# Malliavin Calculus: The Hörmander Theorem

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## Basic assumptions on vector fields

We shall always assume the following conditions on vector fields $X : \mathbb{R}^M \times \mathbb{R}^N \to \mathbb{R}^N$:

1. $X$ are measurable.
2. There is a constant $C$ such that $||X(y, x_1) - X(y, x_2)|| \leq C||x_1 - x_2||$ for all $x_1, x_2 \in \mathbb{R}^N$ and all $y \in \mathbb{R}^M$.
3. The function $||X(y, x)||$ is bounded by a constant polynomial in $||y||$ for all $x \in \mathbb{R}^N$.

# Main E&U theorem

Let $T > 0$ and given a probability space $(\Omega, \mathcal{F}, P)$ together with a $d$-dimensional Brownian motion $(W_t)_{0 \leq t \leq T}$. Let $A, A_1, \ldots, A_d$ be vector fields satisfying the above conditions and assume that there are a continuous, adapted $\mathbb{R}^M$-valued process $(Z_t)_{t \geq 0}$ with

$$\sup_{t \in [0,T]} ||Z_t||_p < \infty$$

for all $p \geq 2$ and a continuous adapted $\mathbb{R}^N$-valued process $(\alpha_t)_{t \geq 0}$ with

$$\sup_{t \in [0,T]} ||\alpha_t||_q < \infty$$

for some $q \geq 2$, then the stochastic differential equation

$$X_t = \alpha_t + \int_0^t A(Z_s, X_s) ds + \sum_{i=1}^{d} \int_0^t A_i(Z_s, X_s) dW_s^i$$

# Main E&U theorem

has a unique continuous adapted solution $(X_t)_{0\leq t\leq T}$ with

$$\sup_{t\in[0,T]} ||X_t||_q < \infty.$$

Furthermore the solution can be constructed as $L^q$-limit of the iteration scheme

$$X_t^{n+1} = \alpha_t + \int_0^t A(Z_s, X_s^n)ds + \sum_{i=1}^d \int_0^t A_i(Z_s, X_s^n)dW_s^i$$
$$X_t^0 = \alpha_t$$

for $0 \leq t \leq T$.

This fundamental result leads to the following observations: given a probability space $(\Omega, \mathcal{F}, P)$ together with a $d$-dimensional Brownian motion $(W_t)_{0\leq t\leq T}$ in its natural filtration, we can ask whether the solution of the stochastic differential equation

$$\begin{aligned} dX_t^x &= V(X_t^x)dt + \sum_{i=1}^{d} V_i(X_t^x)dW_t^i \\ X_0^x &= x \end{aligned}$$

for $x \in \mathbb{R}^N$ lies in $\mathcal{D}^{1,2}(\Omega; \mathbb{R}^N)$. Here we work with the isonormal Gaussian process $W : L^2([0,T], \mathbb{R}^d) \to L^2(\Omega)$

$$W(h) = \sum_{i=1}^{d} \int_0^T h_i(s)dW_s^i.$$

# $C^\infty$-boundedness

We assume the following conditions on the vector fields $V, V_1, ..., V_d : \mathbb{R}^N \to \mathbb{R}^N$:

1. The vector fields are smooth.
2. All derivatives of order higher than 1 are bounded.

These conditions are usually referred to as *$C^\infty$-boundedness conditions*.

# Malliavin derivative of $X_t$

Let $V, V_1, ..., V_d : \mathbb{R}^N \to \mathbb{R}^N$ be vector fields satisfying $C^\infty$-boundedness conditions, then $X_t^x \in \mathcal{D}^\infty := \cap_{p\geq 1, k\geq 1}\mathcal{D}^{k,p}$ for $0 \leq t \leq T$. The first Malliavin derivative satisfies the following stochastic differential equation

$$D_r^k X_t^x = V_k(X_r^x) + \int_r^t dV(X_s^x) D_r^k X_s^x ds + \sum_{i=1}^d \int_r^t dV_i(X_s^x) D_r^k X_s^x dW_s^i$$

for $0 \leq r \leq t \leq T$.

