conjugacy relation between U and J. We will see that we actually get equalities above, plus solvability of the primal as well as the dual problem at the expense of one extra assumption on U (reasonable asymptotic elasticity).

Table of Contents

Introduction

We shall follow here in contents and notation the excellent thesis by Alexander Veruurt which can be found at `https://arxiv.org/abs/1504.02988`.

At the beginning we shall provide a bird's eye perspective on the field and some of its assertions from the point of view of FTAP.

Change of numeraire for local martingales

The only value which does not change nominally under changes of numéraire is zero, whence it makes sense, as done in the setting of stochastic portfolio theory, to consider portfolio wealth processes non-negative and starting at some predefined value x, then change of numeraire does not change the lower bound.

We can either look what change of numeraire does to portfolio wealth processes or – on the dual side – equivalent separating measures Q. If Q is an equivalent local martingale measure for a process S and the numeraire N defines a strictly positive density process, then Q' defined through $\frac{dQ'}{dQ} := N$ is an equivalent local martingale measure for S/N by Bayes formula.

Change of numeraire for portfolios – additive

Let X be a (vector valued) semimartingale and $\varphi \in L(X))$, such that $(\varphi \bullet X) = \sum_i \varphi^i X^i$ and $(\varphi \bullet X)_- = \sum_i \varphi^i X^i_-$. Let furthermore N be a strictly positive (real valued) semimartingale (with N_- being strictly positive as well), then

$$\frac{(\varphi \bullet X)}{N} = (\varphi \bullet \frac{X}{N})$$

holds true. This follows from Ito's formula

$$\begin{aligned}\frac{(\varphi \bullet X)}{N} &= ((\varphi \bullet X)_- \bullet \frac{1}{N}) + (\frac{1}{N_-} \bullet (\varphi \bullet X)) + [(\varphi \bullet X), \frac{1}{N}] \\ &= (\varphi \bullet (X_- \bullet \frac{1}{N}) + (\frac{1}{N_-} \bullet X) + [X, \frac{1}{N}]) \,.\end{aligned}$$

We apply the self-financing condition for the first term.

Change of numeraire for portfolios – multiplicative

Let X be a vector valued semimartingale whose entries are strictly positive together with X_- and let $\frac{\pi}{X_-} \in L(X)$ with $\sum_i \pi^i = 1$. Let furthermore N be a strictly positive (real valued) semimartingale (as well as N_-) starting. The value process V satisfies the equation

$$V = 1 + (\frac{\pi V_-}{X_-} \bullet X) \, .$$

Then

$$\frac{1 + (\frac{\pi V_-}{X_-} \bullet X)}{N} = \frac{1}{N_0} + (\frac{\pi V_-}{X_-} \bullet \frac{X}{N}) \, .$$

The proof is the same as before.

Growth Rates and Excess Growth Rates

Let V be a strictly positive process such that $\log V$ is a special semimartingale such that $\log V = A + M$, then A is called the integrated growth rate of V.

Let X be a vector valued semimartingale whose entries are strictly positive with X_- strictly positive as well, and let $\frac{\pi}{X_-} \in L(X)$ with $\sum_i \pi^i = 1$ be a multiplicative strategy with value process V. Let A^i denote the integrated growth rates of X^i. Then V also has an integrated growth rate B and

$$B = (\pi \bullet A) + \frac{1}{2}(\frac{\pi}{X^2{}_-} \bullet [X, X]) - \frac{1}{2}[(\frac{\pi}{X_-} \bullet X), (\frac{\pi}{X_-} \bullet X)],$$

where the quotient in the second and third part is understood componentwise.

Let X be a vector valued semimartingale whose entries are strictly positive together with X_-, then $N = \sum_i X^i$ is a strictly positive (real valued) semimartingale (again as well as N_-). Consider a C^2 function G, then for $Y := \frac{X}{N}$ we have by Ito's formula

$$
\begin{aligned}
G(\frac{X}{N}) &= (\nabla G(Y)_- \bullet Y) + \frac{1}{2}(D^2 G(Y)_- \bullet [Y, Y]) + \\
&\quad + \sum_{s \le t} (G(Y_s) - G(Y_{s-}) - \nabla G(Y)_- \Delta Y_s - \frac{1}{2} D^2 G(Y)_- \Delta [Y, Y]_S)
\end{aligned}
$$

Assume that $G(Y)$ is strictly positive (as well as $G(Y)_-$), then we can define a multiplicative strategy

$$
\pi^i = Y^i_- (\nabla^i \log G(Y)_- + 1 - \sum_i Y^i_- \nabla^i \log G(Y)_-)
$$

Ito's formula and the master equation

and a finite variation process

$$\mathfrak{G} = -\frac{1}{2}\left(\frac{D^2 G(Y)_-}{G(Y)_-} \bullet [Y, Y]\right) -$$
$$- \sum_{0<s\leq t} \frac{G(Y_s) - G(Y_{s-}) - \nabla G(Y)_- \Delta Y_s - \frac{1}{2} D^2 G(Y)_- \Delta [Y, Y]_S}{G(Y)_{s-}}.$$

This then leads to the following formula, the master equation,

$$\frac{V}{N} = \frac{G(Y)}{G(Y_0)} \frac{\mathcal{E}(\frac{1}{Y}_- \pi \bullet Y)}{\mathcal{E}(\mathfrak{G} + (\frac{1}{Y}_- \pi \bullet Y))},$$

where $\mathcal{E}$ denotes the stochastic exponential and the second part is of finite total variation. Therefore the master equation appears just as a multiplicative version of Ito's formula.

