Definition (Regularity structure)

A *regularity structure* $\mathscr{T} = (A, T, G)$ consists of the following elements:

- An *index set* $A \subset \mathbf{R}$ such that $0 \in A$, A is bounded from below, and A is locally finite.
- A *model space* T, which is a graded vector space $T = \bigoplus_{\alpha \in A} T_\alpha$, with each $(T_\alpha, \|\cdot\|_\alpha)$ a Banach space. Furthermore, $T_0 \approx \mathbf{R}$ and its unit vector is denoted by $\mathbf{1}$.
- A *structure group* G of linear operators acting on T such that, for every $\Gamma \in G$, every $\alpha \in A$, and every $a \in T_\alpha$, one has

  $$\Gamma a - a \in \bigoplus_{\beta < \alpha} T_\beta \; .$$

  Furthermore, $\Gamma \mathbf{1} = \mathbf{1}$ for every $\Gamma \in G$.

Definition (Model)

Given a regularity structure $\mathscr{T}$ and an integer $d \geq 1$, a *model* for $\mathscr{T}$ on $\mathbf{R}^d$ consists of maps

$$\Pi \colon \mathbf{R}^d \to \mathcal{L}\big(T, \mathcal{S}'(\mathbf{R}^d)\big) \qquad \Gamma \colon \mathbf{R}^d \times \mathbf{R}^d \to G$$
$$x \mapsto \Pi_x \qquad (x, y) \mapsto \Gamma_{xy}$$

such that $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$ and $\Pi_x \Gamma_{xy} = \Pi_y$. Furthermore, given $r > |\inf A|$, for any compact set $\mathfrak{K} \subset \mathbf{R}^d$ and constant $\gamma > 0$, there exists a constant C such that the bounds

$$\big|(\Pi_x a)(\varphi_x^\delta)\big| \leq C\delta^\alpha \|a\|_\alpha \; , \qquad \|\Gamma_{xy} a\|_\beta \leq C|x - y|^{\alpha - \beta} \|a\|_\alpha \; ,$$

hold uniformly over all test functions $\varphi : \mathbf{R}^d \to \mathbf{R}$ with support on the unit ball satisfying $\|\varphi\|_{\mathcal{C}_r} \leq 1$, $(x, y) \in \mathfrak{K}$, $\delta \in (0, 1]$, $a \in T_\alpha$ with $\alpha \leq \gamma$, and $\beta < \alpha$. Here, for any test function φ, φ_x^δ is a shorthand for the rescaled function $\varphi_x^\delta(y) = \delta^{-d}\varphi\big(\delta^{-1}(y - x)\big)$.

Fix $\mathscr{T} = (A, T, G)$ and (Π, Γ) model with scaling $\mathfrak{s}$.

Definition (Modelled distributions)

For any $\gamma \in \mathbf{R}$, the space $\mathcal{D}^\gamma$ consists of all $f\colon \mathbf{R}^d \to T_\gamma^-$ such that, for every compact set $\mathfrak{K} \subset \mathbf{R}^d$, one has

$$|\!|\!| f |\!|\!|_{\gamma;\mathfrak{K}} = \sup_{x\in\mathfrak{K}} \sup_{\substack{\beta<\gamma \\ \beta\in A}} \|f(x)\|_\beta + \sup_{\substack{(x,y)\in\mathfrak{K} \\ \|x-y\|_\mathfrak{s}\le 1}} \sup_{\substack{\beta<\gamma \\ \beta\in A}} \frac{\|f(x) - \Gamma_{xy} f(y)\|_\beta}{\|x-y\|_\mathfrak{s}^{\gamma-\beta}} < \infty\;.$$

Definition (Generalized Hölder spaces)

Let $\alpha < 0$ and let $r = -\lfloor \alpha \rfloor$. We say that $\xi \in \mathcal{S}'$ belongs to $\mathcal{C}_\mathfrak{s}^\alpha$ if it belongs to the dual of $\mathcal{C}_0^r$ and, for every compact set $\mathfrak{K}$, there exists a constant C such that the bound

$$\langle \xi, \mathcal{S}_{\mathfrak{s},x}^\delta \eta\rangle \le C\delta^\alpha,$$

holds for all $\eta \in \mathcal{C}^r$ with $\|\eta\|_{\mathcal{C}^r} \le 1$ and $\operatorname{supp}\eta \subset B_\mathfrak{s}(0,1)$, all $\delta \le 1$, and all $x \in \mathfrak{K}$.