For the proof we apply two observations. Given $u \in \mathcal{D}^{1,2}(\Omega; \mathbb{R}^N) \otimes H$ predictable, then for $t \geq r$

$$D_r \int_0^t u_s ds = \int_r^t D_r u_s ds$$

by Riemannian approximations and closedness of the operator, since $D_r u_s = 0$ almost surely if $r > s$.

Given predictable $u = (u_1, \dots, u_d) \in \mathcal{D}^{1,2}(\Omega; \mathbb{R}^N) \otimes H$, then for $t \geq r$

$$D_r^k \int_0^t \sum_{i=1}^d u_i(s) dW_s^i = \int_r^t \sum_{i=1}^d D_r^k u_i(s) dW_s^i + u_k(r),$$

again by Riemannian sums and closedness of the Malliavin derivative operator.

Going to the Picard approximation scheme we can apply these results to obtain a sequence $X_t^n \in L^{\infty-0}$ with $X^n \in \mathcal{D}^{1,p}$ for $p \geq 2$ by induction and the chain rule for $n \geq 0$.

The derivatives converge to the solution of a stochastic differential equation, so we conclude by closedness. The solution of this stochastic differential equation exist due to the previous E&U theorem. For higher derivatives we proceed by induction.

## Semi-martingale notations

We are working with Ito diffusions, i.e. continuous adapted processes $X_t$ of the form

$$X_t = X_0 + \int_0^t v(s)ds + \sum_{i=1}^{d} \int_0^t u_i(s)dW_s^i,$$

where we assume all processes in question to be predictable and satisfy some square integrability assumptions. Notice that this decomposition into a finite variation process and a martingale is unique. For two Ito processes $X$ and $Y$ the quadratic variation process $(\langle X, Y \rangle_t)_{0 \leq t \leq T}$ is a continuous, adapted process given by

$$\langle X, Y \rangle_t = \int_0^t (\sum_{i=1}^{d} u_i^X(s)(u_i^Y(s))^{\mathsf{T}})ds.$$

## Semi-martingale notations

The Stratonovich integral (in the one-dimensional case) is then defined by

$$\int_0^t X_s \circ dY_s := \int_0^t X_t dY_s + \frac{1}{2} \langle X, Y \rangle_t .$$

We can therefore write by Ito's formula for Ito diffusions

$$\begin{aligned} df(X_t) &= (df)(X_t) dX_t + \frac{1}{2}(d^2 f)(X_t)(dX_t)(dX_t) \\ &= (df)(X_t) \circ dX_t. \end{aligned}$$

# Semi-martingale notations

Consequently the Stratonovich calculus is of first order, however, we can only integrate continuous semi-martingales. Given the solution of our SDE, we can transform since integrands are semi-martingales to Stratonovich notation and obtain

$$dX_t^x = V_0(X_t^x)dt + \sum_{i=1}^{d} V_i(X_t^x) \circ dW_t^i,$$

with the Stratonovich drift.

# Representing the Malliavin derivative

In order to find a good representation of the Malliavin derivative, we introduce first variations of the solution of the stochastic differential equation:

$$
\begin{aligned}
dJ_{s\to t}(x) &= dV_0(X_t^x)\cdot J_{s\to t}(x)dt+ \\
&\qquad + \sum_{i=1}^{d} dV_i(X_t^x)\cdot J_{s\to t}(x)\circ dW_t^i, \\
J_{s\to s}(x) &= id_N,
\end{aligned}
$$

for $t \geq s$.