Basic Definitions

We place ourselves in a general continuous-time Itô model without frictions (i.e. there are no transaction costs, trading restrictions, or any other imperfections). Let the price processes $X_i(\cdot)$ of stocks $i = 1, \ldots, n$ under the physical measure $\mathbb{P}$ be given by

$$\mathrm{d}X_i(t) = X_i(t)\Big(b_i(t)\mathrm{d}t + \sum_{\nu=1}^{d} \sigma_{i\nu}(t)\mathrm{d}W_\nu(t)\Big), \qquad i = 1, \ldots, n$$

$$X_i(0) = x_i > 0\,.$$

Here, $W(\cdot) = (W_1(\cdot), \dots, W_d(\cdot))$ is a d-dimensional $\mathbb{P}$-Brownian motion, and we assume $d \geq n$. We furthermore assume our filtration $\mathbb{F}$ to contain the filtration $\mathbb{F}^W$ generated by $W(\cdot)$, and the drift rate processes $b_i(\cdot)$ and matrix-valued volatility process $\sigma(\cdot) = (\sigma_{i\nu}(\cdot))_{i=1,\dots,n,\nu=1,\dots,d}$ to be $\mathbb{F}$-progressively measurable and to satisfy the integrability condition

$$\sum_{i=1}^{n} \int_0^T \left(|b_i(t)| + \sum_{\nu=1}^{d} (\sigma_{i\nu}(t))^2 \right) \mathrm{d}t < \infty \qquad \text{for all } T \in (0, \infty)\,.$$

We define the *(instantanous) covariance process* $a(t) = \sigma(t)\sigma'(t)$, with the apostrophe denoting a transpose. Note that $a(\cdot)$ is a positive semi-definite matrix-valued process. Finally, we assume the existence of a *riskless asset* $X_0(t) \equiv 1, \quad$ for all $t \geq 0$; namely, without loss of generality we assume a zero interest rate, by discounting the stock prices by the bond price.

Basic Definitions

Now, let us consider the log-price processes; by Itô's formula, we have

$$\begin{aligned}\mathrm{d}\log X_i(t) &= \left(b_i(t) - \frac{1}{2}a_{ii}(t)\right)\mathrm{d}t + \sum_{\nu=1}^{d}\sigma_{i\nu}\mathrm{d}W_\nu(t)\\ &= \gamma_i(t)\mathrm{d}t + \sum_{\nu=1}^{d}\sigma_{i\nu}(t)\mathrm{d}W_\nu(t)\,,\end{aligned}$$

where we have defined the *growth rates* $\gamma_i(t) := b_i(t) - \frac{1}{2}a_{ii}(t)$. This name is justified by the fact that

$$\lim_{T\to\infty}\frac{1}{T}\left(\log X_i(T) - \int_0^T \gamma_i(t)\mathrm{d}t\right) = 0 \qquad \mathbb{P}\text{-a.s.}$$

under, e.g., $\frac{1}{T^2}\sum_{i=1}^{n}\sum_{\nu=1}^{d}\int_0^T E[(\sigma_{i\nu}(t))^2]\mathrm{d}t \to 0$ as $T\to\infty$.

Investments

We proceed by defining which investment rules are allowed in our framework.

Definition

Define a *portfolio* as an $\mathbb{F}$-progressively measurable vector process $\pi(\cdot)$, (often) uniformly bounded in (t, ω), where $\pi_i(t)$ represents the proportion of wealth invested in asset i at time t, and satisfying $\sum_{i=1}^{n} \pi_i(t) = 1$ for all $t \geq 0$. We say that $\pi(\cdot)$ is a *long-only* portfolio if $\pi_i(t) \geq 0$ for all $i = 1, \ldots, n$. For future reference, we also define the set

$$\Delta_+^n := \{x \in \mathbb{R}^n : \sum x_i = 1 \text{ and } x_i > 0 \text{ for all } i = 1, \ldots, n\}.$$

We denote the *wealth process* of an investor investing according to portfolio $\pi(\cdot)$, with initial wealth $w > 0$, by $V^{w,\pi}(\cdot)$, if it exists under certain integrability conditions (see next slide).

Note that portfolios are self-financing by definition.

Investments

We also define a more general class of investment rules, which we shall call trading strategies.

Definition

A *trading strategy* is an $\mathbb{F}$-progressively measurable process $h(\cdot)$ that takes values in $\mathbb{R}^n$ and satisfies the integrability condition

$$\sum_{i=1}^{n} \int_0^T \Big(|h_i(t) b_i(t)| + h_i^2(t) a_{ii}(t) \Big) \mathrm{d}t < \infty \qquad \mathbb{P}\text{-a.s.}$$

For any t, $h_i(t)$ is the amount of money invested in stock i. Again, we let $V^{w,h}(\cdot)$ denote the wealth process of an investor following the trading strategy $h(\cdot)$ and starting with initial wealth $w \geq 0$. We write $V^h(\cdot) := V^{1,h}(\cdot)$. We require $h(\cdot)$ to be x-admissible for some $x \geq 0$, written as $h(\cdot) \in \mathcal{A}_x$, meaning that $V^{0,h}(t) \geq -x$ for all $t \in [0, T]$ a.s. We shall write $\mathcal{A} := \mathcal{A}_0$.

Note that each portfolio generates a trading strategy by setting $h_i(t) = \pi_i(t) V^{w,\pi}(t)\ \forall t \in [0, T]$. We assume the admissibility condition to exclude doubling strategies. On the contrary, one can define a trading strategy $h(\cdot) \in \mathcal{A}_x$ by specifying it as the *proportions* invested in stocks at each time, $\pi_i(t) = h_i(t)/V^{w,h}(t)$, provided that $w > x$ and similarly to a portfolio but with the exception that in general now $\sum_{i=1}^{n} \pi_i(t) \neq 1$; that is, there is a non-zero holding of cash $\pi_0(t)$.

Wealth process

The wealth process associated to a portfolio $\pi(\cdot)$ and initial wealth $w \in \mathbb{R}_+$ can be seen to evolve as

$$\frac{\mathrm{d}V^{w,\pi}(t)}{V^{w,\pi}(t)} = \sum_{i=1}^{n} \pi_i(t)\frac{\mathrm{d}X_i(t)}{X_i(t)} = b_\pi(t)\mathrm{d}t + \sum_{\nu=1}^{d} \sigma_{\pi\nu}(t)\mathrm{d}W_\nu(t),$$

with the portfolio's rate of return $b_\pi(t) := \sum_{i=1}^{n} \pi_i(t)b_i(t)$ and its volatility coefficients $\sigma_{\pi\nu}(t) := \sum_{i=1}^{n} \pi_i(t)\sigma_{i\nu}(t)$ (very slightly abusing notation). Hence we have, by Itô's formula, that

$$\begin{aligned}\mathrm{d}\log V^{w,\pi}(t) &= \left(b_\pi(t) - \frac{1}{2}\sum_{\nu=1}^{d}(\sigma_{\pi\nu}(t))^2\right)\mathrm{d}t + \sum_{\nu=1}^{d} \sigma_{\pi\nu}(t)\mathrm{d}W_\nu(t)\\ &= \gamma_\pi(t)\mathrm{d}t + \sum_{\nu=1}^{d} \sigma_{\pi\nu}(t)\mathrm{d}W_\nu(t),\end{aligned}$$

where $\gamma_\pi(t) := b_\pi(t) - \frac{1}{2}\sum_{\nu=1}^{d}(\sigma_{\pi\nu}(t))^2$ is the *growth rate* of the portfolio π.