For $\xi \in \mathcal{C}_\mathfrak{s}^\alpha$ and $\mathfrak{K}$ a compact set, we denote by $\|\xi\|_{\alpha;\mathfrak{K}}$ the seminorm given by

$$\|\xi\|_{\alpha;\mathfrak{K}} := \sup_{x\in\mathfrak{K}} \sup_{\eta\in\mathcal{B}_{\mathfrak{s},0}^r} \sup_{\delta\le 1} \delta^{-\alpha} |\langle \xi, \mathcal{S}_{\mathfrak{s},x}^\delta \eta\rangle|\;.$$

Theorem (Reconstruction theorem, Part 1)

Let $\alpha = \min A$, and let $r > |\alpha|$. Then, for every $\gamma \in \mathbf{R}$, there exists a continuous linear map $\mathcal{R}\colon \mathcal{D}^\gamma \to \mathcal{C}_\mathfrak{s}^\alpha$ with the property that, for every compact set $\mathfrak{K} \subset \mathbf{R}^d$,

$$\big|\big(\mathcal{R}f - \Pi_x f(x)\big)(\mathcal{S}_{\mathfrak{s},x}^\delta \eta)\big| \lesssim \delta^\gamma \|\Pi\|_{\gamma;\bar{\mathfrak{K}}} |\!|\!| f |\!|\!|_{\gamma;\bar{\mathfrak{K}}}\;, \tag{1}$$

uniformly over all test functions $\eta \in \mathcal{B}_{\mathfrak{s},0}^r$, all $\delta \in (0,1]$, all $f \in \mathcal{D}^\gamma$, and all $x \in \mathfrak{K}$. If $\gamma > 0$, then the bound (1) defines $\mathcal{R}f$ uniquely. Here, we denoted by $\bar{\mathfrak{K}}$ the 1-fattening of $\mathfrak{K}$.

Let $\mathscr{T} = (A, T, G)$ be a regularity structure with scaling $\mathfrak{s}$ and two models (Π, Γ) and $(\bar\Pi, \bar\Gamma)$.
For $f \in \mathcal{D}^\gamma(\Gamma)$ and $\bar f \in \mathcal{D}^\gamma(\bar\Gamma)$ we introduce

$$\|f;\bar f\|_{\gamma;\mathfrak{K}} := \|f - \bar f\|_{\gamma;\mathfrak{K}} + \sup_{\substack{(x,y)\in\mathfrak{K} \\ \|x-y\|_\mathfrak{s}\le 1}} \sup_{\substack{\beta<\gamma \\ \beta\in A}} \frac{\|f(x) - \bar f(x) - \Gamma_{xy} f(y) + \bar\Gamma_{xy}\bar f(y)\|_\beta}{\|x-y\|_\mathfrak{s}^{\gamma-\beta}} .$$

Theorem (Reconstruction theorem, Part 2)

Let $\alpha = \min A$, and let $r > |\alpha|$.

