UTILITY OPTIMIZATION IN A FINITE SCENARIO SETTING

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ABSTRACT. We introduce the main concepts of duality theory for utility optimization in a setting of finitely many economic scenarios.

1. UTILITY OPTIMIZATION IN DISCRETE MODELS

Definition 1.1. *A function* $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ *is called* utility function *if a strictly increasing, strictly concave* C^2*-function on its domain* $\operatorname{dom}(u)$*, i.e. the set where it is finitely valued. Furthermore we shall always assume that* $\lim_{x \downarrow \inf \operatorname{dom}(u)} u'(x) = \infty$*.*

We consider the general situation in discrete models, i.e. finite Ω. Given a financial market $(S_t^0, \dots, S_t^d)_{t=0,\dots,T}$ on $(\Omega, \mathcal{F}, P)$ and a utility function u, then we define the *utility optimization problem* as determination of $U(x)$ for $x \in \operatorname{dom}(u)$, i.e.

$$\sup_{\substack{\phi \text{ trading strategy} \\ \xi \text{ self financing} \\ V_0(\phi)=x}} E(u(\frac{1}{S_T^0} V_T(\xi)) =: U(x).$$

We say that the utility optimization problem at $x \in \operatorname{dom}(u)$ is *solvable* if $U(x)$ is finitely valued and if we find an optimal self financing trading strategy $\widehat{\xi}(x)$ for $x \in \operatorname{dom}(u)$ such that

$$U(x) = E(u(\frac{1}{S_N^0} V_N(\widehat{\xi}(x))),$$
$$V_0(\widehat{\xi}(x)) = x.$$

We shall introduce three methods for the solution of the utility optimization problem, where the number of variables involved differs.

We assume that $\mathcal{F} = 2^\Omega$ and $P(\omega) > 0$ for $\omega \in \Omega$. We then have three characteristic dimensions: the dimension of all random variables $|\Omega|$ (the number of paths), then the dimension of discounted outcomes at initial wealth 0, denoted by $\dim \mathcal{K}$, and the number of extremal points of the set of absolutely continuous martingale measures m. We have the basic relation

$$m + \dim \mathcal{K} = |\Omega| \,.$$

- the pedestrian method is an unconstraint extremal value problem in $\dim \mathcal{K}$ variables.
- the Lagrangian method yields an unconstraint extremal value problem in $|\Omega| + m$ variables.
- the duality method (martingale approach) yields an unconstraint extremal value problem in m variables. Additionally one has to transform the dual

value function to the original, which is a one dimensional extremal value problem.

In financial mathematics usually $\dim \mathcal{K} \gg m$, which means that the duality method is of particular importance.

1.1. **Pedestrian's method.** We can understand utility optimization as unrestricted optimization problem. Define $\mathcal{S}$ the vector space of all predictable strategies $(\xi_t)_{t=0,\dots,T}$, then the utility optimization problem for $x \in \operatorname{dom}(u)$ is equivalent to solving the following problem

$$F_x : \begin{cases} \mathcal{S} \to \mathbb{R} \cup \{-\infty\} \\ (\xi_t)_{t=0,\dots,T} \mapsto E(u(x + (\xi \cdot X)_T)) \end{cases}$$

$$\sup_{\xi \in \mathcal{S}} F_x(\xi) = U(x)$$

This is an ordinary extremal value problem for every $x \in \operatorname{dom}(u)$. Let $(\widehat{\xi}_t)_{t=0,\dots,T}$ be an optimal strategy (which is then necessarily in the *interior* of the domain of F_x), then necessarily

$$\operatorname{grad} F_x((\widehat{\xi}_t)_{t=0,\dots,T}) = 0$$

and therefore we can in principle calculate the optimal strategy. From this formulation we take one fundamental conclusion.

Theorem 1.2. *Let the utility optimization problem at $x \in \operatorname{dom}(u)$ be solvable and let $(\widehat{\xi}_t)_{t=0,\dots,T}$ be an optimal strategy, so*

$$\sup_{\phi \in \mathcal{S}} F_x(\xi) = U(x) = F_x(\widehat{\xi}),$$

then the set of equivalent martingale measures $\mathcal{M}^e$ of the discounted price process X is non-empty.

Proof. We calculate the directional derivative with respect to 1_A for $A \in \mathcal{F}_{i-1}$ for $i = 1, \dots, T$,

$$\begin{aligned} &\frac{d}{ds}|_{s=0} E(u(x + (\widehat{\xi} \cdot X)_T + s 1_A \Delta X_i)) \\ &= E(u'(x + (\widehat{\xi} \cdot X)_T) 1_A \Delta X_i). \end{aligned}$$

Since $(\widehat{\xi}_t)_{t=0,\dots,T}$ is an optimizer we necessarily have that the directional derivatives in direction of the elements $1_A \Delta X_i$ vanish. We define

$$\lambda := E(u'(x + (\widehat{\xi} \cdot X)_T)) > 0$$

since $u'(y) > 0$ for $y \in \operatorname{dom}(U)$. Consequently

$$\frac{dQ}{dP} := \frac{1}{\lambda} u'(x + (\widehat{\xi} \cdot X)_T)$$

defines a probability measure equivalent to P. Hence we obtain from the gradient condition that

$$E_Q(1_A(X_i - X_{i-1})) = 0$$

for all $A \in \mathcal{F}_{i-1}$ and $i = 1, \dots, T$, which means

$$E(X_i \mid \mathcal{F}_{i-1}) = X_{i-1}$$

for $i = 1, \dots, T$, therefore $Q \in \mathcal{M}^e$. □

Condition 1.3. *We shall always assume $\mathcal{M}^e \neq \emptyset$.*

Furthermore we can easily formulate a basis existence and regularity result by the pedestrian's method.

Proposition 1.4. *Assume $\mathcal{M}^e \neq \emptyset$, then the utility optimization problem for $x \in \operatorname{dom}(u)$ has a unique solution $\widehat{Y}(x) \in x+\mathcal{K}$, which is also the unique local maximum, and $x \mapsto \widehat{Y}(x)$ is C^1 on $\operatorname{dom}(u)$. If $x \notin \operatorname{dom}(u)$, then $\sup_{\xi \in \mathcal{S}} F_x(\xi) = -\infty$.*

Proof. The functional $Y \mapsto E_P(u(Y))$ is C^2, strictly concave and increasing. Assume that there are two optimizers $\widehat{Y}_1(x) \neq \widehat{Y}_2(x) \in x + \mathcal{K}$, then

$$E_P(u(t\widehat{Y}_1(x) + (1-t)\widehat{Y}_2(x))) > tE_P(u(\widehat{Y}_1(x))) + (1-t)E_P(u(\widehat{Y}_2(x))) = U(x)$$

for $t \in]0,1[$, which is a contradiction. The argument also yields that two local maxima have to coincide. Therefore the optimizer is also the unique local maximum.