Wealth process

Note the disappearance of the drift processes from this expression; since we may write

$$\gamma_\pi(t) = \sum_{i=1}^{n} \pi_i(t) b_i(t) - \frac{1}{2} \sum_{i,j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t) = \sum_{i=1}^{n} \pi_i(t) \gamma_i(t) + \gamma_\pi^*(t),$$

where the *excess growth rate* is defined as

$$\gamma_\pi^*(t) := \frac{1}{2} \left(\sum_{i=1}^{n} \pi_i(t) a_{ii}(t) - \sum_{i,j=1}^{n} \pi_i(t) a_{ij}(t) \pi_j(t) \right),$$

it follows directly that

$$\mathrm{d} \log V^\pi(t) = \gamma_\pi^*(t) \mathrm{d}t + \sum_{i=1}^{n} \pi_i(t) \mathrm{d} \log X_i(t),$$

which also motivates the nomenclature for $\gamma_\pi^*(\cdot)$.

The market portfolio

We define a particular portfolio, the *market portfolio* $\mu(\cdot)$, by

$$\mu_i(t) := \frac{X_i(t)}{X(t)}, \qquad X(t) := \sum_{i=1}^{n} X_i(t).$$

We assume there is only one share per company (or, equivalently, that $X_i(\cdot)$ is the capitalisation process of company i), so $\mu_i(t)$ is the *relative market weight* of company i at time t. The wealth process associated to the market portfolio is

$$\frac{\mathrm{d}V^{w,\mu}(t)}{V^{w,\mu}(t)} = \sum_{i=1}^{n} \mu_i(t) \frac{\mathrm{d}X_i(t)}{X_i(t)} = \sum_{i=1}^{n} \frac{X_i(t)}{X(t)} \frac{\mathrm{d}X_i(t)}{X_i(t)} = \frac{\mathrm{d}X(t)}{X(t)},$$

and hence

$$V^{w,\mu}(t) = \frac{w}{X(0)} X(t)\,.$$

The market portfolio

The wealth resulting from the market portfolio is therefore simply equal to a constant times the total market size: $\mu(\cdot)$ is a buy-and-hold strategy. In Stochastic Portfolio Theory (SPT), one measures the performance of portfolios with respect to the market portfolio (i.e. one uses the market portfolio as a 'benchmark' — this is similar to the approach taken in the Benchmark Approach to finance, developed by Platen and Heath). The market portfolio is therefore of great importance.

First Calculations

We obtain that

$$\mathrm{d}\log V^{w,\mu}(t) = \gamma_\mu(t)\mathrm{d}t + \sum_{\nu=1}^{d} \sigma_{\mu\nu}(t)\mathrm{d}W_\nu(t),$$

which gives that

$$\mathrm{d}\log \mu_i(t) = \left(\gamma_i(t) - \gamma_\mu(t)\right)\mathrm{d}t + \sum_{\nu=1}^{d} \left(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)\right)\mathrm{d}W_\nu(t)\,.$$

Equivalently, the relative market weights evolve as

$$\begin{aligned}
\frac{\mathrm{d}\mu_i(t)}{\mu_i(t)} &= \Big(\gamma_i(t) - \gamma_\mu(t) + \\
&+ \frac{1}{2}\sum_{\nu=1}^{d}\Big(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)\Big)^2\Big)\mathrm{d}t + \sum_{\nu=1}^{d}\Big(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)\Big)\mathrm{d}W_\nu(t) \\
&= \Big(\gamma_i(t) - \gamma_\mu(t) + \frac{1}{2}\tau_{ii}^{\mu}(t)\Big)\mathrm{d}t + \sum_{\nu=1}^{d}\Big(\sigma_{i\nu}(t) - \sigma_{\mu\nu}(t)\Big)\mathrm{d}W_\nu(t)\,.
\end{aligned}$$

First Calculations

Here, we have defined the matrix-valued *covariance process of the stocks relative to the portfolio* $\pi(\cdot)$ as

$$\begin{aligned}\tau^{\pi}_{ij}(t) &:= \sum_{\nu=1}^{d}(\sigma_{i\nu}(t) - \sigma_{\pi\nu}(t))(\sigma_{j\nu}(t) - \sigma_{\pi\nu}(t)) \\ &= (\pi(t) - e_i)' a(t)(\pi(t) - e_j) = a_{ij}(t) - a_{\pi i}(t) - a_{\pi j}(t) + a_{\pi\pi}(t)\,,\end{aligned}$$

where e_i is the i-th unit vector in $\mathbb{R}^n$, and

$$a_{\pi i}(t) := \sum_{j=1}^{n} \pi_j(t) a_{ij}(t), \qquad a_{\pi\pi}(t) := \sum_{i,j=1}^{n} \pi_i(t)\pi_j(t) a_{ij}(t)\,.$$

First Calculations

Note that we have the following relation:

$$\sum_{j=1}^{n} \pi_j(t)\tau_{ij}^{\pi}(t) = \sum_{j=1}^{n} \pi_j(t)a_{ij}(t) - a_{\pi i}(t) - \sum_{j=1}^{n} \pi_j(t)a_{\pi j}(t) + a_{\pi\pi}(t)$$
$$= 0, \qquad i = 1, \ldots, n$$

since the first two and last two terms cancel each other. Finally, note also that

$$\tau_{ij}^{\mu}(t) = \frac{\mathrm{d}\langle \mu_i, \mu_j \rangle (t)}{\mu_i(t)\mu_j(t)\mathrm{d}t}, \qquad 1 \leq i, j \leq n\,.$$

Useful properties

Let us start by defining the *relative returns process* of stock i with respect to portfolio $\pi(\cdot)$ as

$$R_i^\pi(t) := \log\left(\frac{X_i(t)}{V^{w,\pi}(t)}\right)\Bigg|_{w=X_i(0)}.$$

Lemma

We have that $\tau_{ii}^\pi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle R_i^\pi\rangle(t) \geq 0.$