# Representing the Malliavin derivative

A similar equation is satisfied by the Malliavin derivative itself (except for the initial value!). The equation for the inverse is of the same type, namely

$$d(J_{s\to t}(x))^{-1} = -J_{s\to t}(x)^{-1} \cdot dV_0(X_t^x)dt - \\ - \sum_{i=1}^{d} J_{s\to t}(x)^{-1} \cdot dV_i(X_t^x) \circ dW_t^i.$$

# Representing the Malliavin derivative

Calculating the semi-martingale decomposition of $(J_{0\to t}(x))^{-1}J_{0\to t}(x)$ yields the result, namely

$$(J_{0\to t}(x))^{-1}J_{0\to t}(x) = id_N,$$

hence the statement on invertibility is justified.

## Representing the Malliavin derivative

Furthermore, we are able to write the Malliavin derivative,

$$D_s^i X_t^x = J_{0\to t}(x) J_{0\to s}(x)^{-1} V_i(X_s^x) 1_{[0,t]}(s).$$

This is due to the fact that the $\mathbb{R}^N$-valued solution process $(Y_t)_{r\le t\le T}$ of

$$Y_t = V_k(X_r^x) + \int_r^t dV(X_s^x) Y_s ds + \sum_{i=1}^d \int_r^t dV_i(X_s^x) Y_s dW_s^i,$$

is given through

$$Y_t = J_{0\to t}(x) J_{0\to r}(x)^{-1} V_k(X_r^x)$$

for $r \le t$.

# Malliavin Covariance Matrix

We give ourselves a scalar product on $\mathbb{R}^N$, then we can calculate the covariance matrix with respect to a orthonormal basis, i.e.

$$\langle\gamma(X_t^x)\xi,\xi\rangle := \sum_{i=1}^{d}\int_0^t \left\langle J_{0\to t}(x)J_{0\to s}(x)^{-1}V_i(X_s^x),\xi\right\rangle^2 ds.$$

Consequently, the covariance matrix can be calculated via the reduced covariance matrix

$$\langle C_t\xi,\xi\rangle := \sum_{i=1}^{d}\int_0^t \left\langle J_{0\to s}(x)^{-1}V_i(X_s^x),\xi\right\rangle^2 ds,$$
$$\gamma(X_t^x) = J_{0\to t}(x)C^t J_{0\to t}(x)^{\mathsf{T}}.$$

In order to show invertibility of $\gamma(X_t^x)$ it is hence sufficient to show it for $C_t$, since the first variation process $J_{0\to t}(x)$ is invertible.

# Uniform Hörmander Assumptions

$$\langle V_1(x), \dots V_d(x), [V_i, V_k](x)(i, k = 0, \dots, d), \dots \rangle = \mathbb{R}^N$$

for all $x \in \mathbb{R}^N$ in a uniform way, i.e. there exists a finite number of vector fields $X_1, \dots, X_N$ generated by the above procedure through Lie-bracketing and $c > 0$ such that

$$\inf_{\xi \in S^{N-1}} \sum_{k=1}^{N} \langle X_k(x), \xi \rangle^2 \geq c$$

for all $x \in \mathbb{R}^N$. Here we apply again the Stratonovich drift vector field, i.e.

$$V_0(x) := V(x) - \frac{1}{2} \sum_{i=1}^{d} DV_i(x) \cdot V_i(x).$$

# Main Theorem (Malliavin)

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ be a filtered probability space and let $(W_t)_{t\geq 0}$ be a $d$-dimensional Brownian motion adapted to the filtration (which is not necessarily generated by the Brownian motion). Let $V, V_1, \ldots, V_d$, the diffusion vector fields be $C^\infty$-bounded on $\mathbb{R}^N$ and consider the solution $(X_t^x)_{0\leq t\leq T}$ of a stochastic differential equation (in Stratonovich notation). $V_0$ denotes the Stratonovich corrected drift term,

$$dX_t^x = V_0(X_t^x)dt + \sum_{i=1}^{d} V_i(X_t^x) \circ dW_t^i,$$

$$X_0^x = x.$$

# Main Theorem (Malliavin)

Assume uniform Hörmander condition. Then for any $p \geq 1$ we find numbers $\epsilon_0(p) > 0$ and an integer $K(p) \geq 1$ such that for each $0 < s < T$

$$\sup_{\xi \in S^{N-1}} P(\langle C^s \xi, \xi \rangle < \epsilon) \leq \epsilon^p$$

holds true for $0 \leq \epsilon \leq s^{K(p)} \epsilon_0(p)$. The result holds uniformly in $x$.