The space $\mathcal{K}$ of terminal wealth with zero initial investment has the property that for $Y \in L^2(\Omega, \mathcal{F}, P)$

$$Y \in \mathcal{K} \Longleftrightarrow E_Q(Y) = 0$$

for all $Q \in \mathcal{M}^a$, where $\mathcal{M}^a$ denotes the set of absolutely continuous martingale measures. Given an equivalent martingale measure $Q \in \mathcal{M}^e$, then we can prove that for any $x \in \operatorname{dom}(u)$

$$\lim_{\substack{Y \in \mathcal{K} \\ E_Q(|Y|) \to \infty}} E_P(u(x+Y)) = -\infty.$$

The existence of the optimizer follows from a compactness consideration using the previous assertion.

For the regularity assertion we take a basis of $\mathcal{K}$ denoted by $(f_i)_{i=1,\ldots,\dim \mathcal{K}}$ and calculate the derivative with respect to this basis at the unique existing optimizer $\widehat{Y}(x)$,

$$E_P(u'(x + \widehat{Y}(x))f_i) = 0$$

for $i = 1, \ldots, \dim \mathcal{K}$. Calculating the second derivative we obtain the matrix

$$(E_P(u''(x+Y)f_i f_j))_{i,j=1,\ldots,\dim \mathcal{K}}$$

which is invertible for any $Y \in \mathcal{K}$, since u'' is strictly negative. Therefore $x \mapsto \widehat{Y}(x)$ is C^1 on $\operatorname{dom}(u)$. □

1.2. **Duality methods.** Since we have a dual relation between the set of martingale measures and the set $\mathcal{K}$ of claims attainable at price 0, we can formulate the optimization problem as constraint problem: for any $Y \in L^2(\Omega, \mathcal{F}, P)$

$$Y \in \mathcal{K} \Longleftrightarrow E_Q(X) = 0$$

for $Q \in \mathcal{M}^a$ and for any probability measure Q

$$Q \in \mathcal{M}^a \Longleftrightarrow E_Q(Y) = 0$$

for all $Y \in \mathcal{K}$. Therefore we can formulate the problem as constraint optimization problem and apply the method of Lagrangian multipliers.

First we define a function $H : L^2(\Omega, \mathcal{F}, P) \longrightarrow \mathbb{R} \cup \{-\infty\}$ via

$$H(Y) := E_P(u(Y))$$

for a utility function u. For $x \in \operatorname{dom}(u)$ we can formulate the constraints

$$\mathcal{K} + x = \{Y \in L^2(\Omega, \mathcal{F}, P) \text{ such that } E_Q(Y) = x \text{ for } Q \in \mathcal{M}^a\}.$$

Consequently the utility optimization problem reads

$$\sup_{Y \in \mathcal{K}+x} E_P(u(Y)) = U(x)$$

for $x \in \mathrm{dom}(u)$. Hence we can treat the problem by Lagrangian multipliers, i.e. if $\widehat{Y} \in \mathcal{K} + x$ is an optimizer, then

$$\text{(LM)} \qquad u'(\widehat{Y}) - \sum_{i=1}^{m} \widehat{\eta_i} \frac{dQ_i}{dP} = 0$$
$$E_{Q_i}(\widehat{Y}) = x$$

for $i = 1, \ldots, m$, $\mathcal{M}^a = \langle Q_1, \ldots, Q_m \rangle$ and some values $\widehat{\eta_i}$. This result is obtained by taking the gradient of the function

$$Y \mapsto E_P(u(Y) - \sum_{i=1}^{m} \eta_i (\frac{dQ_i}{dP} Y - x))$$

with respect to some basis. We can choose the $\widehat{\eta_i}$ positive, since $u'(\widehat{Y})$ represents a positive multiple of an equivalent martingale measure and *any* equivalent martingale measure is represented by a unique positive linear combination of $Q_1, \ldots, Q_m$. Notice that by assumption $u'(x) > 0$ for all $x \in \mathrm{dom}(u)$, and $u'(\widehat{Y})$ is finitely valued.

Lemma 1.5. *If $(\widehat{Y}, \widehat{\eta_1}, \ldots, \widehat{\eta_m})$ is a solution of the Lagrangian multiplier equation (LM), then the multipliers $\widehat{\eta_i} > 0$ are uniquely determined and $\sum_{i=1}^{m} \widehat{\eta_i} > 0$. Given $x \in \mathrm{dom}(u)$, the map $x \mapsto (\widehat{\eta_i}(x))_{i=1,\ldots,m}$ is C^1.*

Proof. The coefficients $\widehat{\eta_i}$ are uniquely determined and the inverse function theorem together with the previous result yields the C^1-dependence. □

The Lagrangian $\widetilde{L}$ is given through

$$\widetilde{L}(Y, \eta_1, \ldots, \eta_m) = E_P(u(Y)) - \sum_{i=1}^{m} \eta_i (E_{Q_i}(Y) - x)$$

for $Y \in L^2(\Omega, \mathcal{F}, P)$ and $\eta_i \geq 0$. We introduce $y := \eta_1 + \cdots + \eta_m$ and $\mu_i := \frac{\eta_i}{y}$. Therefore

$$L(Y, y, Q) = E_P(u(Y)) - y(E_Q(Y) - x)$$

for $Y \in L^2(\Omega, \mathcal{F}, P)$, $Q = \sum_i \mu_i Q_i \in \mathcal{M}^a$ and $y > 0$. We define

$$\Phi(Y) := \inf_{\substack{y>0 \\ Q \in \mathcal{M}^a}} L(Y, y, Q)$$

for $Y \in L^2(\Omega, \mathcal{F}, P)$ and

$$\psi(y, Q) = \sup_{Y \in L^2(\Omega, \mathcal{F}, P)} L(Y, y, Q)$$

for $y > 0$ and $Q \in \mathcal{M}^a$. We can hope for

$$\sup_{Y \in L^2(\Omega, \mathcal{F}, P)} \Phi(Y) = \inf_{y>0} \inf_{Q \in \mathcal{M}^a} \psi(y, Q) = U(x).$$

by a mini-max consideration.