Proof

We get that

$$\mathrm{d}R_i^\pi(t) = (\gamma_i(t) - \gamma_\pi(t))\,\mathrm{d}t + \sum_{\nu=1}^{d} (\sigma_{i\nu}(t) - \sigma_{\pi\nu}(t))\,\mathrm{d}W_\nu(t).$$

From this and the defining equation, we see that $\tau_{ij}^\pi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle R_i^\pi, R_j^\pi \right\rangle(t)$, and thus

$$\tau_{ii}^\pi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left\langle R_i^\pi \right\rangle(t) \geq 0.$$

Useful properties

Lemma

We have the numéraire-invariance property

$$\gamma_\pi^*(t) = \frac{1}{2}\Big(\sum_{i=1}^n \pi_i(t)\tau_{ii}^\rho(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}^\rho(t) \Big)$$

for any two portfolios $\pi(\cdot)$ and $\rho(\cdot)$. In particular, we have that

$$\gamma_\pi^*(t) = \frac{1}{2}\sum_{i=1}^n \pi_i(t)\tau_{ii}^\pi(t),$$

which is non-negative for any long-only portfolio $\pi(\cdot)$.

Proof

By definition of $\tau_{ij}^{\rho}(t)$ we have that

$$\sum_{i=1}^{n} \pi_i(t)\tau_{ii}^{\rho}(t) = \sum_{i=1}^{n} \pi_i(t)a_{ii}(t) - 2\sum_{i=1}^{n} \pi_i(t)a_{\rho i}(t) + a_{\rho\rho}(t)$$

and

$$\sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\tau_{ij}^{\rho}(t) = \sum_{i=1}^{n} \pi_i(t)\pi_j(t)a_{ij}(t) - 2\sum_{i=1}^{n} \pi_i(t)a_{\rho i}(t) + a_{\rho\rho}(t).$$

Proof

Putting these equations together, we see that

$$\begin{aligned}
&\frac{1}{2}\Big(\sum_{i=1}^{n}\pi_i(t)\tau_{ii}^{\rho}(t)-\sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)\tau_{ij}^{\rho}(t)\Big)=\\
&=\frac{1}{2}\Big(\sum_{i=1}^{n}\pi_i(t)a_{ii}(t)-\sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)a_{ij}(t)\Big)=\\
&=\gamma_\pi^*(t)
\end{aligned}$$

proving the first statement.

Proof

Now, choosing $\rho(\cdot) = \pi(\cdot)$ we see that we may write the excess growth rate as

$$\gamma_\pi^*(t) = \frac{1}{2} \sum_{i=1}^{n} \pi_i(t) \tau_{ii}^\pi(t);$$

that is, as the weighted average of the stocks' variances relative to $\pi(\cdot)$. We conclude that for all long-only portfolios we have $\gamma_\pi^*(t) \geq 0$.

Remark

Note that for $\pi(\cdot) = \mu(\cdot)$, we get that the excess growth rate of the market portfolio is

$$\gamma_\mu^*(t) = \frac{1}{2}\sum_{i=1}^{n} \mu_i(t)\tau_{ii}^{\mu}(t),$$

namely the weighted average of the stocks' variances relative to the market. This is interpreted as a measure of the 'intrinsic volatility' of the market.

Ordered statistics

As this will be useful later, let us introduce some notation:

Definition

We shall use the reverse-order-statistics notation, defined by

$$\theta_{(1)}(t) := \max_{1 \le i \le n} \{\theta_i(t)\}$$

$$\theta_{(i)}(t) := \max \left(\{\theta_1(t), \ldots, \theta_n(t)\} \setminus \{\theta_{(1)}(t), \ldots, \theta_{(i-1)}(t)\} \right), \quad i = 2, \ldots, n$$

for any $\mathbb{R}^n$-valued process $\theta(\cdot)$. Thus we have

$$\theta_{(1)}(t) \ge \theta_{(2)}(t) \ge \ldots \ge \theta_{(n)}(t).$$

Functionally generated portfolios

The biggest advantage of SPT over classical approaches to constructing well-performing portfolios is that in general it does not require estimation of the drifts or volatilities of the stocks. The machinery of SPT, i.e., the way in which virtually all relative arbitrages are constructed, involves what Robert Fernholz has called *functionally generated portfolios* (FGPs).

Definition

Let $U \subset \Delta^n_+$ be a given open set. Call $\mathsf{G} \in \mathcal{C}^2(U, (0,\infty))$ a *generating function* for the portfolio $\pi(\cdot)$ if G is such that $x \mapsto x_i D_i \log \mathsf{G}(x)$ is bounded on U, and if there exists a measurable, adapted process $\mathfrak{g}(\cdot)$ such that

$$\mathrm{d}\log\left(\frac{V^\pi(t)}{V^\mu(t)}\right) = \mathrm{d}\log\mathsf{G}(\mu(t)) + \mathfrak{g}(t)\mathrm{d}t, \qquad \forall t \geq 0, \quad \text{a.s.}$$

Remark

We can interpret the above equation as follows: the process measuring the performance of the portfolio $\pi(\cdot)$ relative to the market can be decomposed into a stochastic part of infinite variation, written as a deterministic function of the market weights process, plus a finite variation part $\mathfrak{g}(t)\mathrm{d}t$.

Master equation

Proposition

Let a function G *as in the previous definition generate the portfolio* $\pi(\cdot)$*. Then we have the following expression, for* $i = 1, \dots, n$*:*

$$\pi_i(t) = \Big(D_i \log \mathrm{G}(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j(t) D_j \log \mathrm{G}(\mu(t)) \Big) \cdot \mu_i(t).$$

Note that this defines a portfolio indeed, in particular, $\sum_{i=1}^{n} \pi(t) = 1$.

Proof

The proof follows from the following lemma:

Lemma

For a portfolio $\pi(\cdot)$ satisfying the above formula, we have that $\pi(\cdot)$ is generated by G, *i.e.*

$$\log\left(\frac{V^{\pi}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathbf{G}(\mu(T))}{\mathbf{G}(\mu(0))}\right) + \int_0^T \mathfrak{g}(t)\mathrm{d}t \quad \text{a.s.},$$

where

$$\mathfrak{g}(t) := \frac{-1}{2\mathbf{G}(\mu(t))} \sum_{i,j=1}^{n} D_{ij}^2 \mathbf{G}(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}^{\mu}(t)$$

is called the drift process.