1. *If $\mathcal{R}$ is the reconstruction operator associated to (Π, Γ) and $\overline{\mathcal{R}}$ to $(\bar\Pi, \bar\Gamma)$, then one has the bound*
$$\big|\big(\mathcal{R}f - \overline{\mathcal{R}}\bar f - \Pi_x f(x) + \bar\Pi_x \bar f(x)\big)(\mathcal{S}^\delta_{\mathfrak{s},x}\eta)\big| \lesssim \delta^\gamma\big(\|\bar\Pi\|_{\gamma;\bar{\mathfrak{K}}}\|f;\bar f\|_{\gamma;\bar{\mathfrak{K}}} + \|\Pi - \bar\Pi\|_{\gamma;\bar{\mathfrak{K}}}\|f\|_{\gamma;\bar{\mathfrak{K}}}\big),$$
uniformly over x and η as before.
2. *Finally, for $0 < \kappa < \gamma/(\gamma - \alpha)$ and for every $C > 0$, one has the bound*
$$\big|\big(\mathcal{R}f - \overline{\mathcal{R}}\bar f - \Pi_x f(x) + \bar\Pi_x \bar f(x)\big)(\mathcal{S}^\delta_{\mathfrak{s},x}\eta)\big| \lesssim \delta^{\bar\gamma}\big(\|f - \bar f\|^\kappa_{\gamma;\bar{\mathfrak{K}}} + \|\Pi - \bar\Pi\|^\kappa_{\gamma;\bar{\mathfrak{K}}} + \|\Gamma - \bar\Gamma\|^\kappa_{\gamma;\bar{\mathfrak{K}}}\big),$$
where we set $\bar\gamma := \gamma - \kappa(\gamma - \alpha)$, and where we assume that $\|f\|_{\gamma;\bar{\mathfrak{K}}}$, $\|\Pi\|_{\gamma;\bar{\mathfrak{K}}}$ and $\|\Gamma\|_{\gamma;\bar{\mathfrak{K}}}$ are bounded by C, and similarly for $\bar f$, $\bar\Pi$ and $\bar\Gamma$.

Theorem (Wavelet analysis)

One has $\langle \psi_x^n, \psi_y^m \rangle = \delta_{n,m}\delta_{x,y}$ for every $n, m \in \mathbb{Z}$ and every $x \in \Lambda_n$, $y \in \Lambda_m$. Furthermore, $\langle \varphi_x^n, \psi_y^m \rangle = 0$ for every $m \geq n$ and every $x \in \Lambda_n$, $y \in \Lambda_m$. Finally, for every $n \in \mathbb{Z}$, the set

$$\{\varphi_x^n \,:\, x \in \Lambda_n\} \cup \{\psi_x^m \,:\, m \geq n\,,\; x \in \Lambda_m\}\;,$$

forms an orthonormal basis of $L^2(\mathbf{R})$.

Extending the construction to $\mathbf{R}^d$

For any given scaling $\mathfrak{s}$ of $\mathbb{R}^d$ and any $n \in \mathbb{Z}$, we thus define

$$\Lambda_n^{\mathfrak{s}} = \left\{ \sum_{j=1}^{d} 2^{-n\mathfrak{s}_j} k_j e_j \,:\, k_j \in \mathbb{Z} \right\} \subset \mathbb{R}^d\;,$$

For every $x \in \Lambda_n^{\mathfrak{s}}$, we then set

$$\varphi_x^{n,\mathfrak{s}}(y) := \prod_{j=1}^{d} \varphi_{x_j}^{n\mathfrak{s}_j}(y_j),$$

with

$$\varphi_{x_j}^{n\mathfrak{s}_j}(y_j) = 2^{n\mathfrak{s}_j/2}\varphi\big(2^{n\mathfrak{s}_j}(y_j - x_j)\big), \quad j = 1, \ldots, d.$$

Similarly, there exists a finite collection Ψ of orthonormal compactly supported functions such that, if we define V_n similarly as before, $V_n^{\perp}$ is given by

$$V_n^{\perp} = \mathrm{span}\{\psi_x^{n,\mathfrak{s}} \,:\, \psi \in \Psi \quad x \in \Lambda_n^{\mathfrak{s}}\}\;.$$

In this expression, given a function $\psi \in \Psi$, we have set

$$\psi_x^{n,\mathfrak{s}} = 2^{-n|\mathfrak{s}|/2}\mathcal{S}_{\mathfrak{s},x}^{2^{-n}}\psi.$$

This collection forms an orthonormal basis of $V_n^{\perp}$.

