

# Lecture Notes: Interest Rate Theory

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Fall 2010

Foreword

Mathematical Finance

Basics on Interest Rate Modeling

Black formulas

Affine LIBOR Models

Markov Processes

The SABR model

HJM-models

References

Catalogue of possible questions for the oral exam

In mathematical Finance we need processes

- ▶ which can model all stylized facts of volatility surfaces and times series (e.g. tails, stochastic volatility, etc)
- ▶ which are analytically tractable to perform efficient calibration.
- ▶ which are numerically tractable to perform efficient pricing and hedging.

# Goals

- ▶ Basic concepts of stochastic modeling in interest rate theory.
- ▶ "No arbitrage" as concept and through examples.
- ▶ Concepts of interest rate theory like yield, forward rate curve, short rate.
- ▶ Spot measure, forward measures, swap measures and Black's formula.
- ▶ Short rate models
- ▶ Affine LIBOR models
- ▶ Fundamentals of the SABR model
- ▶ HJM model
- ▶ Consistency and Yield curve estimation

## Modeling of financial markets

We are describing *models for financial products related to interest rates*, so called interest rate models. We are facing several difficulties, some of the specific for interest rates, some of them true for all models in mathematical finance:

- ▶ stochastic nature: traded prices, e.g.  $\tilde{}$ prices of interest rate related products, are not deterministic!
- ▶ information is increasing: every day additional information on markets appears and this stream of information should enter into our models.
- ▶ stylized facts of markets should be reflected in the model: stylized facts of time series, trading opportunities (portfolio building), etc.

# Mathematical Finance 1

A financial market can be modeled by

- ▶ a filtered (discrete) probability space  $(\Omega, \mathcal{F}, Q)$ ,
- ▶ together with price processes, namely  $K$  risky assets  $(S_n^1, \dots, S_n^K)_{0 \leq n \leq N}$  and one risk-less asset  $S^0$ , i.e.  $\sim S_n^0 > 0$  almost surely (no default risk for at least one asset),
- ▶ all price processes being adapted to the filtration.

*This structure reflects stochasticity of prices and the stream of incoming information.*

A portfolio is a predictable process  $\phi = (\phi_n^0, \dots, \phi_n^K)_{0 \leq n \leq N}$ , where  $\phi_n^i$  represents the number of risky assets one holds at time  $n$ . The value of the portfolio  $V_n(\phi)$  is

$$V_n(\phi) = \sum_{i=0}^K \phi_n^i S_n^i.$$

## Mathematical Finance 2

Self-financing portfolios  $\phi$  are characterized through the condition

$$V_{n+1}(\phi) - V_n(\phi) = \sum_{i=0}^K \phi_{n+1}^i (S_{n+1}^i - S_n^i),$$

for  $0 \leq n \leq N - 1$ , i.e.  $\tilde{V}$  changes in value come from changes in prices, no additional input of capital is required and no consumption is allowed.



Self-financing portfolios can be characterized in *discounted terms*.

$$\widetilde{V}_n(\phi) = (S_n^0)^{-1} V_n(\phi)$$

$$\widetilde{S}_n^i = (S_n^0)^{-1} S_n^i(\phi)$$

$$\widetilde{V}_n(\phi) = \sum_{i=0}^K \phi_n^i \widetilde{S}_n^i$$

for  $0 \leq n \leq N$ , and recover

$$\widetilde{V}_n(\phi) = V_0(\phi) + (\phi \cdot \widetilde{S}) = V_0(\phi) + \sum_{j=1}^n \sum_{i=1}^K \phi_j^i (\widetilde{S}_j^i - \widetilde{S}_{j-1}^i)$$

for self-financing predictable trading strategies  $\phi$  and  $0 \leq n \leq N$ . In words: discounted wealth of a self-financing portfolio is the cumulative sum of discounted gains and losses. *Notice that we apply a generalized notion of “discounting” here, prices  $S^i$  divided by  $S^0$  – only these relative prices can be compared.*

# Fundamental Theorem of Asset Pricing

A minimal condition for modeling financial markets is the *No-arbitrage condition*: there are no self-financing trading strategies  $\phi$  (arbitrage strategies) with

$$V_0(\phi) = 0, V_N(\phi) \geq 0$$

such that  $Q(V_N(\phi) \neq 0) > 0$  holds (NFLVR).

## Fundamental Theorem of Asset Pricing

A minimal condition for financial markets is the no-arbitrage condition: there are no self-financing trading strategies  $\phi$  (arbitrage strategies) with  $V_0(\phi) = 0$ ,  $V_N(\phi) \geq 0$  such that  $Q(V_N(\phi) \neq 0) > 0$  holds (NFLVR).

In other words the set

$$\mathcal{K} = \{ \widetilde{V}_N(\phi) \mid V_0(\phi) = 0, \phi \text{ self-financing} \}$$

intersects  $L_{\geq 0}^0(\Omega, \mathcal{F}, Q)$  only at 0,

$$\mathcal{K} \cap L_{\geq 0}^0(\Omega, \mathcal{F}, Q) = \{0\}.$$

## FTAP

## Theorem

*Given a financial market, then the following assertions are equivalent:*

- 1. (NFLVR) holds.*
- 2. There exists an equivalent measure  $P \sim Q$  such that the discounted price processes are  $P$ -martingales, i.e.*

$$E_P\left(\frac{1}{S_N^0} S_N^i \mid \mathcal{F}_n\right) = \frac{1}{S_n^0} S_n^i$$

*for  $0 \leq n \leq N$ .*

*Main message: Discounted (relative) prices behave like martingales with respect to one martingale measure.*

## What is a martingale?

Formally a martingale is a stochastic process such that today's best prediction of a future value of the process is today's value, i.e.~

$$E[M_n|\mathcal{F}_m] = M_m$$

for  $m \leq n$ , where  $E[M_n|\mathcal{F}_m]$  calculates the best prediction with knowledge up to time  $m$  of the future value  $M_n$ .

Random walks and Brownian motions are well-known examples of martingales. Martingales are particularly suited to describe (discounted) price movements on financial markets, since the prediction of future returns is 0.

## Pricing rules

(NFLVR) also leads to arbitrage-free pricing rules. Let  $X$  be the payoff of a claim  $X$  paying at time  $N$ , then an adapted stochastic process  $\pi(X)$  is called pricing rule for  $X$  if

- ▶  $\pi_N(X) = X$ .
- ▶  $(S^0, \dots, S^N, \pi(X))$  is free of arbitrage.