Proof

We clearly have

$$\mathrm{d}\log\left(\frac{V^{\pi}(T)}{V^{\rho}(T)}\right) = \gamma_{\pi}^{*}(t)\mathrm{d}t + \sum_{i=1}^{n}\pi_i(t)\mathrm{d}\log\left(\frac{X_i(t)}{V^{\rho}(T)}\right).$$

Setting $\rho(\cdot) = \mu(\cdot)$ this becomes

$$\mathrm{d}\log\left(\frac{V^{\pi}(T)}{V^{\mu}(T)}\right) = \gamma_{\pi}^{*}(t)\mathrm{d}t + \sum_{i=1}^{n}\pi_i(t)\mathrm{d}\log\mu_i(t).$$

Now, recall the dynamics of $\log\mu_i(\cdot)$ and $\mu_i(\cdot)$, respectively, and apply the numéraire-invariance property to get

$$\mathrm{d}\log\left(\frac{V^{\pi}(T)}{V^{\mu}(T)}\right) = \sum_{i=1}^{n}\frac{\pi_i(t)}{\mu_i(t)}\mathrm{d}\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^{n}\pi_i(t)\pi_j(t)\tau_{ij}^{\mu}(t)\,\mathrm{d}t.$$

Proof

In order for us to relate $V^\pi(T)/V^\mu(T)$ to $\mathsf{G}(\mu(0))$ and $\mathsf{G}(\mu(T))$, we need to derive a useful expression for the dynamics of $\log \mathsf{G}(\mu(\cdot))$. Note the relation

$$D^2_{ij} \log \mathsf{G}(\mu(t)) = \frac{D^2_{ij}\mathsf{G}(\mu(t))}{\mathsf{G}(\mu(t))} - D_i\mathsf{G}(\mu(t)) \cdot D_j\mathsf{G}(\mu(t))$$

and introduce the notation $g_i(t) := D_i \log \mathsf{G}(\mu(t))$,
$N(t) := 1 - \sum_{j=1}^n \mu_j(t) g_j(t)$; then we have that

$$\mathrm{d} \log \mathsf{G}(\mu(t)) = \sum_{i=1}^n g_i(t) \mathrm{d}\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n D^2_{ij} \log \mathsf{G}(\mu(t))\, \mathrm{d}\left\langle \mu_i, \mu_j \right\rangle(t)$$

$$= \sum_{i=1}^n g_i(t) \mathrm{d}\mu_i(t) + \frac{1}{2} \sum_{i,j=1}^n \left(\frac{D^2_{ij}\mathsf{G}(\mu(t))}{\mathsf{G}(\mu(t))} - g_i(t) g_j(t) \right) \mu_i(t)\mu_j(t)\tau^\mu_{ij}(t)\, \mathrm{d}t.$$

Proof

Finally the defining equation becomes $\pi_i(t) = (g_i(t) + N(t))\mu_i(t)$; we compute

$$\sum_{i=1}^{n} \frac{\pi_i(t)}{\mu_i(t)} \mathrm{d}\mu_i(t) = \sum_{i=1}^{n} g_i(t)\mathrm{d}\mu_i(t) + N(t)\mathrm{d}\Big(\sum_{i=1}^{n} \mu_i(t)\Big) = \sum_{i=1}^{n} g_i(t)\mathrm{d}\mu_i(t)$$

and

$$\begin{aligned}\sum_{i,j=1}^{n} \pi_i(t)\pi_j(t)\tau_{ij}^{\mu}(t) &= \sum_{i,j=1}^{n} (g_i(t) + N(t))(g_j(t) + N(t))\mu_i(t)\mu_j(t)\tau_{ij}^{\mu}(t) \\ &= \sum_{i,j=1}^{n} g_i(t)g_j(t)\mu_i(t)\mu_j(t)\tau_{ij}^{\mu}(t),\end{aligned}$$

hence,

$$\mathrm{d}\log\left(\frac{V^{\pi}(T)}{V^{\mu}(T)}\right) = \sum_{i=1}^{n} g_i(t)\mathrm{d}\mu_i(t) - \frac{1}{2}\sum_{i,j=1}^{n} g_i(t)g_j(t)\mu_i(t)\mu_j(t)\tau_{ij}^{\mu}(t)\,\mathrm{d}t,$$

and the result follows.

A first example: diversity

We call a market model *diverse* on $[0, T]$ if

$$\exists\, \delta \in (0,1) \;\text{ such that }\; \mu_{(1)}(t) < 1 - \delta \quad \text{ for all } \; t \in [0, T] \quad \mathbb{P}\text{-a.s.}$$

A model is called *weakly diverse* on $[0, T]$ if

$$\exists\, \delta \in (0,1) \;\text{ such that }\; \frac{1}{T}\int_0^T \mu_{(1)}(t)\mathrm{d}t < 1 - \delta \qquad \mathbb{P}\text{-a.s.}$$

A first example: diversity

A natural question to ask is whether there exists an Itô model that fits our diversity framework at all. Let $\delta \in (1/2, 1)$, $d = n$, and let $\sigma(\cdot) \equiv \sigma$ be a constant matrix satisfying the non-degeneracy condition. Let $g_1, \ldots, g_n \geq 0$; then, for $t \in [0, T]$, set

$$\mathrm{d} \log X_i(t) = \gamma_i(t)\mathrm{d}t + \sum_{\nu=1}^{d} \sigma_{i\nu} \mathrm{d}W_\nu(t) \quad i = 1, \ldots, n,$$

where, for some constant $M > 0$,

$$\gamma_i(t) := g_i 1_{\{X_i(t) \neq X_{(1)}(t)\}} - \frac{M}{\delta} \frac{1_{\{X_i(t) = X_{(1)}(t)\}}}{\log\big((1-\delta)X(t)/X_i(t)\big)}.$$

This system of SDEs has a unique strong solution, and that the diversity holds.

A first example: diversity

We introduce the following non degeneracy condition:

$$\exists\,\varepsilon > 0 \ \text{ such that } \ \xi' a(t)\xi \geq \varepsilon ||\xi||^2, \qquad \text{for all } \xi \in \mathbb{R}^n, t \geq 0 \qquad \mathbb{P}\text{-a.s.}$$

By easy calculations we see that

$$\exists\,\zeta > 0 \ \text{ such that } \ \gamma^*_\mu(t) \geq \zeta \qquad \text{for all } \ t \in [0, T] \quad \mathbb{P}\text{-a.s.}$$

holds true under diversity.