The last statement implies that $\frac{1}{det(\gamma(X_t^x))} \in L^{\infty - 0}$ and hence due to the fact that $X_t^x \in \mathcal{D}^\infty$ for $t > 0$ the law of $X_t^x$ has a density with respect to the Lebesgue measure, which is Schwarz.

Take $t > 0$. We have to form the Malliavin covariance matrix $\gamma_t$, which is done by well-known formulas on the first variation. The covariance matrix can be decomposed into

$$\gamma(X_t^x) = J_{0\to t}(x) C_t J_{0\to t}(x)^{\mathsf{T}},$$

where $C_t$, the reduced covariance matrix, is defined via

$$\langle y, C_t y \rangle = \sum_{p=1}^{d} \int_0^t \left\langle y, J_{0\to s}(x)^{-1} \cdot V_p(X_s^x) \right\rangle^2 ds.$$

We first show that $C_t$ is a positive operator. We denote the kernel of $C_t$ by $K_t \subset \mathbb{R}^N$ and get a decreasing sequence of closed random subspaces of $\mathbb{R}^N$. $V = \cup_{t>0} K_t$ is a deterministic subspace by the Blumenthal zero-one law, i.e. there exists a null set $N$ such that $V$ is deterministic on $N^c$. We shall do the following calculus on $N^c$.

We fix $y \in V$, then we consider the stopping time

$$\theta := \inf\{s, \quad q_s > 0\}$$

with respect to the continuous semi-martingale

$$q_s = \sum_{p=1}^{d} \left\langle y, J_{0\to s}(x)^{-1} \cdot V_p(X_s^x) \right\rangle^2,$$

Then $\theta > 0$ almost surely and $q_{s\wedge\theta} = 0$ for $s \geq 0$.

Now, a continuous $L^2$-semi-martingale with values in $\mathbb{R}$

$$M_s - M_0 = \sum_{k=1}^{d} \int_0^s \alpha_k(u) dW_u^k + \int_0^s \beta(u) du$$

for $s \geq 0$, which vanishes up to the stopping time $\theta$, satisfies – due to the Doob-Meyer decomposition –

$$\alpha_k(s \wedge \theta) = 0$$

for $k = 1, \ldots, d$, and $\beta(s \wedge \theta) = 0$, for $s \geq 0$.

We shall apply this consideration for the continuous semi-martingales $m_s := \left\langle y, J_{0\to s}(x)^{-1} \cdot V_p(X_s^x) \right\rangle$ on $[0, t]$ for $p = 1, \ldots, d$. Therefore we need to calculate the Doob-Meyer decomposition of $(m_s)_{0\leq s\leq t}$.

$$
\begin{aligned}
dm_s &= -\left\langle y, J_{0\to s}(x)^{-1} dV_0(X_s^x) V_p(X_s^x)\right\rangle ds - \\
&- \sum_{i=1}^{d} \left\langle y, J_{0\to s}(x)^{-1} dV_i(X_s^x) V_p(X_s^x)\right\rangle \circ dW_s^i + \\
&+ \left\langle y, J_{0\to s}(x)^{-1} dV_p(X_s^x) \cdot V_0(X_s^x)\right\rangle ds + \\
&+ \sum_{i=1}^{d} \left\langle y, J_{0\to s}(x)^{-1} dV_p(X_s^x) \cdot V_i(X_s^x)\right\rangle \circ dW_s^i \\
&= \left\langle y, J_{0\to s}(x)^{-1} [V_p, V_0](X_s^x)\right\rangle ds + \\
&+ \sum_{i=1}^{d} \left\langle y, J_{0\to s}(x)^{-1} [V_p, V_i](X_s^x)\right\rangle \circ dW_s^i \,,
\end{aligned}
$$

where we denote by $d$ the stochastic differential of $m$ and the first derivative of $V_i$.