Fix $\mathscr{T} = (A, T, G)$ and (Π, Γ) model with scaling $\mathfrak{s}$.

Proposition (Characterising $\mathcal{C}^\alpha_\mathfrak{s}$ by wavelet coefficients)

Let $\alpha < 0$ and $\xi \in \mathcal{S}'(\mathbf{R}^d)$. Consider a wavelet analysis with a compactly supported scaling function $\varphi \in \mathcal{C}^r$ for some $r > |\alpha|$.
Then, $\xi \in \mathcal{C}^\alpha_\mathfrak{s}$ iff ξ belongs to the dual of $\mathcal{C}^r_0$ and, for every compact set $\mathfrak{K} \subset \mathbf{R}^d$, the bounds

$$|\langle \xi, \psi^{n,\mathfrak{s}}_x \rangle| \lesssim 2^{-\frac{n|\mathfrak{s}|}{2} - n\alpha}\;, \qquad |\langle \xi, \varphi^0_y \rangle| \lesssim 1\;,$$

hold uniformly over $n \geq 0$, every $\psi \in \Psi$, every $x \in \Lambda^\mathfrak{s}_n \cap \mathfrak{K}$, and every $y \in \Lambda^\mathfrak{s}_0 \cap \mathfrak{K}$.

Theorem (Convergence criterion in $\mathcal{C}^\alpha_\mathfrak{s}$)

Let $\mathfrak{s}$ be a scaling of $\mathbf{R}^d$, let $\alpha < 0 < \gamma$, and fix a wavelet basis with regularity $r > |\alpha|$. For every $n \geq 0$, let $x \mapsto A^n_x$ be a function on $\mathbf{R}^d$ satisfying the bounds

$$|A^n_x| \leq \|A\| 2^{-\frac{n\mathfrak{s}}{2} - \alpha n}\;, \qquad |\delta A^n_x| \lesssim \|A\| 2^{-\frac{n\mathfrak{s}}{2} - \gamma n}\;,$$

for some constant $\|A\|$, uniformly over $n \geq 0$ and $x \in \mathbf{R}^d$.
Then, the sequence $\{f_n\}_{n\geq 0}$ given by $f_n = \sum_{x \in \Lambda^\mathfrak{s}_n} A^n_x \varphi^{n,\mathfrak{s}}_x$ converges in $\mathcal{C}^{\bar\alpha}_\mathfrak{s}$ for every $\bar\alpha < \alpha$ and its limit f belongs to $\mathcal{C}^\alpha_\mathfrak{s}$. Furthermore, the bounds

$$\|f - f_n\|_{\bar\alpha} \lesssim \|A\| 2^{-(\alpha - \bar\alpha)n}\;, \qquad \|\mathcal{P}_n f - f_n\|_\alpha \lesssim \|A\| 2^{-\gamma n}\;,$$

hold for $\bar\alpha \in (\alpha - \gamma, \alpha)$, where $\mathcal{P}_n$ is given by

$$\mathcal{P}_n f := \sum_{x \in \Lambda_n} \langle f, \varphi^n_x \rangle \varphi^n_x.$$

Suppose there exists a family $x \mapsto \zeta_x \in \mathcal{S}'(\mathbf{R}^d)$ of distributions such that the sequence f_n is given by

$$f_n = \sum_{x \in \Lambda_n^{\mathfrak{s}}} A_x^n \varphi_x^{n,\mathfrak{s}},$$

with $A_x^n = \langle \varphi_x^{n,\mathfrak{s}}, \zeta_x \rangle$.