Lemma 1.6. *We have*

$$\sup_{Y \in L^2(\Omega, \mathcal{F}, P)} \Phi(Y) = U(x)\,.$$

Proof. We can easily prove the following facts:

$$\Phi(Y) = -\infty \text{ if } E_Q(Y) > x$$

for at least one $Q \in \mathcal{M}^a$. Furthermore

$$\Phi(Y) = E_P(u(Y)) \text{ if } E_Q(Y) \leq x$$

for all $Q \in \mathcal{M}^a$. Consequently

$$\sup_{Y \in L^2(\Omega,\mathcal{F},P)} \Phi(Y) = \sup_{\substack{Y \in L^2(\Omega,\mathcal{F},P) \\ E_Q(Y) \leq x \text{ for } Q \in \mathcal{M}^a}} E_P(u(Y)) = U(x)$$

since u is increasing. □

For the proof of the minimax statement we need to calculate ψ, which is done in the next lemma. Therefore we assume the generic conditions for conjugation as stated in the Appendix.

Lemma 1.7. *The function*

$$\psi(y, Q) = \sup_{Y \in L^2(\Omega,\mathcal{F},P)} L(Y, y, Q)$$

can be expressed by the conjugate function v of u,

$$\psi(y, Q) = E_P(v(y\frac{dQ}{dP})) + yx.$$

Proof. By definition we have

$$\begin{aligned} L(Y, y, Q) &= E_P(u(Y)) - y(E_Q(Y) - x) \\ &= E_P(u(Y) - y\frac{dQ}{dP}Y) + yx. \end{aligned}$$

If we fix $Q \in \mathcal{M}^a$ and $y > 0$, then the calculation of the supremum over all random variables yields

$$\begin{aligned} &\sup_{Y \in L^2(\Omega,\mathcal{F},P)} E_P(u(Y) - y\frac{dQ}{dP}Y) \\ &= E_P(\sup_{Y \in L^2(\Omega,\mathcal{F},P)} u(Y) - y\frac{dQ}{dP}Y) \\ &= E_P(v(y\frac{dQ}{dP})) \end{aligned}$$

by definition of the conjugate function. □

Definition 1.8. *Given the above setting we call the optimization problem*

$$V(y) := \inf_{Q \in \mathcal{M}^a} E_P(v(y\frac{dQ}{dP}))$$

the dual problem and V the dual value function for $y > 0$.

Next we formulate that the dual optimization problem has a solution.

Lemma 1.9. *Let u be a utility function under the above assumptions and assume $\mathcal{M}^e \neq \emptyset$, then there is a unique optimizer $\widehat{Q}(y)$ such that*

$$V(y) = \inf_{Q \in \mathcal{M}^a} E_P(v(y\frac{dQ}{dP})) = E_P(v(y\frac{d\widehat{Q}(y)}{dP})).$$

Furthermore

$$\inf_{y>0}(V(y)+xy) = \inf_{\substack{y>0\\ Q\in\mathcal{M}^a}} (E_P(v(y\frac{dQ}{dP}))+xy)\,.$$

Proof. Since v is strictly convex, C^2 on $]0,\infty[$ and $v'(0)=-\infty$ we obtain by compactness the existence of an optimizer $\widehat{Q}(y)$ and by $v'(0)=-\infty$ that the optimizer is an equivalent martingale measure (since one can decrease the value of $v(y\frac{dQ}{dP})$ by moving away from the boundary). By strict convexity the optimizer is also unique. The gradient condition for $\widehat{Q}(y)$ reads as follows

$$E_P(v'(\widehat{Q}(y))(\frac{d\widehat{Q}(y)}{dP}-\frac{dQ}{dP}))=0$$

for all $Q\in\mathcal{M}^a$. The function V shares the same qualitative properties as v and therefore we can define the concave conjugate. Fix $x\in\operatorname{dom}(u)$ and take the optimizer $\widehat{y}=\widehat{y}(x)>0$, then

$$\begin{aligned}\inf_{y>0}(V(y)+xy) = V(\widehat{y})+x\widehat{y} &\le \inf_{Q\in\mathcal{M}^a} E_P(v(y\frac{dQ}{dP}))+xy\\ &\le E_P(v(y\frac{dQ}{dP}))+xy\end{aligned}$$

for all $Q\in\mathcal{M}^a$ and $y>0$, so

$$\inf_{y>0}(V(y)+xy) \le \inf_{\substack{y>0\\ Q\in\mathcal{M}^a}} (E_P(v(y\frac{dQ}{dP}))+xy).$$

Take $y_1>0$ and $Q_1\in\mathcal{M}^e$ for some $\epsilon>0$ such that

$$\begin{aligned}\inf_{y>0}(V(y)+xy)+2\epsilon &\ge V(y_1)+xy_1+\epsilon\\ &\ge E_P(v(y_1\frac{dQ_1}{dP}))+xy_1\\ &\ge \inf_{\substack{y>0\\ Q\in\mathcal{M}^a}} (E_P(v(y\frac{dQ}{dP}))+xy).\end{aligned}$$

Since this holds for every $\epsilon>0$ we can conclude. □

Theorem 1.10. *Let u a utility function with the above properties for an arbitrage-free financial market, then*

$$U(x) = \inf_{\substack{y>0\\ Q\in\mathcal{M}^a}} (E_P(v(y\frac{dQ}{dP}))+xy)$$

and the mini-max assertion holds.

Proof. Fix $x\in\operatorname{dom}(u)$ and take an optimizer $\widehat{Y}$, then there are Lagrangian multipliers $\widehat{\eta_1},\dots,\widehat{\eta_m}\ge 0$ such that $\widehat{y}:=\sum_{i=1}^m\widehat{\eta_i}>0$ and

$$\widetilde{L}(\widehat{Y},\widehat{\eta_1},\dots,\widehat{\eta_m})=U(x),$$

and the constraints are satisfied so $E_{Q_i}(\widehat{Y})=x$ and $\widehat{Y}$ is an optimizer. We define a measure $\widehat{Q}$ via

$$u'(\widehat{Y})=\widehat{y}\frac{d\widehat{Q}}{dP}.$$

Since

$$u'(\widehat{Y}) - \widehat{y}\sum_{i=1}^{m}\frac{\widehat{\eta}_i}{\widehat{y}}\frac{dQ_i}{dP} = 0$$

by the Lagrangian multipliers method, we see that

$$\widehat{y}\frac{d\widehat{Q}}{dP} = \widehat{y}\sum_{i=1}^{m}\frac{\widehat{\eta}_i}{\widehat{y}}\frac{dQ_i}{dP}$$

and therefore $\widehat{Q} \in \mathcal{M}^e$ (its Radon-Nikodym derivative is strictly positive). Furthermore

$$E_P(v(\widehat{y}\frac{d\widehat{Q}}{dP})) + x\widehat{y} = \inf_{Q\in\mathcal{M}^a}(E_P(v(\widehat{y}\frac{dQ}{dP})) + x\widehat{y}),$$