From the Doob-Meyer decomposition this leads to

$$\left\langle y, J_{0\to s}(x)^{-1} \cdot [V_p, V_i](X_s^x)\right\rangle = 0$$
$$\left\langle y, J_{0\to s}(x)^{-1} \cdot [V_p, V_0](X_s^x)\right\rangle = 0$$

for $i = 1, \ldots, d$, $p = 1, \ldots, d$ and $0 \le s \le \theta$.

Consequently the above equation leads by iterative application to

$$\left\langle y, J_{0\to s}(x)^{-1} \cdot \mathcal{D}(X_s^x) \right\rangle = 0$$

for $s \leq \theta$, where $\mathcal{D}(x)$ is the set of Lie brackets at $x$. Evaluation at $s = 0$ yields $y = 0$, since $\mathcal{D}(x)$ spans $\mathbb{R}^N$, hence $C_t$ is invertible.

Therefore we obtain that there is a null set $N$, such that on $N^c$ the matrix $C_t$ has an empty kernel. Hence the law is absolutely continuous with respect to Lebesgue measure, since $J_{0\to t}(x)$ is invertible and therefore $\gamma_t$ has vanishing kernel.

# Smoothness – Step 1

Consider the random quadratic form

$$\langle C_s \xi, \xi \rangle = \sum_{i=1}^{d} \int_0^s \left\langle J_{0\to u}(x)^{-1} V_i(X_u^x), \xi \right\rangle^2 du.$$

We define

$$\begin{aligned}
\Sigma_0' &:= \{V_1, \ldots, V_d\} \\
\Sigma_n' &:= \Big\{ [V_k, V], k = 1, \ldots, d, V \in \Sigma_{n-1}'; [V_0, V] + \\
&\quad + \frac{1}{2} \sum_{i=1}^{d} [V_i, [V_i, V]], V \in \Sigma_{n-1}' \Big\}
\end{aligned}$$

for $n \geq 1$.

Then we know that there exists $j_0$ such that

$$\inf_{\xi \in S^{M-1}} \sum_{j=0}^{j_0} \sum_{V \in \Sigma_j'} \langle V(x), \xi \rangle^2 \geq c$$

uniformly in $x \in \mathbb{R}^M$.

# Smoothness – Step 2

We define $m(j) := 2^{-4j}$ for $0 \le j \le j_0$ and the sets

$$E_j := \{ \sum_{V \in \Sigma'_j} \int_0^s \left\langle J_{0\to u}(x)^{-1} V(X_u^x), \xi \right\rangle^2 du \le \epsilon^{m(j)} \}.$$

We consider the decomposition

$$E_0 = \{\langle C^s \xi, \xi\rangle \le \epsilon\} \subset (E_0 \cap E_1^c) \cup (E_1 \cap E_2^c) \cup \cdots \cup (E_{j_0 - 1} \cap E_{j_0}^c) \cup F,$$

$$F = E_0 \cap \cdots \cap E_{j_0}.$$

and proceed with

$$P(F) \le C \epsilon^{\frac{q\beta}{2}},$$

for $\epsilon \le \epsilon_1$. Furthermore $0 < \beta < m(j_0)$, any $q \ge 2$, a constant $C$ depending on $q$ and the norms of the derivatives of the vector fields $V_0, \ldots, V_d$. The number $\epsilon_1$ is determined by the following two (!) equations