A first example: diversity

Proposition

If a model is diverse and non degeneracy holds, then

$$\exists \zeta > 0 \;\; \text{such that} \;\; \gamma_\mu^*(t) \geq \zeta \quad \textit{for all} \;\; t \in [0, T] \quad \mathbb{P}\text{-a.s.}\,.$$

Lemma

If non degeneracy holds, then for any long-only portfolio $\pi(\cdot)$ we have

$$\frac{\varepsilon}{2}(1 - \pi_{(1)}(t)) \leq \gamma_\pi^*(t) \quad \text{a.s.}\,.$$

A first example: diversity

By definition of $\tau^{\pi}_{ij}(t)$ and by non degeneracy we have the inequality

$$\begin{aligned}\tau^{\pi}_{ii}(t) &= (\pi(t) - e_i)'\, a(t)\, (\pi(t) - e_i) \geq \varepsilon ||\pi(t) - e_i||^2 \\ &= \varepsilon\Big((1 - \pi_i(t))^2 + \sum_{j \neq i} \pi_j^2(t)\Big).\end{aligned}$$

Whence we conclude that

$$\begin{aligned}\gamma^*_{\pi}(t) &\geq \frac{\varepsilon}{2} \sum_{i=1}^{n} \pi_i(t)\Big((1 - \pi_i(t))^2 + \sum_{j \neq i} \pi_j^2(t)\Big) \\ &= \frac{\varepsilon}{2}\Big(\sum_{i=1}^{n} \pi_i(t)(1 - \pi_i(t))^2 + \sum_{j=1}^{n} \pi_j^2(t)(1 - \pi_j(t)\Big) \\ &= \frac{\varepsilon}{2} \sum_{i=1}^{n} \pi_i(t)(1 - \pi_i(t)) \geq \frac{\varepsilon}{2}(1 - \pi_{(1)}(t)).\end{aligned}$$

A first example: diversity

Definition

Define the *diversity-weighted* portfolio $\mu^{(p)}(\cdot)$ with parameter $p \in (0,1)$ by

$$\mu_i^{(p)}(t) := \frac{(\mu_i(t))^p}{\sum_{j=1}^n (\mu_j(t))^p} \quad i = 1, \dots, n.$$

A first example: diversity

One can check that this portfolio is generated by the function

$$\mathsf{G}_p : x \mapsto \Big(\sum_{i=1}^{n} x_i^p\Big)^{1/p}.$$

We compute, for $\mu \in \mathbb{R}^n$ and $i, j = 1, \ldots, n$,

$$D^2_{ij}\mathsf{G}_p(\mu) = \begin{cases} (1-p)(\mathsf{G}_p(\mu))^{1-2p}\mu_i^{p-2}\Big(\mu_i^p - (\mathsf{G}_p(\mu))^p\Big) & (i = j) \\ (1-p)(\mathsf{G}_p(\mu))^{1-2p}(\mu_i\mu_j)^{p-1} & (i \neq j) \end{cases}$$

and the bounds

$$1 = \sum_{i=1}^{n} \mu_i(t) \leq \sum_{i=1}^{n} \mu_i^p(t) \leq \sum_{i=1}^{n} \Big(\frac{1}{n}\Big)^p = n^{1-p}.$$

A first example: diversity

By the master equation this implies that the drift process equals

$$\begin{aligned}\mathfrak{g}(t) &= -\frac{1-p}{2\mathsf{G}_p(\mu)}\left[-\sum_{i=1}^{n}(\mathsf{G}_p(\mu))^{1-p}\mu_i^p\tau_{ii}^{\mu} + \sum_{i,j=1}^{n}(\mathsf{G}_p(\mu))^{1-2p}\mu_i^p\mu_j^p\tau_{ij}^{\mu}\right] \\ &= \frac{1-p}{2}\left[\sum_{i=1}^{n}\mu_i^{(p)}\tau_{ii}^{\mu} - \sum_{i,j=1}^{n}\mu_i^{(p)}\mu_j^{(p)}\tau_{ij}^{\mu}\right] \\ &= (1-p)\gamma_{\mu^{(p)}}^{*}(t)\end{aligned}$$

and therefore that

$$\log\left(\frac{V^{\mu^p}(T)}{V^{\mu}(T)}\right) = \log\left(\frac{\mathsf{G}_p(\mu(T))}{\mathsf{G}_p(\mu(0))}\right) + (1-p)\int_0^T \gamma_{\mu^{(p)}}^{*}(t)\mathrm{d}t \quad \text{a.s.}$$

Now we get the lower bound

$$\log\left(\frac{\mathsf{G}_p(\mu(T))}{\mathsf{G}_p(\mu(0))}\right) \geq -\frac{1-p}{p}\log n,$$

which implies that $V^{\mu^p}(T)/V^{\mu}(T) \geq n^{-(1-p)/p}$, $\mathbb{P}$-a.s., since $\gamma^*_{\mu^{(p)}}(\cdot)$ is a non-negative process for the long-only portfolio $\mu^{(p)}(\cdot)$.

By the previous lemma and together with the observation that $\mu_{(1)}^{(p)}(t) \le \mu_{(1)}(t)$, to get

$$\int_0^T \gamma^*_{\mu^{(p)}}(t)\mathrm{d}t \ge \frac{\varepsilon}{2}\int_0^T (1-\mu_{(1)}^{(p)}(t))\mathrm{d}t \ge \frac{\varepsilon}{2}\int_0^T (1-\mu_{(1)}(t))\mathrm{d}t > \frac{1}{2}\varepsilon\delta T.$$

We conclude that

$$\log\left(\frac{V^{\mu^p}(T)}{V^{\mu}(T)}\right) > (1-p)\left(\frac{\varepsilon\delta T}{2} - \frac{\log n}{p}\right) \quad \text{a.s.}\,.$$

A first example: diversity

Therefore, if we have

$$T \geq 2 \log n / p\varepsilon\delta,$$

(i.e., if T is big enough) we get that

$$\mathbb{P}(V^{\mu^{(p)}}(T) > V^{\mu}(T)) = 1.$$

Therefore, the diversity-weighted portfolio is a relative arbitrage with respect to the market over long enough time horizons, under the conditions of weak diversity and non-degeneracy. Note that this is a portfolio and therefore invests only in the stocks, since $\sum_i \mu_i^{(p)}(\cdot) = 1$.