since $v'(y) = -(u')^{-1}(y)$ and $Q_* \in \mathcal{M}^e$ is a minimum if and only if

$$E_P(v'(y\frac{dQ_*}{dP})(\frac{dQ_*}{dP} - \frac{dQ}{dP})) = 0$$

for all $Q \in \mathcal{M}^a(\widetilde{S})$. This is satisfied by $\widehat{y}$ and $\widehat{Q}$. By definition of v we obtain

$$\begin{aligned} E_P(v(\widehat{y}\frac{d\widehat{Q}}{dP})) + x\widehat{y} &= \sup_{Y\in L^2(\Omega,\mathcal{F},P)} L(Y,\widehat{y},\widehat{Q}) \\ &= L(\widehat{Y},\widehat{y},\widehat{Q}), \end{aligned}$$

since $u'(\widehat{Y}) = \widehat{y}\frac{d\widehat{Q}}{dP}$, $v(y) = u((u')^{-1}(y) - y(u')^{-1}(y)$, so $v(\widehat{y}\frac{d\widehat{Q}}{dP}) = u(\widehat{Y}) - \frac{d\widehat{Q}}{dP}\widehat{y}\widehat{Y}$. However $L(\widehat{Y},\widehat{y},\widehat{Q}) = U(x)$ by assumption on optimality of $\widehat{Y}$. Therefore

$$E_P(v(\widehat{y}\frac{d\widehat{Q}}{dP})) + x\widehat{y} = U(x)$$

and $\widehat{y}$ is the minimizer since

$$E_P(v'(\widehat{y}\frac{d\widehat{Q}}{dP})\frac{d\widehat{Q}}{dP}) = -x$$

by assumption. Calculating with the formulas for v yields

$$\begin{aligned} \inf_{\substack{y>0\\ Q\in\mathcal{M}^a}}(E_P(v(y\frac{dQ}{dP}))) + xy) &= \inf_{y>0}(E_P(v(y\frac{d\widehat{Q}}{dP})) + xy) \\ &= U(x) \\ &= E_P(u(\widehat{Y})) \end{aligned}$$

by definition. □

This Theorem enables us to formulate the following duality relation. Given a utility optimization problem for $x \in \operatorname{dom}(u)$

$$\sup_{Y\in\mathcal{K}} E_P(u(x+Y)) = U(x),$$

then we can associate a dual problem, namely

$$\inf_{Q\in\mathcal{M}^a} E_P(v(y\frac{dQ}{dP})) = V(y)$$

for $y > 0$. The main assertion of the minimax considerations is that

$$\inf_{y>0}(V(y) + xy) = U(x),$$

so the concave conjugate of V is U and since V shares the same regularity as U, also U is the convex conjugate of V. First we solve the dual problem (which is much easier) and obtain $y \mapsto \widehat{Q}(y)$. For given $x \in \mathrm{dom}(u)$ we can calculate $\widehat{y}(x)$ and obtain

$$V(\widehat{y}(x)) + x\widehat{y}(x) = U(x)$$

$$u'(\widehat{Y}(x)) = \widehat{y}(x)\frac{d\widehat{Q}(\widehat{y}(x))}{dP}.$$

2. Appendix: methods from convex analysis

In this chapter basic duality methods from convex analysis are discussed. We shall also apply the notions of dual normed vector spaces in finite dimensions. Let V be a real vector space with norm and real dimension $\dim V < \infty$, then we can define the *pairing*

$$\langle .,. \rangle : V \times V' \to \mathbb{R}$$

$$(v, l) \mapsto l(v)$$

where V' denotes the dual vector space, i.e. the space of continuous linear functionals $l : V \to \mathbb{R}$. The dual space carries a natural dual norm namely

$$||l|| := \sup_{||v|| \leq 1} |l(v)|.$$

We obtain the following duality relations:

- If for some $v \in V$ it holds that $\langle v, l \rangle = 0$ for all $l \in V'$, then $v = 0$.
- If for some $l \in V'$ it holds that $\langle v, l \rangle = 0$ for all $v \in V$, then $l = 0$.
- There is a natural isomorphism $V \to V''$ and the norms on V and V'' coincide (with respect to the previous definition).

If V is an euclidean vector space, i.e. there is a scalar product $\langle .,. \rangle : V \times V \to \mathbb{R}$, which is symmetric and positive definite, then we can identify V' with V and every linear functional $l \in V'$ can be uniquely represented $l = \langle ., x \rangle$ for some $x \in V$.

Definition 2.1. *Let V be a finite dimensional vector space. A subset $C \subset V$ is called* convex *if for all $v_1, v_2 \in C$ also $tv_1 + (1-t)v_2 \in C$ for $t \in [0, 1]$.*

Since the intersection of convex sets is convex, we can define the convex hull of any subset $M \subset V$, which is denoted by $\langle M \rangle_{conv}$. We also define the closed convex hull $\overline{\langle M \rangle_{conv}}$, which is the smallest closed, convex subset of V containing M. If M is compact the convex hull $\langle M \rangle_{conv}$ is already closed and therefore compact.

Definition 2.2. *Let C be a closed convex set, then $x \in C$ is called* extreme point *of C if for all $y, z \in C$ with $x = ty + (1-t)z$ and $t \in [0, 1]$, we have either $t = 0$ or $t = 1$. This is equivalent to saying that there are no two different points $x_1 \neq x_2$ such that $x = \frac{1}{2}(x_1 + x_2)$.*

First we treat a separation theorem, which is valid in a fairly general context and known as Hahn-Banach Theorem.

Theorem 2.3. *Let C be a closed convex set in an euclidean vector space V, which does not contain the origin, i.e. $0 \notin C$. Then there exists a linear functional $\xi \in V'$ and $\alpha > 0$ such that for all $x \in C$ we have $\xi(x) \geq \alpha$.*

Proof. Let r be a radius such that the closed ball $B(r)$ intersects C. The continuous map $x \mapsto ||x||$ achieves a minimum $x_0 \neq 0$ on $B(r) \cap C$, which we denote by x_0, since $B(r) \cap C$ is compact. We certainly have for all $x \in C$ the relation $||x|| \geq ||x_0||$. By convexity we obtain that $x_0 + t(x - x_0) \in C$ for $t \in [0,1]$ and hence

$$||x_0 + t(x - x_0)||^2 \geq ||x_0||^2.$$

This equation can be expanded for $t \in [0,1]$,

$$\begin{aligned} ||x_0||^2 + 2t \langle x_0, x - x_0 \rangle + t^2 ||(x - x_0)||^2 &\geq ||x_0||^2, \\ 2t \langle x_0, x - x_0 \rangle + t^2 ||(x - x_0)||^2 &\geq 0. \end{aligned}$$