# Smoothness – Step 3

We obtain furthermore with $n(j) = \#\Sigma'_j$

$$
\begin{aligned}
& P(E_j \cap E^c_{j+1}) \\
& \leq \sum_{V \in \Sigma'_j} P \Bigg( \int_0^s \left\langle J_{0\to u}(x)^{-1} V(X^x_u), \xi \right\rangle^2 du \leq \epsilon^{m(j)}, \\
& \sum_{k=1}^d \int_0^s \left\langle J_{0\to u}(x)^{-1} [V_k, V](X^x_u), \xi \right\rangle^2 du + \\
& + \int_0^s \left\langle J_{0\to u}(x)^{-1} \left( [V_0, V] + \frac{1}{2} \sum_{i=1}^d [V_i, [V_i, V]] \right) (X^x_u), \xi \right\rangle^2 du \\
& > \frac{\epsilon^{m(j+1)}}{n(j)} \Bigg),
\end{aligned}
$$

Since we can find the bounded variation and the quadratic variation part of the martingale $(\langle J_{0\to u}(x)^{-1}V(X_u^x), \xi\rangle)_{0\leq u\leq s}$ in the above expression, we are able to apply Norris' Lemma. We observe that $8m(j+1) < m(j)$, hence we can apply it with $q = \frac{m(j)}{m(j+1)}$.

# Smoothness – Step 4

We obtain for $p \geq 2$ – still by the Norris' Lemma – the estimate

$$P(E_j \cap E_{j+1}^c) \leq d_1 \left( \frac{\epsilon^{m(j+1)}}{n(j)} \right)^{rp} + d_2 \exp \left( -( \frac{\epsilon^{m(j+1)}}{n(j)} )^{-\nu} \right)$$

for $\epsilon \leq \epsilon_2$. Furthermore $r, \nu > 0$ with $18r + 9\nu < q - 8$, the numbers $d_1, d_2$ depend on the vector fields $V_0, \dots, V_d$, and on $p$, $T$. The number $\epsilon_2$ can be chosen like $\epsilon_2 = \epsilon_3 s^{k_1}$, where $\epsilon_3$ does not depend on $s$ anymore.

# Smoothness – Step 5

Putting all together we take the minimum of $\epsilon_1$and $\epsilon_2$ to obtain the desired dependence on $s$ by applying the following lemma:

Given a random matrix $\gamma \in \cap_{p\geq 1} L^p(\Omega)$ and assume that for $p \geq 1$ there is $\epsilon_0(p)$ such that

$$\sup_{\xi \in S^{M-1}} P(\langle \gamma\xi, \xi\rangle < \epsilon) \leq \epsilon^p$$

for $0 \leq \epsilon \leq \epsilon_0(p)$, then $\frac{1}{\det(\gamma)} \in \cap_{p\geq 1} L^p(\Omega)$.

# Definition

Let $(X_t^x)_{t\geq 0}$ denote the solution of our SDE and assume the uniform Hörmander condition. Fix $t > 0$ and $x \in \mathbb{R}^N$. Fix a direction $v \in \mathbb{R}^N$. We define a set of Skorohod-integrable processes

$$\mathbb{A}_{t,x,v} = \{a \in \mathrm{dom}(\delta) \text{ such that } \sum_{i=1}^{d} \int_0^t J_{0\to s}(x)^{-1} V_i(X_s^x) a_s^i ds = v\}.$$

Let $(X_t^x)_{t \geq 0}$ denote the unique solution of our SDE and assume $d = N$. Fix $t > 0$ and $x \in \mathbb{R}^N$. Assume furthermore uniform ellipticity, i.e., there is $c > 0$ such that

$$\inf_{\xi \in S^{M-1}} \sum_{k=1}^{N} \langle V_k(x), \xi \rangle^2 \geq c.$$

Then $\mathbb{A}_{t,x,v} \neq \emptyset$ and there exists a real valued random variable $\pi$ (which depends linearly on $v$) such that for all bounded random variables $f$ we obtain