Proposition

In the above situation, assume that the family ζ_x is such that, for some constants K_1 and K_2 and exponents $\alpha < 0 < \gamma$, the bounds

$$|\langle \varphi_x^{n,\mathfrak{s}}, \zeta_x - \zeta_y \rangle| \leq K_1 \|x - y\|_{\mathfrak{s}}^{\gamma-\alpha} 2^{-\frac{n|\mathfrak{s}|}{2} - \alpha n},$$
$$|\langle \varphi_x^{n,\mathfrak{s}}, \zeta_x \rangle| \leq K_2 2^{-\alpha n - \frac{n|\mathfrak{s}|}{2}},$$

hold uniformly over all x, y such that $2^{-n} \leq \|x - y\|_{\mathfrak{s}} \leq 1$. Here, as before, φ is the scaling function for a wavelet basis of regularity $r > |\alpha|$.

Then, the $\lim_{n\to\infty} f_n = f$ exists and the limit distribution $f \in \mathcal{C}_{\mathfrak{s}}^{\alpha}$ satisfies the bound

$$|(f - \zeta_x)(\mathcal{S}_{\mathfrak{s},x}^{\delta} \eta)| \lesssim K_1 \delta^{\gamma},$$

uniformly over $\eta \in \mathcal{B}_{\mathfrak{s},0}^{r}$. Here, the proportionality constant only depends on the choice of wavelet basis, but not on K_2.

Classical multiplication

- $\mathcal{C}^\alpha \times \mathcal{C}^\beta \to \mathcal{C}^{\alpha\wedge\beta}$ continuous for $\alpha + \beta > 0$.
- Not continuous for $\alpha + \beta \leq 0, \alpha \notin \mathbb{N}$.

Multiplication of modelled distributions

- Algebraic structure: need product on T.
- Get $\mathcal{D}^{\gamma_1}_{\alpha_1} \times \mathcal{D}^{\gamma_2}_{\alpha_2} \to \mathcal{D}^{(\gamma_1+\alpha_2)\wedge(\gamma_2+\alpha_1)}_{\alpha_1+\alpha_2}$ continuous.
- Note: $(\mathcal{R}f_1)(\mathcal{R}f_2) \neq \mathcal{R}(f_1 f_2)$ in general, even when this makes sense in the classical way.
- However, the formalism is flexible enough for products that encode some renormalisation procedure.

Constructing products on T

- Constructing products on Hopf algebras T.
- Example: Polynomial regularity structure.
- Example: Regularity structure of rough paths.

Composition of functions

- $G \circ f \in \mathcal{D}^\gamma(V)$ if $G : \mathbb{R}^n \to \mathbb{R}^n$ is smooth, $f \in \mathcal{D}^\gamma(V)$, and $V \subseteq T$ is function-like.

Definition ($\mathscr{D}^\gamma$)

Given a regularity structure $\mathscr{T}$ equipped with a model (Π, Γ) over $\mathbf{R}^d$, the space $\mathscr{D}^\gamma$ is given by the set of functions $f \colon \mathbf{R}^d \to \bigoplus_{\alpha<\gamma} T_\alpha$ such that, for every compact set $\mathfrak{K}$ and every $\alpha < \gamma$, the exists a constant C with

$$\|f(x) - \Gamma_{xy} f(y)\|_\alpha \leq C|x-y|^{\gamma-\alpha}$$

uniformly over $x, y \in \mathfrak{K}$.

Definition ($\mathscr{D}^\gamma_\alpha$)

$\mathscr{D}^\gamma_\alpha$ denotes those elements $f \in \mathscr{D}^\gamma$ such that

$$f(x) \in T^+_\alpha \equiv \bigoplus_{\beta \geq \alpha} T_\beta, \qquad \forall x.$$

Definition ($\mathcal{C}^\alpha$)

Let (A, T, G) be the polynomial regularity structure. A function $f \colon \mathbf{R}^d \to \mathbf{R}$ is of class $\mathcal{C}^\alpha$ with $\alpha > 0$ if and only if the Taylor expansion

$$F(x) = \sum_{|k|_\mathfrak{s} < \alpha} \frac{X^k}{k!} D^k f(x) \; .$$

is of class $\mathscr{D}^\alpha$.