Take now small t and assume $\langle x_0, x - x_0 \rangle < 0$ for some $x \in C$, then there appears a contradiction in the previous inequality, hence we obtain

$$\langle x_0, x - x_0 \rangle \geq 0$$

and consequently $\langle x, x_0 \rangle \geq ||x_0||^2$ for $x \in C$, so we can choose $\xi = \langle ., x_0 \rangle$. □

As a corollary we have that each subspace $V_1 \subset V$, which does not intersect with a convex, compact and non-empty subset $K \subset V$ can be separated from K, i.e. there is $\xi \in V'$ such that $\xi(V_1) = 0$ and $\xi(x) > 0$ for $x \in K$. This is proved by considering the set

$$C := K - V := \{w - v \text{ for } v \in V \text{ and } w \in K\},$$

which is convex and closed, since V, K are convex and K is compact, and which does not contain the origin. By the above theorem we can find a separating linear functional $\xi \in V'$ such that $\xi(w - v) \geq \alpha$ for all $w \in K$ and $v \in V$, which means in particular that $\xi(w) > 0$ for all $w \in K$. Furthermore we obtain from $\xi(w) - \xi(v) \geq \alpha$ for all $v \in V$ that $\xi(v) = 0$ for all $v \in V$ (replace v by λv, which is possible since V is a vector space, and lead the assertion to a contradiction in case that $\xi(v) \neq 0$).

Theorem 2.4. *Let C be a compact convex non-empty set, then C is the convex hull of all its extreme points.*

Proof. We have to show that there is an extreme point. We take a point $x \in C$ such that the distance $||x||^2$ is maximal, then x is an extreme point. Assume that there are two different points x_1, x_2 such that $x = \frac{1}{2}(x_1 + x_2)$, then

$$\begin{aligned} ||x||^2 = ||\frac{1}{2}(x_1 + x_2)||^2 &< \frac{1}{2}(||x_1||^2 + ||x_2||^2) \\ &\leq \frac{1}{2}(||x||^2 + ||x||^2) = ||x||^2, \end{aligned}$$

by the parallelogram law $\frac{1}{2}(||y||^2 + ||z||^2) = ||\frac{1}{2}(y+z)||^2 + ||\frac{1}{2}(y-z)||^2$ for all $y, z \in V$ and the maximality of $||x||^2$. This is a contradiction. Therefore we obtain at least one extreme point.

The set of all extreme points is a compact set, since it lies in C and is closed. Take now the convex hull of all extreme points, which is a closed convex subset S of C and hence compact. If there is $x \in C \setminus S$, then we can separate by a hyperplane l the point x and S such that $l(x) \geq \alpha > l(y)$ for $y \in S$. The set $\{l \geq \alpha\} \cap C$ is compact, convex, nonempty and has therefore an extreme point z, which is also an extreme point of C. So $z \in S$, which is a contradiction. □

Next we treat basic duality theory in the finite dimensional vector space V with euclidean structure. We identify the dual space V' with V by the above representation.

Definition 2.5. *A subset $C \subset V$ is called convex cone if for all $v_1, v_2 \in C$ the sum $v_1 + v_2 \in C$ and $\lambda v_1 \in C$ for $\lambda \geq 0$. Given a cone C we define the polar C^0*

$$C^0 := \{l \in V \text{ such that } \langle l, v\rangle \leq 0 \text{ for all } v \in C\}.$$

The intersection of convex cones is a convex cone and therefore we can speak of the smallest convex cone containing an arbitrary set $M \subset V$, which is denoted by $\langle M\rangle_{cone}$. We want to prove the bipolar theorem for convex cones.

Theorem 2.6 (Bipolar Theorem)**.** *Let $C \subset V$ be a convex cone, then $C^{00} \subset V$ is the closure of C.*

Proof. We show both inclusions. Take $v \in \overline{C}$, then $\langle l, v\rangle \leq 0$ for all $l \in C^0$ by definition of C^0 and therefore $v \in C^{00}$. If there were $v \in C^{00} \setminus \overline{C}$, where $\overline{C}$ denotes the closure of C, then for all $l \in C^0$ we have that $\langle l, v\rangle \leq 0$ by definition. On the other hand we can find $l \in V$ such that $\langle l, \overline{C}\rangle \leq 0$ and $\langle l, v\rangle > 0$ by the separation theorem since $\overline{C}$ is a closed cone. By assumption we have $l \in C^0$, however this yields a contradiction since $\langle l, v\rangle > 0$ and $v \in C^{00}$. □

Definition 2.7. *A convex cone C is called* polyhedral *if there is a finite number of linear functionals $l_1, \ldots, l_m$ such that*

$$C := \cap_{i=1}^{n} \{v \in V | \langle l_i, v\rangle \leq 0\}.$$

In particular a polyhedral cone is closed as intersection of closed sets.

Lemma 2.8. *Given $e_1, \ldots, e_n \in V$. For the cone $C = \langle e_1, \ldots, e_n\rangle_{con}$ the polar can be calculated as*

$$C^0 = \{l \in V \text{ such that } \langle l, e_i\rangle \leq 0 \text{ for all } i = 1, \ldots, n\}.$$

Proof. The convex cone $C = \langle e_1, \ldots, e_n\rangle_{cone}$ is given by

$$C = \{\sum_{i=1}^{n} \alpha_i e_i \text{ for } \alpha_i \geq 0 \text{ and } i = 1, \ldots, n\}.$$

Given $l \in C^0$, the equation $\langle l, e_i\rangle \leq 0$ necessarily holds and we have the inclusion $\subset$. Given $l \in V$ such that $\langle l, e_i\rangle \leq 0$ for $i = 1, \ldots, n$, then for $\alpha_i \geq 0$ the equation $\sum_{i=1}^{n} \alpha_i \langle l, e_i\rangle \leq 0$ holds and therefore $l \in C^0$ by the explicit description of C as $\sum_{i=1}^{n} \alpha_i e_i$ for $\alpha_i \geq 0$. □

Corollary 2.9. *Given $e_1, \ldots, e_n \in V$, the cone $C = \langle e_1, \ldots, e_n\rangle_{con}$ has a polar which is polyhedral and therefore closed.*

Proof. The polyhedral cone is given through

$$\begin{aligned} C^0 &= \{l \in V \text{such that } \langle l, e_i\rangle \leq 0 \text{ for all } i = 1, \ldots, n\} \\ &= \cap_{i=1}^{n} \{l \in V | \langle l, e_i\rangle \leq 0\}. \end{aligned}$$

□

Lemma 2.10. *Given a finite set of vectors $e_1, \ldots, e_n \in V$ and the convex cone $C = \langle e_1, \ldots, e_n\rangle_{con}$, then C is closed.*