This is equivalent to the existence of one equivalent martingale measure  $P$  such that

$$E_P\left(\frac{X}{S_N^0} \mid \mathcal{F}_n\right) = \frac{\pi_n(X)}{S_n^0}$$

holds true for  $0 \leq n \leq N$ .

## Proof of FTAP

The proof is an application of separation theorems for convex sets: we consider the euclidean vector space  $L^2(\Omega, \mathbb{R})$  of real valued random variables with scalar product

$$\langle X, Y \rangle = E(XY).$$

Then the convex set  $K$  does not intersect the positive orthant  $L^2_{\geq 0}(\Omega, \mathbb{R})$ , hence we can find a vector  $R$ , which is strictly positive and which is orthogonal to all elements of  $K$  (draw it!). We are free to choose  $E(R) = 1$ . We can therefore define a measure  $Q$  on  $\mathcal{F}$  via

$$Q(A) = E(1_A R)$$

and this measure has the same nullsets as  $P$  by strict positivity.



## Proof of FTAP

By construction we have that every element of  $K$  has vanishing expectation with respect to  $Q$  since  $R$  is orthogonal to  $K$ . Since  $K$  consists of all stochastic integrals with respect to  $\tilde{S}$  we obtain by Doob's optional sampling that  $\tilde{S}$  is a  $Q$ -martingale, which completes the proof.

## One step binomial model

We model one asset in a zero-interest rate environment just before the next tick. We assume two states of the world: up, down. The riskless asset is given by  $S^0 = 1$ . The risky asset is modeled by

$$S_0^1 = S_0, S_1^1 = S_0 * u > S_0 \text{ or } S_1^1 = S_0 * 1/u = S_0 * d$$

where the events at time one appear with probability  $q$  and  $1 - q$  ("physical measure"). The martingale measure is apparently given through  $u * p + (1 - p)d = 1$ , i.e.  $\tilde{p} = \frac{1-d}{u-d}$ .

Pricing a European call option at time one in this setting leads to fair price

$$E[(S_1^1 - K)_+] = p * (S_0 u - K)_+ + (1 - p) * (S_0 d - K)_+.$$

## Black-Merton-Scholes model 1

We model one asset with respect to some numeraire by an exponential Brownian motion. If the numeraire is a bank account with constant rate we usually speak of the Black-Merton-Scholes model, if the numeraire some other traded asset, for instance a zero-coupon bond, we speak of Black's model. Let us assume that  $S_0 = 1$ , then

$$S_t^1 = S_0 \exp\left(\sigma B_t - \frac{\sigma^2 t}{2}\right)$$

with respect to the martingale measure  $P$ . In the physical measure  $Q$  a drift term is added in the exponent, i.e.~

$$S_t^1 = S_0 \exp\left(\sigma B_t - \frac{\sigma^2 t}{2} + \mu t\right).$$

## Black-Merton-Scholes model 2

Our theory tells that the price of a European call option on  $S^1$  at time  $T$  is priced via

$$E[(S_T^1 - K)_+] = S_0\Phi(d_1) - K\Phi(d_2)$$

yielding the Black-Scholes formula, where  $\Phi$  is the cumulative distribution function of the standard normal distribution and

$$d_{1,2} = \frac{\log \frac{S_0}{K} \pm \frac{\sigma^2 T}{2}}{\sigma\sqrt{T}}.$$

Notice that this price corresponds to the value of a portfolio mimicking the European option at time  $T$ .

## Some general facts

- ▶ Fixed income markets (i.e. interest rate related products) form a large scale market in any major economy, for instance swaping fixed against floating rates.
- ▶ Fixed income markets, in contrast to stock markets, consist of products with a finite life time (i.e. zero coupon bonds) and strong dependencies (zero coupon bonds with close maturities are highly dependent).
- ▶ mathematically highly challenging structures can appear in interest rate modeling.

# Interest Rate mechanics 1

Prices of zero-coupon bonds (ZCB) with maturity  $T$  are denoted by  $P(t, T)$ . Interest rates are given by a market of (default free) zero-coupon bonds. We shall always assume the nominal value  $P(T, T) = 1$ .

- ▶  $T$  denotes the *maturity* of the bond,  $P(t, T)$  its price at a time  $t$  before maturity  $T$ .
- ▶ The yield

$$Y(t, T) = -\frac{1}{T-t} \log P(t, T)$$

describes the compound interest rate p.~a. for maturity  $T$ .

- ▶ The *forward rate curve*  $f$  of the bond market is defined via

$$P(t, T) = \exp\left(-\int_t^T f(t, s) ds\right)$$

for  $0 \leq t \leq T$ .

## Interest rate models

An interest rate model is a collection of adapted stochastic processes  $(P(t, T))_{0 \leq t \leq T}$  on a stochastic basis  $(\Omega, \mathcal{F}, P)$  with filtration  $(\mathcal{F}_t)_{t \geq 0}$  such that

- ▶  $P(T, T) = 1$  (nominal value is normalized to one),
- ▶  $P(t, T) > 0$  (default free market)

holds true for  $0 \leq t \leq T$ .

## Interest Rate mechanics 2

- ▶ The *short rate process* is given through  $R_t = f(t, t)$  for  $t \geq 0$  defining the “bank account process”

$$(B(t))_{t \geq 0} := \left( \exp\left( \int_0^t R_s ds \right) \right)_{t \geq 0}.$$

- ▶ The existence of forward rates and short rates is an assumption on regularity with respect to maturity  $T$ . A sufficient conditions for the existence of Yields and forward rates is that bond prices are continuously differentiable with respect with  $T$ .



- ▶ Notice that the market to model consists only of ZCB, apparently the bank account has to be formed from ZCB via a roll-over-portfolio.
- ▶ The roll-over-portfolio consists of investing one unit of currency into a  $T_1$ -ZCB, then reinvesting at time  $T_1$  into a  $T_2$ -ZCB, etc. Given an increasing sequence  $\mathbb{T} = 0 < T_1 < T_2 < \dots$  yields the wealth at time  $t$

$$B^{\mathbb{T}}(t) = \prod_{T_i \leq t} \frac{1}{P(t, T_i)}$$

- ▶ We speak of a (generalized) “bank account process” of  $B^{\mathbb{T}}$  allows for limiting – this is in particular the case of we have a short rate process with some integrability properties.