Definition ($\mathcal{C}^{-\alpha}$)

For each $\alpha > 0$, we denote by $\mathcal{C}^{-\alpha}$ the space of all Schwartz distributions η such that η belongs to the dual of $\mathcal{C}^r$ with $r = \lceil \alpha \rceil$ and such that

$$\left|\eta(\varphi^\lambda_x)\right| \lesssim \lambda^{-\alpha} \; ,$$

uniformly over all $\varphi \colon \mathbf{R}^d \to \mathbf{R}$ with $\|\varphi\|_{\mathcal{C}^r} \leq 1$ supported in the unit ball around the origin, and $\lambda \in (0,1]$, and locally uniformly in x.

Theorem (Classical multiplication)

If $\beta > \alpha$, then there is a continuous bilinear map $B\colon \mathcal{C}^{-\alpha} \times \mathcal{C}^{\beta} \to \mathcal{S}'(\mathbf{R}^d)$ such that $B(f,g) = fg$ for any two continuous functions f and g.

Definition (Sector)

Given a regularity structure (T, A, G) we say that a subspace $V \subset T$ is a *sector* if it is invariant under the action of the structure group G and if it can furthermore be written as $V = \bigoplus_{\alpha \in A} V_\alpha$ with $V_\alpha \subset T_\alpha$.

Definition (Multiplication in T)

Given a regularity structure (T, A, G) and two sectors $V, \bar V \subset T$, a *product* on $(V, \bar V)$ is a bilinear map $\star\colon V \times \bar V \to T$ such that, for any $\tau \in V_\alpha$ and $\bar\tau \in \bar V_\beta$, one has $\tau \star \bar\tau \in T_{\alpha+\beta}$ and such that, for any element $\Gamma \in G$, one has $\Gamma(\tau \star \bar\tau) = \Gamma\tau \star \Gamma\bar\tau$. Furthermore, $\star : V_\alpha \times \bar V_\beta \to T_{\alpha+\beta}$ is continuous.

Theorem (Multiplication of modeled distributions)

Let $f_1 \in \mathscr{D}^{\gamma_1}_{\alpha_1}(V)$, $f_2 \in \mathscr{D}^{\gamma_2}_{\alpha_2}(\bar V)$, and let $\star$ be a product on $(V, \bar V)$. Then, the function f given by $f(x) = f_1(x) \star f_2(x)$ belongs to $\mathscr{D}^{\gamma}_{\alpha}$ with

$$\alpha = \alpha_1 + \alpha_2\;, \qquad \gamma = (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1)\;.$$

Remark

If $\Pi_x\tau$ happens to be a continuous function for every $\tau \in T$ and the product satisfies $\Pi_x(a \star b) = \Pi_x(a)\Pi_x(b)$ we also have

$$\mathcal{R}(f_1 \star f_2)(x) = \Pi_x\big(f_1(x) \star f_2(x)\big)(x) = \Pi_x\big(f_1(x)\big)(x)\Pi_x\big(f_2(x)\big)(x) = \mathcal{R}f_1(x)\,\mathcal{R}f_2(x).$$

This holds for example if $f_i \in \mathscr{D}^{\gamma}_0(V)$ with $\gamma > 0$. Note however, that even if both $\mathcal{R}f_1$ and $\mathcal{R}f_2$ happen to be continuous functions, this does *not* in general imply that $\mathcal{R}(f_1 \star f_2)(x) = (\mathcal{R}f_1)(x)\,(\mathcal{R}f_2)(x)$!