Proof. Assume that $C = \langle e_1, \dots, e_n \rangle_{con}$ for vectors $e_i \in V$. If the e_i are linearly independent, then C is closed by the argument, that any $x \in C$ can be uniquely written as $x = \sum_{i=1}^n \alpha_i e_i$. Suppose next that there is a non-trivial linear combination $\sum_{i=1}^n \beta_i e_i = 0$ with $\beta \in \mathbb{R}^n$ non-zero. We can write $x \in C$ as

$$x = \sum_{i=1}^n \alpha_i e_i = \sum_{i=1}^n (\alpha_i + t(x)\beta_i) e_i = \sum_{j \neq i(x)} \alpha'_i e_i$$

with

$$i(x) \in \{i \text{ such that } |\frac{\alpha_i}{\beta_i}| = \max_{\beta_j < 0} |\frac{\alpha_j}{\beta_j}|\},$$
$$t(x) = -\frac{\alpha_{i(x)}}{\beta_{i(x)}}$$

Then $\alpha'_j \geq 0$ by definition. Consequently we can construct by variation of x a decomposition

$$C = \cup_{i=1}^{n'} C_i$$

where C_i are cones generated by $n-1$ vectors from the set $e_1, \dots, e_n$. By induction on the number of generators n we can conclude by the statement on linearly independent generators. □

Proposition 2.11. *Let $C \subset V$ be a convex cone generated by $e_1, \dots, e_n$ and $\mathcal{K}$ a subspace, then $\mathcal{K} - C$ is closed convex.*

Proof. First we prove that $\mathcal{K} - C$ is a convex cone. Taking $v_1, v_2 \in \mathcal{K} - C$, then $v_1 = k_1 - c_1$ and $v_2 = k_2 - c_2$, therefore

$$v_1 + v_2 = k_1 + k_2 - (c_1 + c_2) \in \mathcal{K} - C,$$
$$\lambda v_1 = \lambda k_1 - \lambda c_1 \in \mathcal{K} - C.$$

In particular $0 \in \mathcal{K} - C$. The convex cone is generated by a generating set $e_1, \dots, e_n$ for C and a basis $f_1, \dots, f_p$ for $\mathcal{K}$, which has to be taken with $-$ sign, too. So

$$\mathcal{K} - C = \langle -e_1, \dots, -e_n, f_1, \dots, f_p, -f_1, \dots, -f_p \rangle_{con}$$

and therefore $\mathcal{K} - C$ is closed by Lemma 2.10. □

Lemma 2.12. *Let C be a polyhedral cone, then there are finitely many vectors $e_1, \dots, e_n \in V$ such that*

$$C = \langle e_1, \dots, e_n \rangle_{con}.$$

Proof. By assumption $C = \cap_{i=1}^p \{v \in V | \langle l_i, v \rangle \leq 0\}$ for some vectors $l_i \in V$. We intersect C with $[-1,1]^m$ and obtain a convex, compact set. This set is generated by its extreme points. We have to show that there are only finitely many extreme points. Assume that there are infinitely many extreme points, then there is also an adherence point $x \in C$. Take a sequence of extreme points $(x_n)_{n \geq 0}$ such that $x_n \to x$ as $n \to \infty$ with $x_n \neq x$. We can write the defining inequalities for $C \cap [-1,1]^m$ by

$$\langle k_j, v \rangle \leq a_j$$

for $j = 1, \dots, r$ and we obtain $\lim_{n \to \infty} \langle k_j, x_n \rangle = \langle k_j, x \rangle$. Define

$$\epsilon := \min_{\langle k_j, x \rangle < a_j} a_j - \langle k_j, x \rangle > 0.$$

Take n_0 large enough such that $|\langle k_j, x_{n_0}\rangle - \langle k_j, x\rangle| \leq \frac{\epsilon}{2}$, which is possible due to convergence. Then we can look at $x_{n_0} + t(x - x_{n_0}) \in C$ for $t \in [0,1]$. We want to find a continuation of this segment for some $\delta > 0$ such that $x_{n_0} + t(x - x_{n_0}) \in C$ for $[-\delta, 1]$. Therefore we have to check three cases:

- If $\langle k_j, x_{n_0}\rangle = \langle k_j, x\rangle = a_j$, then we can continue for all $t \leq 0$ and the inequality $\langle k_j, x_{n_0} + t(x - x_{n_0})\rangle = a_j$ remains valid.
- If $\langle k_j, x\rangle = a_j$ and $\langle k_j, x_{n_0}\rangle < a_j$, we can continue for all $t \leq 0$ and the inequality $\langle k_j, x_{n_0} + t(x - x_{n_0})\rangle \leq a_j$ remains valid.
- If $\langle k_j, x\rangle < a_j$, then we define $\delta = 1$ and obtain that for $-1 \leq t \leq 1$ the inequality $\langle k_j, x_{n_0} + t(x - x_{n_0})\rangle \leq a_j$ remains valid.

Therefore we can find δ and continue the segment for small times. Hence x_n cannot be an extreme point, since it is a nontrivial convex combination of $x_{n_0} - \delta(x - x_{n_0})$ and x, which is a contradiction. Therefore $C \cap [-1,1]^m$ is generated by finitely many extreme points $e_1, \ldots, e_n$and so

$$C = \langle e_1, \ldots, e_n\rangle_{con}$$

by dilatation. □

3. Appendix: optimization Theory

We shall first consider general principles in optimization theory related to analysis and proceed to special functionals.

Definition 3.1. *Let $U \subset \mathbb{R}^m$ be a subset with $U \subset V$, where V is open in $\mathbb{R}^m$. Let $F : V \to \mathbb{R}$ be a C^2-function. A point $x \in U$ is called local maximum (local minimum) of F on U if there is a neighborhood W_x of x in V such that for all $y \in U \cap W_x$*

$$F(y) \leq F(x)$$

or respectively $F(y) \geq F(x)$.

Lemma 3.2. *Let $U \subset \mathbb{R}^m$ be a subset with $U \subset V$, where V is open in $\mathbb{R}^m$ and let $F : V \to \mathbb{R}$ be a C^2-function. Given a local maximum (or local minimum) $x \in U$ of F on U and a C^2-curve $c :]-1,1[\to V$ such that $c(0) = x$ and $c(t) \in U$ for $t \in]-1,1[$, the following necessary condition holds true,*

$$\frac{d}{dt}|_{t=0} F(c(t)) = \langle \operatorname{grad} F(x), c'(0)\rangle = 0.$$

Proof. The function $t \mapsto F(c(t))$ has a local extremum at $t = 0$ and therefore the first derivative at $t = 0$ must vanish. □

We shall now prove a version of the Lagrangian multiplier theorem for affine subspaces $U \subset \mathbb{R}^m$. We take a affine subspace $U \subset \mathbb{R}^m$ and an open neighborhood $V \subset \mathbb{R}^m$ such that $U \cap V \neq \emptyset$, where a C^2-function $F : V \to \mathbb{R}$ is defined.