## Simple forward rates – LIBOR rates

Consider a bond market  $(P(t, T))_{t \leq T}$  with  $P(T, T) = 1$  and  $P(t, T) > 0$ . Let  $t \leq T \leq T^*$ . We define the simple forward rate through

$$F(t; T, T^*) := \frac{1}{T^* - T} \left( \frac{P(t, T)}{P(t, T^*)} - 1 \right).$$

We abbreviate

$$F(t, T) := F(t; t, T).$$

Apparently  $P(t, T^*)F(t; T, T^*)$  is the fair value at time  $t$  of a contract paying  $F(T, T^*)$  at time  $T^*$ , in the sense that there is a self-financing portfolio with value  $P(t, T^*)F(t; T, T^*)$  at time  $t$  and value  $F(T, T^*)$  at time  $T^*$ .

Indeed, note that

$$P(t, T^*)F(t; T, T^*) = \frac{P(t, T) - P(t, T^*)}{T^* - T},$$

$$F(T, T^*) = \frac{1}{T^* - T} \left( \frac{1}{P(T, T^*)} - 1 \right).$$

We can build a self-financing portfolio at time  $t$  at price  $\frac{P(t, T) - P(t, T^*)}{T^* - T}$  yielding  $F(T, T^*)$  at time  $T^*$ :

- ▶ Buying a ZCB with maturity  $T$  at time  $t$  costs  $P(t, T)$ , selling a ZCB with maturity  $T^*$  amounts all together to  $P(t, T) - P(t, T^*)$ .
- ▶ at time  $T$  we have to rebalance the portfolio by buying with the maturing ZCB another bond with maturity  $T^*$ , precisely an amount  $1/P(T, T^*)$ .
- ▶ at time  $T^*$  we receive  $1/P(T, T^*) - 1$ .

# Caps

In the sequel, we fix a number of future dates

$$T_0 < T_1 < \dots < T_n$$

with  $T_i - T_{i-1} \equiv \delta$ .

Fix a rate  $\kappa > 0$ . At time  $T_i$  the holder of the cap receives

$$\delta(F(T_{i-1}, T_i) - \kappa)^+.$$

Let  $t \leq T_0$ . We write

$$\text{Cpl}(t; T_{i-1}, T_i), \quad i = 1, \dots, n$$

for the time  $t$  price of the  $i$ th caplet, and

$$\text{Cp}(t) = \sum_{i=1}^n \text{Cpl}(t; T_{i-1}, T_i)$$

for the time  $t$  price of the cap.

## Floors

At time  $T_i$  the holder of the floor receives

$$\delta(\kappa - F(T_{i-1}, T_i))^+.$$

Let  $t \leq T_0$ . We write

$$\text{Fl}(t; T_{i-1}, T_i), \quad i = 1, \dots, n$$

for the time  $t$  price of the  $i$ th floorlet, and

$$\text{Fl}(t) = \sum_{i=1}^n \text{Fl}(t; T_{i-1}, T_i)$$

for the time  $t$  price of the floor.

# Swaps

Fix a rate  $K$  and a nominal  $N$ . The cash flow of a payer swap at  $T_i$  is

$$(F(T_{i-1}, T_i) - K)\delta N.$$

The total value  $\Pi_p(t)$  of the payer swap at time  $t \leq T_0$  is

$$\Pi_p(t) = N \left( P(t, T_0) - P(t, T_n) - K\delta \sum_{i=1}^n P(t, T_i) \right).$$

The value of a receiver swap at  $t \leq T_0$  is

$$\Pi_r(t) = -\Pi_p(t).$$

The swap rate  $R_{\text{swap}}(t)$  is the fixed rate  $K$  which gives  $\Pi_p(t) = \Pi_r(t) = 0$ . Hence

$$R_{\text{swap}}(t) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^n P(t, T_i)}, \quad t \in [0, T_0].$$

# Swaptions

A *payer (receiver) swaption* is an option to enter a payer (receiver) swap at  $T_0$ . The payoff of a payer swaption at  $T_0$  is

$$N\delta(R_{\text{swap}}(T_0) - K)^+ \sum_{i=1}^n P(T_0, T_i),$$

and of a receiver swaption

$$N\delta(K - R_{\text{swap}}(T_0))^+ \sum_{i=1}^n P(T_0, T_i).$$



Note that it is very cumbersome to write models which are analytically tractable for both swaptions and caps/floors.

- ▶ Black's model is a lognormal model for one bond price with respect to a particular numeraire. If we change the numeraire the lognormal property gets lost.
- ▶ The change of numeraire between swap and forward measures is a rational function, which usually destroys analytic tractability properties of given models.

## Spot measure

The spot measure is defined as a martingale measure for the ZCB prices discounted by their own bank account process

$$\frac{P(t, T)}{B(t)}$$

for  $T \geq 0$ . This leads to the following fundamental formula of interest rate theory

$$P(t, T) = E(\exp(-\int_t^T R_s ds) | \mathcal{F}_t)$$

for  $0 \leq t \leq T$  with respect to the spot measure.

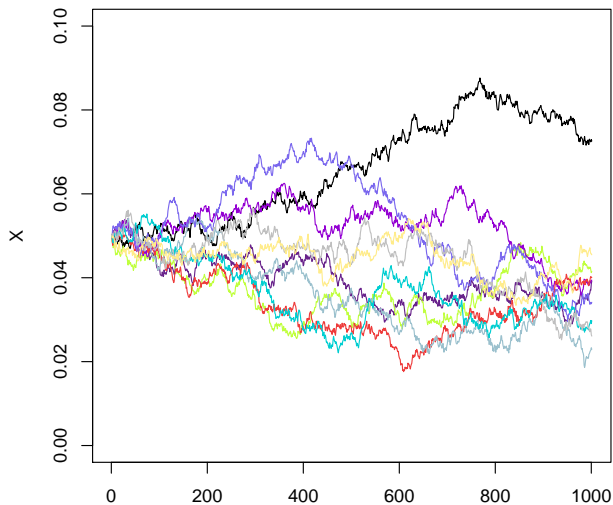
## Short rate models

We can assume several dynamics with respect to the spot measure:

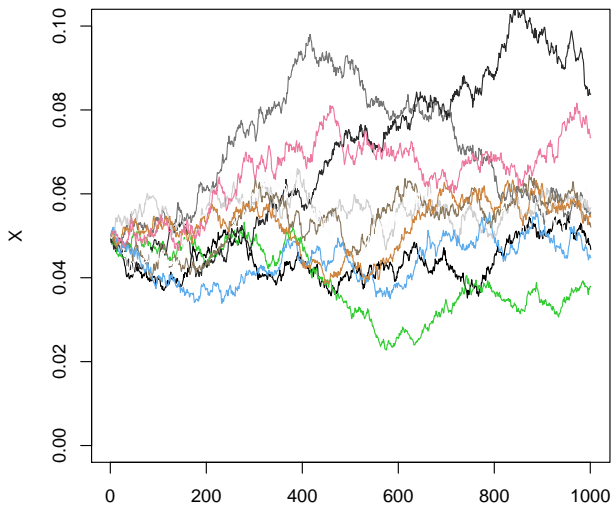
- ▶ Vasicek model:  $dR_t = (\beta R_t + b)dt + 2\alpha dW_t$ .
- ▶ CIR model:  $dR_t = (\beta R_t + b)dt + 2\alpha\sqrt{R_t}dW_t$ .

In the following two slides typical Vasicek and CIR trajectories are simulated.

## Vasicek-trajectories



CIR-trajectories



## Forward measures

For  $T^* > 0$  define the  $T^*$ -forward measure  $P^{T^*}$  such that for any  $T > 0$  the discounted bond price process

$$\frac{P(t, T)}{P(t, T^*)}, \quad t \in [0, T]$$

is a  $P^{T^*}$ -martingale.

## Forward measures

For any  $T < T^*$  the simple forward rate

$$F(t; T, T^*) = \frac{1}{T^* - T} \left( \frac{P(t, T)}{P(t, T^*)} - 1 \right)$$

is a  $\mathbb{P}^{T^*}$ -martingale.

For any time  $T$  derivative  $X \in \mathcal{F}_T$  we have that the fair value via “martingale pricing” is given through

$$P(t, T)\mathbb{E}^T[X|\mathcal{F}_t].$$

The fair price of the  $i$ th caplet is therefore given by

$$\text{Cpl}(t; T_{i-1}, T_i) = \delta P(t, T_i)\mathbb{E}^{T_i}[(F(T_{i-1}, T_i) - \kappa)^+|\mathcal{F}_t].$$

By the martingale property we obtain therefore

$$\mathbb{E}^{T_i}[F(T_{i-1}, T_i)|\mathcal{F}_t] = F(t; T_{i-1}, T_i),$$

what was proved by trading arguments before.



## Swap measures

For  $T < T_1 < \dots < T_n$  define the *swap measure*  $\mathbb{P}^{T; T_1, \dots, T_n}$  by the property that for any  $S > 0$  the process

$$\frac{P(t, S)}{\sum_{i=1}^n P(t, T_i)}, \quad t \in [0, S \wedge T]$$

is a  $\mathbb{P}^{T; T_1, \dots, T_n}$ -martingale.

## Swap measure

In particular the swap rate

$$R_{\text{swap}}(t) = \frac{P(t, T_0) - P(t, T_n)}{\delta \sum_{i=1}^n P(t, T_i)}, \quad t \in [0, T_0]$$

is a  $\mathbb{P}^{T_0; T_1, \dots, T_n}$ -martingale.

For any  $X \in \mathcal{F}_T$  we have that the fair price is given by

$$\left( \sum_{i=1}^n P(t, T_i) \right) \mathbb{E}_t^{T_0; T_1, \dots, T_n} [X].$$

- ▶ Black formulas are applications of the lognormal Black-Scholes theory to model LIBOR rates or swap rates.
- ▶ Black formulas are **not** constructed from one model of lognormal type for all modeled quantities (LIBOR rates, swap rates, forward rates, etc).
- ▶ Generically only of the following quantities is lognormal with respect to one particular measure: one LIBOR rate or one swap rate, each for a certain tenor.

## Black formula

Let  $X \sim N(\mu, \sigma^2)$  and  $K \in \mathbb{R}$ . Then we have

$$\mathbb{E}[(e^X - K)^+] = e^{\mu + \frac{\sigma^2}{2}} \Phi\left(-\frac{\log K - (\mu + \sigma^2)}{\sigma}\right) - K \Phi\left(-\frac{\log K - \mu}{\sigma}\right),$$

$$\mathbb{E}[(K - e^X)^+] = K \Phi\left(\frac{\log K - \mu}{\sigma}\right) - e^{\mu + \frac{\sigma^2}{2}} \Phi\left(\frac{\log K - (\mu + \sigma^2)}{\sigma}\right).$$

## Black formula for caps and floors

Let  $t \leq T_0$ . From our previous results we know that

$$\text{Cpl}(t; T_{i-1}, T_i) = \delta P(t, T_i) \mathbb{E}_t^{T_i} [(F(T_{i-1}, T_i) - \kappa)^+],$$

$$\text{Flr}(t; T_{i-1}, T_i) = \delta P(t, T_i) \mathbb{E}_t^{T_i} [(\kappa - F(T_{i-1}, T_i))^+],$$

and that  $F(t; T_{i-1}, T_i)$  is an  $P^{T_i}$ -martingale.

## Black formula for caps and floors

We assume that under  $P^{T_i}$  the forward rate  $F(t; T_{i-1}, T_i)$  is an exponential Brownian motion

$$F(t; T_{i-1}, T_i) = F(s; T_{i-1}, T_i) \exp \left( -\frac{1}{2} \int_s^t \lambda(u, T_{i-1})^2 du + \int_s^t \lambda(u, T_{i-1}) dW_u^{T_i} \right)$$

for  $s \leq t \leq T_{i-1}$ , with a function  $\lambda(u, T_{i-1})$ .