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} E(f(X_t^{x+\epsilon v})) = E(f(X_t^x)\pi).$$

Here the proof is particularly simple, since we can take a matrix $\sigma(x) := (V_1(x), \ldots, V_N(x))$, which is uniformly invertible with bounded inverse. We define

$$a_s := \frac{1}{t}\sigma(X_s^x)^{-1} \cdot J_{0\to s}(x)v$$

for $0 \leq s \leq t$ and obtain that $a \in \mathbb{A}_{t,x,v}$. Furthermore

$$\pi = \sum_{i=1}^{d} \int_0^t a_s^i dW_s^i,$$

since the Skorohod integrable process $a$ is in fact adapted, continuous and hence Ito-integrable.

Let $(X_t^x)_{t\geq 0}$ denote the unique solution of our SDE and assume uniform Hörmander condition. Fix $t > 0$ and $x \in \mathbb{R}^N$. Fix a direction $v \in \mathbb{R}^N$. Then $\mathbb{A}_{t,x,v} \neq \emptyset$ and there exists a real valued random variable $\pi$ (which depends linearly on $v$) such that for all bounded random variables $f$ we obtain

$$\frac{d}{d\epsilon}\bigg|_{\epsilon=0} E(f(X_t^{x+\epsilon v})) = E(f(X_t^x)\pi).$$

We can choose $\pi$ to be the Skorohod integral of any element $a \in \mathbb{A}_{t,x,v} \neq \emptyset$.

We take $f$ bounded with bounded first derivative, then we obtain

$$\left.\frac{d}{d\epsilon}\right|_{\epsilon=0} E(f(X_t^{x+\epsilon v})) = E(df(X_t^x)J_{0\to t}(x)\cdot v).$$

If there is $a \in \mathbb{A}_{t,x,v}$, we obtain

$$\begin{aligned} & E(df(X_t^x)J_{0\to t}(x)\cdot v) \\ &= E(df(X_t^x)\sum_{i=1}^{d}\int_0^t J_{0\to t}(x)J_{0\to s}(x)^{-1}V_i(X_s^x)a_s^i ds) \\ &= E(\sum_{i=1}^{d}\int_0^t df(X_t^x)J_{0\to t}(x)J_{0\to s}(x)^{-1}V_i(X_s^x)a_s^i ds) \\ &= E(\sum_{i=1}^{d}\int_0^t D_s^i f(X_t^x)a_s^i ds) = E(f(X_t^x)\delta(a)). \end{aligned}$$

Here we cannot assert that the strategy is Ito-integrable, since it will anticipative in general. In order to see that $\mathbb{A}_{t,x,v} \neq \emptyset$ we construct an element, namely

$$a_s^i := \left\langle J_{0\to s}(x)^{-1} V_i(X_s^x), (C^t)^{-1} v \right\rangle ,$$

where $C^t$ denotes the reduced covariance matrix.

Indeed

$$\sum_{i=1}^{d}\left\langle \int_0^t J_{0\to s}(x)^{-1} V_i(X_s^x) a_s^i ds, \xi \right\rangle$$
$$= \sum_{i=1}^{d} \int_0^t \left\langle J_{0\to s}(x)^{-1} V_i(X_s^x), \xi \right\rangle \left\langle J_{0\to s}(x)^{-1} V_i(X_s^x), (C^t)^{-1} v \right\rangle ds$$
$$= \left\langle \xi, C^t (C^t)^{-1} v \right\rangle = \langle \xi, v \rangle$$

for all $\xi \in \mathbb{R}^N$, since $C^t$ is a symmetric random operator defined via

$$\left\langle \xi, C^t \xi \right\rangle = \sum_{i=1}^{d} \int_0^t \left\langle J_{0\to s}(x)^{-1} V_i(X_s^x), \xi \right\rangle^2 ds$$

for $\xi \in \mathbb{R}^N$.

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