Theorem 3.3. *Let x be a local maximum (local minimum) of F on $U \cap V$ and assume that there are $k := m - \dim U$ vectors $l_1, \ldots, l_k \in \mathbb{R}^m$ and real numbers $a_1, \ldots, a_k \in \mathbb{R}$ such that*

$$U = \{x \in V \text{ with } \langle l_i, x\rangle = a_i \text{ for } i = 1, \ldots, k\}.$$

Then

$$\operatorname{grad} F(x) \in \langle l_1, \ldots, l_k\rangle$$

or in other words there are real numbers $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ *such that*

$$\operatorname{grad} F(x) = \lambda_1 l_1 + \cdots + \lambda_k l_k.$$

Proof. Take a C^2-curve $c :]-1, +1[\to V$, then c takes values in U if and only if

$$c(0) \in U$$

and

$$\langle l_i, c'(t) \rangle = 0$$

for $i = 1, \ldots, k$ and $t \in]-1, 1[$. The proof is simply done by Taylor's formula. Fix $t \in]-1, 1[$ and take

$$c(t) = c(0) + \int_0^t c'(s) ds.$$

By definition $c(t) \in U$ if and only if $\langle l_i, c(t) \rangle = a_i$, but

$$\begin{aligned} \langle l_i, c(t) \rangle &= \langle l_i, c(0) \rangle + \int_0^t \langle l_i, c'(s) \rangle \, ds \\ &= a_i \end{aligned}$$

by assumption for $i = 1, \ldots, k$. We denote the span of $l_1, \ldots, l_k$ by T and can consequently state that a C^2-curve $c :]-1, +1[\to V$ takes values in U if and only if $c(0) \in U$ and $c'(t) \in T^0$ for all $t \in]-1, 1[$. Furthermore we can say that T^0 is generated by all derivatives of C^2-curves $c :]-1, +1[\to V$ taking values in U at time $t = 0$ (simply take a line with direction a vector in T^0 through some point of U).

By the previous lemma we know that for all C^2-curves $c :]-1, +1[\to V$ with $c(0) = x$ the relation

$$\langle \operatorname{grad} F(x), c'(0) \rangle = 0$$

holds. Therefore $\operatorname{grad} F(x) \in T^{00}$. By the bipolar theorem we know that $T^{00} = T = \langle l_1, \ldots, l_k \rangle$, which proves the result. □

Remark 3.4. This leads immediately to the receipt of Lagrangian multipliers as it is well known from basic calculus: a necessary condition for an extremal point of $F : V \to \mathbb{R}$ subject to the conditions $\langle l_i, x \rangle = a_i$ for $i = 1, \ldots, k$ is to solve the extended problem with the Lagrangian L

$$L(x, \lambda_1, \ldots, \lambda_k) = F(x) - \sum_{i=1}^{k} \lambda_i (\langle l_i, x \rangle - a_i).$$

Taking the gradients leads to the system of equations

$$\begin{aligned} \operatorname{grad} F(x) - \sum_{i=1}^{k} \lambda_i l_i &= 0 \\ \langle l_i, x \rangle &= a_i \end{aligned}$$

for $i = 1, \ldots, k$, which necessarily has a solution if there is an extremal point at x.

Remark 3.5. How to calculate a gradient? The gradient of a C^1-function $F : V \to \mathbb{R}$ on a finite dimensional vector space V is defined through

$$\langle \operatorname{grad} F(x), w \rangle = \frac{d}{ds}|_{s=0} F(x + sw),$$

for $x \in V$ and $w \in \mathbb{R}^n$ (and a scalar product!). This can be calculated with respect to any basis and gives a coordinate representation. The derivative of F is understood as element of the dual space

$$dF(x)(w) := \frac{d}{ds}|_{s=0} F(x + sw)$$

for $x \in V$ and $w \in \mathbb{R}^n$ (even without scalar product!). The derivative can be calculated with respect to a basis $(e_i)_{i=1,\ldots,\dim V}$. That means that it simply represents a collection of directional derivatives of a function, i.e.

$$\operatorname{grad}_{(e_i)} F(x) := (\frac{d}{ds}|_{s=0} F(x + se_i))_{i=1,\ldots,\dim V}$$

for $x \in V$.

4. Appendix: conjugate functions

Given a concave, increasing function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, which usual conventions for the calculus with $-\infty$. We denote by $\operatorname{dom}(u)$ the set $\{u > -\infty\}$ and assume that the closure of $\operatorname{dom}(u)$ is either $[a, \infty[$ or $\mathbb{R}$. We shall always assume that u is strictly concave and C^2 on $\operatorname{dom}(u)$.

In this and more general cases we can define the conjugate function

$$v(y) := \sup_{x \in \mathbb{R}} (u(x) - yx)$$

for $y > 0$.

Since the function $x \mapsto u(x) - yx$ is strictly concave for every $y > 0$, there is some hope for a maximum. If there is one, let's say $\widehat{x}$, then it satisfies

$$u'(\widehat{x}) = y. \tag{4.1}$$

Since the second derivative exists and is strictly negative, $\widehat{x}$ is a local maximum if the above equation is satisfied. By strict concavity the local maximum is unique and global, too.

We need basic assumptions for the existence and regularity of the conjugate function:

(1) If the interior of $\operatorname{dom}(u)$ equals $]a, \infty[$ (wealth below a not allowed), then we assume

$$\lim_{x \downarrow a} u'(x) = \infty,$$
$$\lim_{x \to \infty} u'(x) = 0 \text{ (marginal utility tends to 0).}$$

(2) If $\operatorname{dom}(u) = \mathbb{R}$ (negative wealth allowed), then we assume

$$\lim_{x \downarrow -\infty} u'(x) = \infty,$$
$$\lim_{x \to \infty} u'(x) = 0 \text{ (marginal utility tends to 0).}$$

Under these assumptions we can state the following theorem on existence and convexity of v.

Theorem 4.1. *Let $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ be a concave function satisfying the above assumptions, then the conjugate function is strictly convex and C^2 on* $\mathrm{dom}(v) =]0, \infty[$. *Additionally for* $\mathrm{dom}(u) =]0, \infty[$ *we have*

$$v'(0) := \lim_{y \downarrow 0} v'(y) = -\infty,$$
$$\lim_{y \to \infty} v'(y) = 0$$

and for $\mathrm{dom}(u) = \mathbb{R}$

$$v'(0) := \lim_{y \downarrow 0} v'(y) = -\infty,$$
$$\lim_{y \to \infty} v'(y) = \infty$$

Furthermore the inversion formula

$$u(x) = \inf_{y>0} (v(y) + xy)$$

holds true.