Definition (Composition with smooth functions)

Let V be a function-like sector (i.e., $V_\alpha = 0$ if $\alpha < 0$ and $V_0 = \mathbf{R}$) endowed with a product $\star\colon V \times V \to V$. For any smooth function $G\colon \mathbf{R} \to \mathbf{R}$ and any $f \in \mathscr{D}^\gamma(V)$ with $\gamma > 0$, we can then *define* $G(f)$ to be the V-valued function given by

$$\big(G \circ f\big)(x) = \sum_{k \geq 0} \frac{G^{(k)}(\bar f(x))}{k!} \tilde f(x)^{\star k} \;,$$

where we have set

$$\bar f(x) = \langle \mathbf{1}, f(x)\rangle \;, \qquad \tilde f(x) = f(x) - \bar f(x)\mathbf{1} \;.$$

Here, $G^{(k)}$ denotes the kth derivative of G and $\tau^{\star k}$ denotes the k-fold product $\tau \star \cdots \star \tau$. We also used the usual conventions $G^{(0)} = G$ and $\tau^{\star 0} = \mathbf{1}$.

Proposition (Regularity of composition with smooth function)

In the same setting as above, provided that G is of class $\mathcal{C}^k$ with $k > \gamma/\alpha_0$, the map $f \mapsto G \circ f$ is continuous from $\mathscr{D}^\gamma(V)$ into itself. If $k > \gamma/\alpha_0 + 1$, then it is locally Lipschitz continuous.

Multiplication

Hopf algebras

Definition

- Algebra (T, ∇, e) over $\mathbb{R}$: unital, associative, commutative.
- Coalgebra (T, Δ, ϵ) over $\mathbb{R}$: counital, coassociative.
- Compatibility: for all $p, q \in T$,

$$\Delta(pq) = (\Delta p)(\Delta q), \quad \Delta e = e \otimes e, \quad \epsilon(pq) = \epsilon(p)\epsilon(q), \quad \epsilon(e) = 1.$$

- Grading: $T = \bigoplus_{k \in \mathbb{Z}_+^d} T_k$ with $\dim T_k < \infty$ such that

$$\nabla \colon T_k \times T_\ell \to T_{k+\ell}, \qquad \Delta \colon T_k \to \bigoplus_{\ell+m=k} T_\ell \otimes T_m.$$

- Connectedness: $T_0 = \mathrm{span}_{\mathbb{R}}\{e\}$.
- Antipode: linear mapping $\mathcal{A} : T_k \to T_k$ such that

$$\begin{array}{ccccc} T \otimes T & \xrightarrow[\mathrm{id} \otimes \mathcal{A}]{\mathcal{A} \otimes \mathrm{id}} & & & T \otimes T \\ \Delta \uparrow & & & & \downarrow \nabla \\ T & \xrightarrow{\epsilon} & \mathbb{R} & \xrightarrow{e} & T \end{array}$$

Constructing a group acting on T

- Dual Hopf algebra $(T^*, \nabla^*, e^*, \Delta^*, \epsilon^*, \mathcal{A}^*)$.
- Primitive elements $P(T^*) = \{f \in T^* \colon \Delta^* f = e^* \otimes f + f \otimes e^*\}$ form a Lie algebra with universal enveloping algebra T^* (Milnor-Moore Theorem).
- Define $G = \exp(P(T^*)) \subset T^*$. Then $\Delta^* g = g \otimes g$ holds, for all $g \in G$.
- Group action 1: $\langle f, \Gamma_g p \rangle = \langle fg, p \rangle$, for all $f \in T^*, g \in G, p \in T$.
- Group action 2: $\langle f, \Gamma_g p \rangle = \langle (\mathcal{A}^* g) f, p \rangle$, for all $f \in T^*, g \in G, p \in T$.

Properties of the group action

- If $p \in T_\gamma$, then $\Gamma_g p - p \in T_\gamma^-$.
- Multiplication on T is *regular*:

$$\Gamma_g(pq) = \Gamma_g(p)\Gamma_g(q), \qquad \forall g \in G,$$

as a consequence of $\Delta^* g = g \otimes g$.

Multiplication

Polynomial regularity structure as a Hopf algebra

Definition

- $A = \mathbb{N}_0$, $T = \mathbb{R}[X_1, \dots, X_d]$, $G = \mathbb{R}^d$.
- Group action $(\Gamma_g p)(X) = p(X + g)$.