We define the volatility  $\sigma^2(t)$  as

$$\sigma^2(t) := \frac{1}{T_{i-1} - t} \int_t^{T_{i-1}} \lambda(s, T_{i-1})^2 ds.$$

The  $P^{T_i}$ -distribution of  $\log F(T_{i-1}, T_i)$  conditional on  $\mathcal{F}_t$  is  $N(\mu, \sigma^2)$  with

$$\begin{aligned}\mu &= \log F(t; T_{i-1}, T_i) - \frac{\sigma^2(t)}{2}(T_{i-1} - t), \\ \sigma^2 &= \sigma^2(t)(T_{i-1} - t).\end{aligned}$$

In particular

$$\begin{aligned}\mu + \frac{\sigma^2}{2} &= \log F(t; T_{i-1}, T_i), \\ \mu + \sigma^2 &= \log F(t; T_{i-1}, T_i) + \frac{\sigma^2(t)}{2}(T_{i-1} - t).\end{aligned}$$



We have

$$\begin{aligned} \text{Cpl}(t; T_{i-1}, T_i) &= \delta P(t, T_i)(F(t; T_{i-1}, T_i)\Phi(d_1(i; t)) - \kappa\Phi(d_2(i; t))), \\ \text{Flt}(t; T_{i-1}, T_i) &= \delta P(t, T_i)(\kappa\Phi(-d_2(i; t)) - F(t; T_{i-1}, T_i)\Phi(-d_1(i; t))). \end{aligned}$$

where

$$d_{1,2}(i; t) = \frac{\log\left(\frac{F(t; T_{i-1}, T_i)}{\kappa}\right) \pm \frac{1}{2}\sigma(t)^2(T_{i-1} - t)}{\sigma(t)\sqrt{T_{i-1} - t}}.$$

# Proof

We just note that

$$\text{Cpl}(t; T_{i-1}, T_i) = \delta P(t, T_i) \mathbb{E}[(e^X - \kappa)^+],$$

$$\text{Flt}(t; T_{i-1}, T_i) = \delta P(t, T_i) \mathbb{E}[(\kappa - e^X)^+]$$

with  $X \sim N(\mu, \sigma^2)$ .

## Black's formula for swaptions

Let  $t \leq T_0$ . From our previous results we know that

$$\text{Swpt}_p(t) = N\delta \sum_{i=1}^n P(t, T_i) \mathbb{E}_t^{\mathbb{P}^{T_0; T_1, \dots, T_n}} [(R_{\text{swap}}(T_0) - K)^+],$$

$$\text{Swpt}_r(t) = N\delta \sum_{i=1}^n P(t, T_i) \mathbb{E}_t^{\mathbb{P}^{T_0; T_1, \dots, T_n}} [(K - R_{\text{swap}}(T_0))^+],$$

and that  $R_{\text{swap}}$  is an  $\mathbb{P}^{T_0; T_1, \dots, T_n}$ -martingale.

## Black's formula for swaptions

We assume that under  $\mathbb{P}^{T_0; T_1, \dots, T_n}$  the swap rate  $R_{\text{swap}}$  is an exponential Brownian motion

$$R_{\text{swap}}(t) = R_{\text{swap}}(s) \exp \left( -\frac{1}{2} \int_s^t \lambda(u)^2 ds + \int_s^t \lambda(u) dW_u \right)$$

for  $s \leq t \leq T_0$ , with a function  $\lambda(u)$ .

We define the implied volatility  $\sigma^2(t)$  as

$$\sigma^2(t) := \frac{1}{T_0 - t} \int_t^{T_0} \lambda(s)^2 ds.$$

The  $\mathbb{P}^{T_0; T_1, \dots, T_n}$ -distribution of  $\log R_{\text{swap}}(T_0)$  conditional on  $\mathcal{F}_t$  is  $N(\mu, \sigma^2)$  with

$$\begin{aligned}\mu &= \log R_{\text{swap}}(t) - \frac{\sigma^2(t)}{2}(T_0 - t), \\ \sigma^2 &= \sigma^2(t)(T_0 - t).\end{aligned}$$

In particular

$$\begin{aligned}\mu + \frac{\sigma^2}{2} &= \log R_{\text{swap}}(t), \\ \mu + \sigma^2 &= \log R_{\text{swap}}(t) + \frac{\sigma^2(t)}{2}(T_0 - t).\end{aligned}$$

We have

$$\text{Swpt}_p(t) = N\delta(R_{\text{swap}}(t)\Phi(d_1(t)) - K\Phi(d_2(t))) \sum_{i=1}^n P(t, T_i),$$

$$\text{Swpt}_r(t) = N\delta(K\Phi(-d_2(t)) - R_{\text{swap}}(t)\Phi(-d_1(t))) \sum_{i=1}^n P(t, T_i),$$

with

$$d_{1,2}(t) = \frac{\log\left(\frac{R_{\text{swap}}(t)}{K}\right) \pm \frac{1}{2}\sigma(t)^2(T_0 - t)}{\sigma(t)\sqrt{T_0 - t}}.$$

## Market Models

- ▶ Let  $0 = T_0 < \dots < T_N = T$  be a discrete tenor structure of maturity dates.
- ▶ We shall assume that  $T_{k+1} - T_k \equiv \delta$ .
- ▶ Our goal is to model the LIBOR market

$$L(t, T_k, T_{k+1}) = \frac{1}{\delta} \left( \frac{P(t, T_k)}{P(t, T_{k+1})} - 1 \right).$$

## Three Axioms

The following axioms are motivated by economic theory, arbitrage pricing theory and applications:

- ▶ *Axiom 1*: Positivity of the LIBOR rates

$$L(t, T_k, T_{k+1}) \geq 0.$$

- ▶ *Axiom 2*: Martingale property under the corresponding forward measure

$$L(t, T_k, T_{k+1}) \in \mathcal{M}(\mathbb{P}^{T_{k+1}}).$$

- ▶ *Axiom 3*: Analytical tractability.



## Known Approaches

Here are some known approaches:

- ▶ Let  $L(t, T_k, T_{k+1})$  be an exponential Brownian motion. Then analytical tractability not completely satisfied (“Freezing the drift”).
- ▶ Let  $\frac{P(t, T_k)}{P(t, T_{k+1})}$  be an exponential Brownian motion. Then positivity of the LIBOR rates is not satisfied.
- ▶ We will study affine LIBOR models.

## Affine Processes

Let  $X = (X_t)_{0 \leq t \leq T}$  be a conservative, time-homogeneous, stochastically continuous process taking values in  $D = \mathbb{R}_{\geq 0}^d$ .