Proof. By formula 4.1 and our assumptions we see that for every $y > 0$ there is exactly one $\widehat{x}$, since u' is strictly decreasing and C^1. We denote the inverse of u' by $(u')^{-1}$. Therefore v is well-defined and at least C^1, since the inverse is C^1. Furthermore

$$\begin{aligned} v(y) &= u((u')^{-1}(y)) - y \cdot (u')^{-1}(y) \\ v'(y) &= u'((u')^{-1}(y))((u')^{-1})'(y) - (u')^{-1}(y) - y((u')^{-1})'(y) \\ &= -(u')^{-1}(y) \\ v''(y) &= -((u')^{-1})'(y) = -\frac{1}{u''((u')^{-1}(y))} > 0 \end{aligned}$$

Hence v is C^2 on $]0, \infty[$ and a fortiori, by $v'' > 0$, strictly convex.

We know that u' is positive and strictly decreasing from ∞ to 0 by the previous assumptions, hence the two limiting properties for v, since $v'(y) = -(u')^{-1}(y)$.

Replacing v by $-v$, we can apply the same reasoning for existence of the concave conjugate of v. Take $\widehat{y} > 0$ such that $\inf_{y>0}(v(y) + xy)$ takes the infimum, then necessarily

$$v'(\widehat{y}) = -x,$$

hence $-(u')^{-1}(\widehat{y}) = -x$ and therefore $\widehat{y}(x) = u'(x)$. Inserting yields

$$\begin{aligned} v(u'(x)) + x\widehat{y}(x) &= u((u')^{-1}(u'(x))) - u'(x)(u')^{-1}(u'(x)) + xu'(x) \\ &= u(x), \end{aligned}$$

which is the desired relation. □

5. Catalogue of questions for the oral exam

For the oral exam I shall choose randomly three questions from the following list, from which you have the right to select two for your exam. The exam is "open book", i.e. you can use the book of Föllmer-Schied as well as my lecture notes (however, no Musterlösungen!) during the preparation of the answers. To guarantee equal conditions we shall provide the 'books' in my office, no personal copies are allowed. You will have about 10 minutes of time for each question after about 10 minutes of preparation. I expect you to speak about the question like in a seminar, i.e. explaining the structure of the answer and important details such that a good mathematician, who does not know precisely about the topic could in principle follow.

(1) (Chapter 5 of FS) Explain the multi-period market model, portfolios, self-financing portfolios, discounting, Prop 5.7.
(2) (Chapter 5 of FS) Arbitrage and time-localization of arbitrage: Def 5.10 and Prop. 5.11.
(3) (Chapter 5 of FS) characterization of martingale measures: Theorem 5.14. (a) to (b) to (c).
(4) (Chapter 5 of FS) characterization of martingale measures: Theorem 5.14. (c) to (d) to (a).
(5) (Chapter 5 of FS) Theorem 5.16 with the knowledge of Theorem 1.55 (FTAP result in the one period case).
(6) (Chapter 1 of FS) Theorem 1.55 with the knowledge of the Lemmata; explain where the problem lies.
(7) (Chapter 1 of FS) Lemma 1.57 and Lemma 1.58.
(8) (Chapter 1 of FS) Lemma 1.59 with the help of the Hahn-Banach theorem, explain its meaning and idea of the proof of the Hahn-Banach theorem.
(9) (Chapter 1 of FS) Lemma 1.60 and Theorem 1.62 (explain its meaning).
(10) (Chapter 1 of FS) Lemma 1.64 and its meaning for the proof of FTAP.
(11) (Chapter 1 of FS) Lemma 1.66 and its meaning for the proof of FTAP.
(12) (Chapter 1 of FS) Lemma 1.68 and its meaning for the proof of FTAP.
(13) (Chapter 5 of FS) Prop. 5.17 (change of numeraire theorem) with Remark 5.18.
(14) (Chapter 5 of FS) attainable claims and Theorem 5.25.
(15) (Chapter 5 of FS) Def 5.28 and Theorem 5.29 (arbitrage free prices of claims).
(16) (Chapter 5 of FS) Theorem 5.32 (characterization of attainable claims by arbitrage-free prices).
(17) (Chapter 5 of FS) Theorem 5.32 (characterization of non-attainable claims upper and lower bounds).
(18) (Chapter 5 of FS) CRR model and Theorem 5.39.
(19) (Chapter 5 of FS) Prop 5.41 and Prop 5.44 (pricing and hedging in CRR model)
(20) (Chapter 6 of FS) American Options, exercise strategy, Prop 6.1 (Doob decomposition), Def 5.8 (Snell envelope) [complete case].
(21) (Chapter 6 of FS) Prop 6.10 (Snell envelope is smallest supermartingale dominating the payoff) [complete case].
(22) (Chapter 6 of FS) Theorem 6.11 [complete case].
(23) (Chapter 6 of FS) Theorem 6.15 (optional stopping theorem).
(24) (Chapter 6 of FS) Theorem 6.18 (smallest optimal stopping time).
(25) (Chapter 6 of FS) Prop. 6.20 (characterization of optimal stopping times) and Proposition 6.21 (maximal optimal stopping time).
(26) (Chapter 6 of FS) Def 6.29 and Theorem 6.31 under the assumption that one can interchange inf (sup) and sup [incomplete case].
(27) (Chapter 6 of FS) define lower and upper Snell envelope and explain in a sketch why one can interchange inf (sup) and sup.

(28) explain the notion of super-hedging of European claims and connect it to the bi-polar theorem, prove the existence of a super-hedging strategy.
(29) Exercise 9.2
(30) Exercise 11.1
(31) Exercise 11.2
(32) Exericse 11.3
(33) (Lecture Notes) Theorem 1.2 and Prop 1.4.
(34) (Lecture Notes) Lemma 1.6 and Lemma 1.7.
(35) (Lecture Notes) Lemma 1.9 and its meaning for the theory.
(36) (Lecture Notes) Theorem 1.10.
(37) (Chapter 4 of FS) define convex and coherent risk measures, prove Lemma 4.3 and Prop 4.6.
(38) (Chapter 4 of FS) Prop. 4.15 (which coincides with Prop. 2.83).
(39) (Chapter 4 of FS) Theorem 4.16.
(40) (Chapter 4 of FS) Remark 4.18 and the relationship to Fenchel-Legendre transforms.

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