A second example: entropy

Definition

Define the *entropy-weighted* portfolio $\pi^c(\cdot)$ with parameter $c > 0$ to be the portfolio generated by a version of the Shannon entropy function

$$\mathbf{H}_c(x) := c + \mathbf{H}(x) := c - \sum_{i=1}^{n} x_i \log x_i.$$

Here, $\mathbf{H}$ is the standard Shannon entropy function. One can check that

$$\pi_i^c(t) = \frac{\mu_i(t)(c - \log \mu_i(t))}{\sum_{j=1}^{n} \mu_j(t)(c - \log \mu_j(t))}, \quad i = 1, \dots, n.$$

A second example: entropy

Once again, we compute for general $\mu \in \mathbb{R}^n$

$$D^2_{ij}\mathbf{H}_c(\mu) = -\frac{1}{\mu_i}\delta_{ij} \quad i,j = 1,\dots,n,$$

with δ_{ij} the Kronecker-delta, which implies for the drift process

$$\mathfrak{g}(t) = \frac{1}{2\mathbf{H}_c(\mu(t))}\sum_{i=1}^{n}\mu_i(t)\tau^{\mu}_{ii}(t) = \frac{\gamma^*_\mu(t)}{\mathbf{H}_c(\mu(t))},$$

where we have used properties of excess growth rate.

A second example: entropy

The last thing we need for the construction of a relative arbitrage is the bound

$$c < \mathbf{H}_c(x) \leq c + \log n;$$

using this together with the master equation we get that

$$\begin{aligned}\log\left(\frac{V^{\mu^p}(T)}{V^{\mu}(T)}\right) &= \log\left(\frac{\mathbf{H}_c(\mu(T))}{\mathbf{H}_c(\mu(0))}\right) + \int_0^T \frac{\gamma_\mu^*(t)}{\mathbf{H}_c(\mu(t))}\mathrm{d}t \\ &> -\log\left(\frac{\mathbf{H}_c(\mu(0))}{c}\right) + \frac{\zeta T}{c + \log n} \quad \text{a.s.}\end{aligned}$$

A second example: entropy

We conclude that, if

$$T > \mathcal{T}_*(c) := \frac{1}{\zeta}(c + \log n) \log\left(\frac{c + \mathbf{H}(\mu(0))}{c}\right),$$

or, alternatively,

$$T > \mathcal{T}_* := \frac{1}{\zeta}\mathbf{H}(\mu(0)) = \lim_{c \to \infty} \mathcal{T}_*(c), \text{ and } c > 0 \text{ is chosen sufficiently large,}$$

then the entropy-weighted portfolio $\pi^c(\cdot)$ is a relative arbitrage with respect to the market portfolio over the time horizon $[0, T]$.

An interesting model class

Consider stochastic process models for discounted prices (actually capitalizations) of the from

$$dX_i(t) = \delta\sigma^2 dt + 2\sqrt{X_i(t)}\sigma \sum_{j=1}^{n} v_{ij} dB^j(t)$$

where $X_i(0) > 0$, the matrix v is just invertible, $\sigma > 0$ and $\delta \geq 0$.

Depending on δ (see Revuz-Yor, Chapter IX) the solutions behave quite differently but do exist strongly as non-negative semi-martingales:

- In case $\delta = 0$ the solution is a martingale, which touches 0 almost surely and stays there.
- In case $\delta \geq 2$ the solution does not reach 0.
- In case $\delta \leq 1$ the point 0 is reached almost surely.

Furthermore the process is affine, i.e. its Fourier-Laplace transform is exponentially affine in its initial value (for each i separately, not jointly!).

Arbitrage properties

The market price of risk $(\lambda_i(t))$ is given by

$$\frac{1}{2\sqrt{X_i(t)}}\sum_j (v^{-1})_{ij}(\delta\sigma^2)$$

which does *never* define an equivalent measure change, because the resulting process would correspond to $\delta = 0$ and does not behave the same way and it is only well defined for $\delta \geq 2$, i.e. pathwise square integrable in time.

Whence the market $(1, X_1, \ldots, X_n)$ has a supermartingale deflator for $\delta \geq 2$, i.e. we do not have unbounded profits with bounded risk. However, arbitrages exist in the market, since (NFLVR) is not satisfied. We are actually not interested in these types of arbitrages which need the bank account, but rather in relative arbitrages with respect to the market portfolio.

The market portfolio

The process $X(t) = \sum_i X_i(t)$ (sometimes scaled to 1) is called the market portfolio and is a valid strictly positive portfolio.

The market consisting of portfolios formed by trading in a self-financing way in $X_1, \ldots, X_n$ has a super-martingale deflator Y being a continuous super-martingale itself.

Whence also the market formed by $\frac{X_1}{X_1+\ldots+X_n}, \ldots, \frac{X_n}{X_1+\ldots+X_n}$ has a super-martingale deflator namely $Y(X_1 + \ldots + X_n)$ being a super-martingale itself. We expect arbitrages to exist, even if only trading in the stocks: an arbitrage is seen relative to the marketportfolio.

Absence of arbitrage

We now give the definition of a relative arbitrage:

Definition

(Relative arbitrage) Let $h(\cdot)$ and $k(\cdot)$ be trading strategies. Then $h(\cdot)$ is called a *relative arbitrage* (RA) over $[0, T]$ with respect to $k(\cdot)$ if their associated wealth processes satisfy

$$V^h(T) \geq V^k(T) \ \text{ a.s.}, \qquad \mathbb{P}(V^h(T) > V^k(T)) > 0.$$

Usually, we will only consider and construct relative arbitrages using portfolios that do not invest in the riskless asset at all. However, it is also possible to create a RA using a trading strategy that has a non-trivial position in the riskless asset, as we show in the following example, which uses results from Johannes Ruf on hedging European claims in Markovian markets where NA is allowed to fail.