Setting

$$\mathcal{I}_T := \{u \in \mathbb{R}^d : \mathbb{E}[e^{\langle u, X_T \rangle}] < \infty\},$$

we assume that:

- ▶  $0 \in \mathcal{I}_T^\circ$ ;
- ▶ there exist functions  $\phi : [0, T] \times \mathcal{I}_T \rightarrow \mathbb{R}$  and  $\psi : [0, T] \times \mathcal{I}_T \rightarrow \mathbb{R}^d$  such that

$$\mathbb{E}[e^{\langle u, X_T \rangle}] = \exp(\phi_t(u) + \langle \psi_t(u), X_0 \rangle)$$

for all  $0 \leq t \leq T$  and  $u \in \mathcal{I}_T$ .

## Some Properties of Affine Processes

- ▶ For all  $0 \leq s \leq t \leq T$  and  $u \in \mathcal{I}_T$  we have

$$\mathbb{E}[e^{\langle u, X_T \rangle} | \mathcal{F}_s] = \exp(\phi_{t-s}(u) + \langle \psi_{t-s}(u), X_0 \rangle).$$

- ▶ Semiflow property: For all  $0 \leq t + s \leq T$  and  $u \in \mathcal{I}_T$  we have

$$\begin{aligned}\phi_{t+s}(u) &= \phi_t(u) + \phi_s(\psi_t(u)), \\ \psi_{t+s}(u) &= \psi_s(\psi_t(u)).\end{aligned}$$

- ▶ Order-preserving: For  $(t, u), (t, v) \in [0, T] \times \mathcal{I}_T$  with  $u \leq v$  we have

$$\phi_t(u) \leq \phi_t(v) \quad \text{and} \quad \psi_t(u) \leq \psi_t(v).$$

## Constructing Martingales $\geq 1$

For  $u \in \mathcal{I}_T$  we define  $M^u = (M_t^u)_{0 \leq t \leq T}$  as

$$M_t^u := \exp(\phi_{T-t}(u) + \langle \psi_{T-t}(u), X_t \rangle).$$

Then the following properties are valid:

- ▶  $M^u$  is a martingale.
- ▶ For  $u \in \mathbb{R}_{\geq 0}^d$  and  $X_0 \in \mathbb{R}_{\geq 0}^d$  we have  $M_t^u \geq 1$ .

## Constructing the Affine LIBOR Model

- ▶ We fix  $u_1 > \dots > u_N$  from  $\mathcal{I}_T \cap \mathbb{R}_{\geq 0}^d$  and set

$$\frac{P(t, T_k)}{P(t, T_N)} = M_t^{u_k}, \quad k = 1, \dots, N.$$

- ▶ Obviously, we set

$$u_N = 0 \Leftrightarrow \frac{P(0, T_N)}{P(0, T_N)} = 1.$$

## Positivity

Then we have

$$\frac{P(t, T_k)}{P(t, T_{k+1})} = \exp(A_k + \langle B_k, X_t \rangle),$$

where we have defined

$$A_k := A_{T-t}(u_k, u_{k+1}) := \phi_{T-t}(u_k) - \phi_{T-t}(u_{k+1}),$$

$$B_k := B_{T-t}(u_k, u_{k+1}) := \psi_{T-t}(u_k) - \psi_{T-t}(u_{k+1}).$$

Note that  $A_k, B_k \geq 0$  by the order-preserving property of  $\phi_t(\cdot)$  and  $\psi_t(\cdot)$ . Thus, the LIBOR rates are *positive*:

$$L(t, T_k, T_{k+1}) = \frac{1}{\delta} \left( \frac{P(t, T_k)}{P(t, T_{k+1})} - 1 \right) = \frac{1}{\delta} \underbrace{(\exp(A_k + \langle B_k, X_t \rangle))}_{\geq 1} - 1 \geq 0.$$

## Martingale Property

- ▶ We define the equivalent probability measures

$$\frac{d\mathbb{P}^{T_k}}{d\mathbb{P}^{T_N}} \Big|_{\mathcal{F}_t} := \frac{M_t^{u_k}}{M_0^{u_k}}, \quad t \in [0, T_k].$$

- ▶ By Bayes' rule these are forward measures:

$$M_t^{u_j} = \frac{P(t, T_j)}{P(t, T_N)} \in \mathcal{M}(\mathbb{P}^{T_N}) \Rightarrow \frac{P(t, T_j)}{P(t, T_k)} = \frac{M_t^{u_j}}{M_t^{u_k}} \in \mathcal{M}(\mathbb{P}^{T_k}).$$

- ▶ We deduce the *martingale property*

$$L(t, T_k, T_{k+1}) = \frac{1}{\delta} \left( \frac{P(t, T_k)}{P(t, T_{k+1})} - 1 \right) \in \mathcal{M}(\mathbb{P}^{T_{k+1}}).$$

## Analytical Tractability

- ▶  $X$  is a time-homogeneous affine process under any forward measure:

$$\mathbb{E}^{T_k} [e^{\langle v, X_t \rangle}] = \exp(\phi_t^k(v) + \langle \psi_t^k(v), X_0 \rangle).$$

- ▶ The functions  $\phi^k$  and  $\psi^k$  are given by

$$\phi_t^k(v) := \phi_t(\psi_{T-t}(u_k) + v) - \phi_t(\psi_{T-t}(u_k)),$$

$$\psi_t^k(v) := \psi_t(\psi_{T-t}(u_k) + v) - \psi_t(\psi_{T-t}(u_k)).$$



## Option Pricing

The price of a caplet with reset date  $T_k$ , settlement date  $T_{k+1}$  and strike rate  $K$  is given by

$$\text{Cpl}(T_k, K) = P(0, T_{k+1}) \mathbb{E}^{T_{k+1}} \left[ \left( e^{A_k + \langle B_k, X_{T_k} \rangle} - \mathcal{K} \right)^+ \right],$$

where  $\mathcal{K} = 1 + \delta K$ . By applying Fourier methods, we obtain

$$\text{Cpl}(T_k, K) = \frac{P(0, T_{k+1})}{2\pi} \int_{\mathbb{R}} \frac{\mathbb{E}^{T_{k+1}} [e^{(R-iv)(A_k + \langle B_k, X_{T_k} \rangle)}] \mathcal{K}^{1+iv-R}}{(iv - R)(1 + iv - R)} dv,$$

where  $R \in (1, \infty)$ .