Hopf algebra structure on T

- Multiplication ∇ as usual; unit $e = 1$.
- Comultiplication Δ is the unique homomorphism satisfying $\Delta X_i = 1 \otimes X_i + X_i \otimes 1$ ("divided powers"); counit ϵ is evaluation at zero.
- Antipode $\mathcal{A}$ is the unique antihomomorphism satisfying $\mathcal{A} X_i = -X_i$.

Dual structure on T^*

- T^* identified with formal differential operators $\sum_{n \geq 0} a_{i_1,\dots,i_n} \frac{\partial}{\partial X^{i_1} \dots \partial X^{i_n}}$ with constant coefficients.
- Pairing with T given by differentiation and evaluation at zero.
- Multiplication ∇^* is composition of differential operators.
- Comultiplication Δ^* is the unique homomorphism satisfying $\Delta^* \frac{\partial}{\partial X^i} = \frac{\partial}{\partial X^i} \otimes \mathrm{id} + \mathrm{id} \otimes \frac{\partial}{\partial X^i}$.

Group and group action

- Primitive elements $P(T^*)$ are first order differential operators because $\Delta^* f = e^* \otimes f + f \otimes e^* \Leftrightarrow \langle f, pq \rangle = p(0)\langle f, q \rangle + \langle f, p \rangle q(0)$.
- $G = \exp(P(T^*))$ are translations.
- This is group action 1: $\langle f, \Gamma_g p \rangle = \langle fg, p \rangle$.

Standard model

- $(\Pi_x X_k)(y) = (y - x)^k$
- $\Gamma_{xy} = x - y \in G$

Multiplication

Regularity structure of rough paths as a Hopf algebra

Definition

- $\gamma > 0, E = \mathbb{R}^d$.
- $A = \gamma\mathbb{N}_0$, $T = \bigoplus_{k=0}^{\infty} T_{\gamma k}$ with $T_{\gamma k} = (E^*)^{\otimes k}$, $G = \exp(\mathrm{Lie}(E))$.
- Group action $\langle f, \Gamma_g p\rangle = \langle g^{-1}f, p\rangle$, for all $f \in \prod_{k=0}^{\infty} E^{\otimes k}, g \in G, p \in T$.

Hopf algebra structure on T

- Multiplication $\nabla = ⧢$ is the shuffle product; unit $e = 1 \in \mathbb{R}$.
- Comultiplication Δ obtained by duality from multiplication on T^*, i.e., $\langle f \otimes g, \Delta p\rangle = \langle fg, p\rangle$; counit ϵ extracts the $\mathbb{R}$-component.
- Antipode $\mathcal{A}$ is the unique antihomomorphism satisfying $\mathcal{A}x = -x, \forall x \in E^*$.

Dual structure on T^*

- $T^* = \prod_{k=0}^{\infty} E^{\otimes k}$ is the (pre-)dual of T.
- Multiplication ∇^* is concatenation (alias tensorisation); unit $e^* = 1 \in \mathbb{R}$.
- Comultiplication Δ^* obtained by duality from multiplication on T, i.e., $\langle \Delta^* f, p \otimes q\rangle = \langle f, p ⧢ q\rangle$; counit ϵ^* extracts the $\mathbb{R}$-component.

Group and group action

- Primitive elements $P(T^*) = \mathrm{Lie}(E) \subset T^*$.
- $G = \exp(P(T^*)) \subset T^*$ has the property $\mathcal{A}g = g^{-1}, \forall g \in G$.
- This is group action 2: $\langle f, \Gamma_g p\rangle = \langle(\mathcal{A}g)f, p\rangle = \langle g^{-1}f, p\rangle$.

Standard model

- $(\Pi_s a)(t) = \langle \boldsymbol{X}_{st}, a\rangle$
- $\Gamma_{st} = \boldsymbol{X}_{st} \in G$