An example

Define an auxiliary process $R(\cdot)$ as a Bessel process with drift $-c$, i.e.

$$\mathrm{d}R(t) = \Big(\frac{1}{R(t)} - c\Big)\mathrm{d}t + \mathrm{d}W(t)$$

for $t \in [0, T]$, $c \geq 0$ constant and $W(\cdot)$ a BM. We have that the Bessel process $R(\cdot)$ is strictly positive. Define a stock price process by

$$\mathrm{d}S(t) = \frac{1}{R(t)}\mathrm{d}t + \mathrm{d}W(t), \qquad S(0) = R(0) > 0$$

for $t \in [0, T]$, so $S(t) = R(t) + ct > 0$ forall $t \in [0, T]$. The market price of risk is $\theta(t, s) = 1/(s - ct)$ for $(t, s) \in [0, T] \times \mathbb{R}_+$ with $s > ct$.

An example

The reciprocal $1/Z^{\theta}(\cdot)$ of the local martingale deflator hits zero exactly when $S(t)$ hits ct. For a general payoff function p, and $(t,s) \in [0,T] \times \mathbb{R}_+$ with $s > ct$, it is true that a claim paying $p(S(T))$ at time $t = T$ has value function

$$
\begin{aligned}
h^p(t,s) :&= \mathbb{E}^{t,s}[\tilde{Z}^{\theta,t,s}(T)p(S(T))] \\
&= \mathbb{E}^{\mathbb{Q}}[p(S(T))\mathbf{1}_{\{\min_{t \le u \le T}\{S(u)-cu\}>0\}} \big| \mathcal{F}(t)] \Big|_{S(t)=s} \\
&= \int_{\frac{cT-s}{\sqrt{T-t}}}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} p(z\sqrt{T-t}+s)\mathrm{d}z \\
&\qquad - e^{2c(s-ct)} \int_{\frac{cT-2ct+s}{\sqrt{T-t}}}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} p(z\sqrt{T-t}-s+2ct)\mathrm{d}z \,.
\end{aligned}
$$

An example

Now define another stock price process by

$$\mathrm{d}\tilde{S}(t) = -\tilde{S}^2(t)\mathrm{d}W(t),$$

so $\mathbb{P}$ is already a martingale measure for $\tilde{S}(\cdot)$. We have $\tilde{S}(\cdot) = 1/S(\cdot)$, with $c = 0$, and also $\tilde{\theta}(\cdot) \equiv 0$, so $Z^{\tilde{\theta}}(\cdot) \equiv 1$. Applying Itô' formula, note that

$$\mathrm{d} \log \tilde{S}(t) = -\tilde{S}(t)\mathrm{d}W(t) - \frac{1}{2}\tilde{S}^2(t)\mathrm{d}t = \mathrm{d} \log Z^{\theta}(t);$$

hence $\tilde{S}(t) = \tilde{S}(0)Z^{\theta}(t)$ and

$$\tilde{Z}^{\theta,t,s}(T) = \left.\frac{\tilde{S}(T)}{\tilde{S}(t)}\right|_{\tilde{S}(t)=1/s}.$$

An example

Thus we may compute the hedging price of one unit of this stock as

$$\begin{aligned}
\nu^1(t,s) :&= \mathbb{E}^{t,s}[\tilde{Z}^{\tilde{\theta},t,s}(T)\tilde{S}(T)] = \mathbb{E}^{t,s}[\tilde{S}(T)] \\
&= \mathbb{E}^{t,s}[\tilde{Z}^{\theta,t,1/s}(T)\tilde{S}(t)] = s\mathbb{E}^{t,s}[\tilde{Z}^{\theta,t,1/s}(T)] \\
&= s \cdot \left(\int_{\frac{-1/s}{\sqrt{T-t}}}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \mathrm{d}z - \int_{\frac{1/s}{\sqrt{T-t}}}^{\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}} \mathrm{d}z \right) \\
&= 2s\Phi\left(\frac{1}{s\sqrt{T-t}} \right) - s < s.
\end{aligned}$$

An example

In other words, this stock has a "bubble". The corresponding optimal strategy (expressed in the *number* of stocks the investor holds) is the derivative of the hedging price with respect to s, i.e.

$$\eta^1(t,s) = 2\Phi\left(\frac{1}{s\sqrt{T-t}}\right) - 1 - \frac{2}{s\sqrt{T-t}}\varphi\left(\frac{1}{s\sqrt{T-t}}\right) < 1$$

for $(t,s) \in [0,T) \times \mathbb{R}_+$.

An example

Now $\eta^1(\cdot,\cdot)$ is a relative arbitrage with respect to $\eta^2(t,s) := 1$ (i.e. just holding the stock). Namely, define $\bar{\nu} := \nu^1(0,\tilde{S}_0)$; since

$$V^{\bar{\nu},\eta^2}(T) = \bar{\nu}\tilde{S}(T) < \tilde{S}(T) = V^{\bar{\nu},\eta^1}(T) \text{ a.s.},$$

we see that $\eta^1(\cdot,\cdot)$ is a relative arbitrage with respect to $\eta^2(\cdot,\cdot)$. However, it is *not* a 'real' arbitrage, since for $\hat{\eta}(\cdot,\cdot) := \eta^1(\cdot,\cdot) - \eta^2(\cdot,\cdot)$ we have $V^{\bar{\nu},\hat{\eta}}(0) = 0$ and $V^{\bar{\nu},\hat{\eta}}(T) = (1-\bar{\nu})\tilde{S}(T) > 0$, but since $\eta^1(\cdot,\cdot) < 1$ for $t \in [0,T)$, we get that $\hat{\eta}(\cdot,\cdot) < 0$ for $t \in [0,T)$ and thus the wealth process is unbounded below; i.e. $\hat{\eta}$ is not admissible.

An example

The holding in the riskless asset $\varphi(\cdot)$ corresponding to strategy $\eta^1(\cdot,\cdot)$ can be computed using the self-financing equation $\mathrm{d}V = \varphi \mathrm{d}B + \eta^1 \mathrm{d}\tilde{S} = \eta^1 \mathrm{d}\tilde{S}$ and $V = \varphi B + \eta \tilde{S}$, which gives that

$$\varphi(t) = V(t) - \eta^1(t, \tilde{S}(t))\tilde{S}(t) = \int_0^t \eta^1(u, \tilde{S}(u))\mathrm{d}\tilde{S}(u) - \eta^1(t, \tilde{S}(t))\tilde{S}(t),$$

which can, given the history up to time t, be computed. Note that $\varphi(\cdot)$ is not Markovian, and is in general non-zero.