## Definition of a Markov Process

- ▶ A family of adapted  $\mathbb{R}^d$ -valued stochastic processes  $(X_t^x)_{t \geq 0, x \in S}$  is called *time-homogenous Markov process with state space  $S$*  if for all  $s \leq t$  and  $B \in \mathcal{B}(\mathbb{R}^d)$  we have

$$\mathbb{P}(X_t^x \in B \mid \mathcal{F}_s) = \mathbb{P}(X_{t-s}^y \in B) \Big|_{y=X_s}.$$

In particular Markov processes with state space  $S$  take values in  $S$  almost surely.

- ▶ We can define the associated Markov kernels  $\mu_{s,t} : \mathbb{R}^d \times \mathcal{B}(\mathbb{R}^d) \rightarrow [0, 1]$  with

$$\mu_{s,t}(y, B) = \mathbb{P}(X_{t-s}^y \in B).$$

- ▶ They satisfy the Chapman-Kolmogorov equation

$$\mu_{s,u}(x, B) = \int_{\mathbb{R}^d} \mu_{t,u}(y, B) \mu_{s,t}(x, dy).$$

for  $s < u$  and Borel sets  $B$ .

# Feller Processes

- ▶ For  $t \geq 0$  and  $f \in C_0(\mathbb{R}^d)$  we define

$$T_t f(x) := \int_{\mathbb{R}^d} f(y) \mu_t(x, dy), \quad x \in \mathbb{R}^d.$$

- ▶  $X$  is a *Feller Process* if  $(T_t)_{t \geq 0}$  is a  $C_0$ -semigroup of contractions on  $C_0(\mathbb{R}^d)$ .
- ▶ We define the infinitesimal generator

$$Af := \lim_{t \rightarrow 0} \frac{T_t f - f}{t}, \quad f \in \mathcal{D}(A),$$

which coincides with the concept of infinitesimal generator from functional analysis.

# Stochastic Differential Equations as Markov processes

- ▶ Let  $b : \mathbb{R}^d \rightarrow \mathbb{R}^d$  and  $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times m}$ . Consider the SDE

$$dX_t^x = b(X_t)dt + \sigma(X_t)dW_t, X_0^x = x$$

- ▶ We assume that the solution exists for all times and any initial value in some state space  $S$  as a Feller-Markov process.
- ▶ Set  $a := \sigma\sigma^\top$ . We have  $C_0^2(\mathbb{R}^d) \subset \mathcal{D}(A)$  and

$$Af(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \langle b(x), \nabla f(x) \rangle, \quad f \in C_0^2(\mathbb{R}^d).$$

- ▶ *Example:* For a Brownian motion  $W$  we have  $A = \frac{1}{2}\Delta$ .

## Kolmogorov Backward Equation

We assume there exist transition densities  $p(t, x, y)$  such that

$$\mathbb{P}(X_t \in B \mid X_s = x) = \int_B p(t - s, x, y) dy.$$

- ▶ Recall that the generator is given by

$$Af(x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} f(x) + \sum_{i=1}^d b_i(x) \frac{\partial}{\partial x_i} f(x).$$

- ▶ *Kolmogorov backward equation*: For fixed  $y \in \mathbb{R}^d$  we have

$$\frac{\partial}{\partial t} p(t, x, y) = A p(t, x, y),$$

i.e. the equation acts on the backward (initial) variables. It also holds in the sense of distribution for the expectation functional  $(t, x) \mapsto E(f(X_t^x))$ .

## Kolmogorov Forward Equation

- ▶ The adjoint operator is given by

$$A^* f(y) = \frac{1}{2} \sum_{i,j=1}^d \frac{\partial^2}{\partial y_i \partial y_j} (a_{ij}(y) f(y)) - \sum_{i=1}^d \frac{\partial}{\partial y_i} (b_i(y) f(y)).$$

- ▶ *Kolmogorov forward equation*: For fixed  $x \in \mathbb{R}^d$  we have

$$\frac{\partial}{\partial t} p(t, x, y) = A^* p(t, x, y),$$

i.e. the equation acts on the forward variables. It also holds in the sense of distributions for the Markov kernels  $p(\cdot, x, \cdot)$  for any initial value  $x \in S$ .

## Example: Brownian Motion

- ▶ The Brownian motion  $W$  has the transition densities

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(y-x)^2}{2t}}.$$

- ▶ The infinitesimal generator is given by the Laplace operator  $A = \frac{1}{2}\Delta$ .
- ▶ *Kolmogorov backward equation*: For fixed  $y \in \mathbb{R}^d$  we have

$$\frac{d}{dt} p(t, x, y) = \frac{1}{2} \frac{\partial^2}{\partial x^2} p(t, x, y).$$

- ▶ *Kolmogorov forward equation*: For fixed  $x \in \mathbb{R}^d$  we have

$$\frac{d}{dt} p(t, x, y) = \frac{1}{2} \frac{\partial^2}{\partial y^2} p(t, x, y).$$

## Example: The SABR Model

- ▶ The SABR model for  $\beta = 0, \alpha = 1, \rho = 0$  is given through

$$dX_1(t) = X_2(t)dW_1(t),$$

$$dX_2(t) = X_2(t)dW_2(t).$$

- ▶ Its infinitesimal generator equals therefore

$$A = \frac{x_2^2}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right).$$

- ▶ *Kolmogorov backward equation*: For fixed  $y \in \mathbb{R}^2$  we have

$$\frac{d}{dt} p(t, x, y) = \frac{x_2^2}{2} \left( \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} \right) p(t, x, y).$$



## Example: The SABR model

- ▶ *Kolmogorov forward equation:* For fixed  $x \in \mathbb{R}^2$  we have

$$\frac{d}{dt}p(t, x, y) = \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right) \frac{y_2^2}{2} p(t, x, y).$$

- ▶ Notice that in general the forward and backward equation are different.

## Construction of finite factor models

By the choice of a finite dimensional Markov process  $(X^1, \dots, X^n)$  and the choice of an expression

$$R_t = H(X_t)$$

for the short rate, one can construct – due to the Markov property – consistent finite factor model

$$E\left(\exp\left(-\int_t^T H(X_s) ds\right)\right) = P(t, T) = \exp\left(-\int_t^T G(t, r, X_t) dr\right),$$

for all  $0 \leq t \leq T$ ,  $G$  satisfies a certain P(I)DE.

Vasiček's model is due to its Gaussian nature relatively simple: We apply the parametrization  $x = T - t$  for  $0 \leq t \leq T$ :

$$\Lambda(x) = \frac{1}{\beta}(1 - \exp(-\beta x))$$

$$A_0(t, x) = r^*(x + t) + \frac{\rho^2}{2}\Lambda(x + t)^2 - \frac{\rho^2}{2}\Lambda(x)^2 -$$

$$- (\Lambda'(x))^2 r^*(0) - \Lambda'(x) \int_0^t e^{-\beta(t-s)} b(s) ds$$

$$b(t) = \frac{d}{dt}r^*(t) + \beta r^*(t) + \frac{\rho^2}{2\beta}(1 - \exp(-2\beta t))$$

$$A_1(t, x) = \Lambda'(x)$$

$$dR_t = (b(t) - \beta R_t)dt + \rho dW_t$$

for real constants  $\beta$  and  $\rho$  and an "arbitrary" initial value  $r^*$ . This solves the HJM equation with volatility  $\sigma(r, x) = \rho \exp(-\beta x)$  and initial value  $r^*$ .

The CIR analysis is more involved:

$$\begin{aligned}
 A_0(t, x) &= g(t, x) - c(t)\Lambda'(x) \\
 g(t, x) &= r^*(t + x) + \\
 &+ \rho^2 \int_0^t g(t - s, 0)(\Lambda\Lambda')(x + t - s)ds \\
 c(t) &= g(t, 0), \quad b(t) = \frac{d}{dt}c(t) + \beta c(t) \\
 A_1(t, x) &= \Lambda'(x) \\
 dR_t &= (b(t) - \beta R_t)dt + \rho R_t^{\frac{1}{2}} dW_t
 \end{aligned}$$

for real constants  $\beta$  and  $\rho$  and an "arbitrary" initial value  $r^*$ . This solves the HJM equation with volatility  $\sigma(r, x) = \rho(\text{ev}_0(r))^{\frac{1}{2}}\Lambda(x)$  and initial value  $r^*$ .

- ▶ The SABR model combines an explicit expression for implied volatility with attractive dynamic properties for implied volatilities.
- ▶ In contrast to affine models stochastic volatility is a lognormal random variable.
- ▶ it is a beautiful piece of mathematics.
- ▶ all important details can be found in [2].

We consider a model for forward prices  $F$  and their stochastic volatility  $\Sigma$

$$dF_t = \Sigma_t C(F_t) dW_t \quad (1)$$

$$d\Sigma_t = v \Sigma_t dZ_t \quad (2)$$

with two correlated Brownian motions  $W, Z$  with  $\langle W, Z \rangle_t = \rho t$ . We assume that  $C$  is smooth of 0 and that

$$\int_0^x \frac{du}{C(u)} < \infty$$

for  $x > 0$ . For instance  $C(x) = x^\beta$  for  $0 \leq \beta < 1$ . Notice that usually the SABR price  $F$  is symmetrically extended to the whole real line and some (inner) boundary conditions of Dirichlet, Neuman or mixed type are considered (see the discussion in [2]).

We aim to calculate the transition distribution at time  $G_{F,\Sigma}(\tau, f, \sigma)dF d\Sigma$ , when the process starts from initial value  $(f, \sigma)$  and evolves for some time  $\tau > 0$ . This is done by relating the general SABR model via an invertible map to

$$dX_t = Y_t dW_t, \quad dY_t = Y_t dZ_t,$$

with decorrelated Brownian motions  $W$  and  $Z$ . The latter stochastic differential equation is related to the Poincare halfplane and its hyperbolic geometry. We shall refer to it as Brownian motion on the Poincare halfplane.

## Poincare halfplane

Consider the set of points  $\mathbb{H}_2 := \mathbb{R} \times \mathbb{R}_{>0}$  and the Riemannian metric with matrix  $\frac{1}{y^2}$  id at  $(x, y) \in \mathbb{H}_2$ . Then one can calculate the geodesic distance on  $\mathbb{H}_2$ , i.e. the length of the shortest path connecting to points, via

$$\cosh(d(x, y, X, Y)) = 1 + \frac{(x - X)^2 + (y - Y)^2}{2yY}.$$

Furthermore one can calculate in terms of the geodesic distance  $d$  the heat kernel on  $\mathbb{H}_2$ . A derivation is shown in [2].



Applying the invertible map  $\phi$

$$(f, \sigma) \mapsto \left( \frac{1}{\sqrt{1-\rho^2}} \left( \int_0^f \frac{du}{C(u)} - \rho \text{sigma} \right), \sigma \right)$$

to the equation

$$dF_t = \Sigma_t C(F_t) dW_t, \quad d\Sigma_t = \Sigma_t dZ_t$$

leads to the Poincare halfplane's Brownian motion perturbed by a drift term. From the point of view of the SABR model one has to add a drift such that the  $\phi^{-1}$  transformation of it is precisely the Brownian motion of the Poincare halfplane.

## Regular perturbation techniques

Since we are interested in the original SABR model we have to calculate the influence of the drift term appearing when transforming from the Poincare halfplane to the SABR model, this is done by regular perturbation techniques:

- ▶ consider two linear operators  $A$ ,  $B$ , where  $B$  is considered small in comparison to  $A$ .
- ▶ consider the variation of constants formula ansatz for

$$\begin{aligned} \frac{d}{dt} \exp(t(A + \epsilon B)) &= A \exp(t(A + \epsilon B)) + \epsilon B \exp(t(A + \epsilon B)) \\ &= A \exp(t(A + \epsilon B)) + f(t). \end{aligned}$$

- ▶ this leads to

$$\exp(t(A + \epsilon B)) = \exp(tA) + \epsilon \int_0^t \exp((t-s)A) B \exp(s(A + \epsilon B)) ds,$$

and by iteration to.

$$\exp(A + \epsilon B) = \sum_{k=0}^{\infty} \epsilon^k \int_{0 \leq s_1 \dots \leq s_k \leq 1} \exp(s_1 \operatorname{ad}_A) B \exp(s_2 \operatorname{ad}_A) B \times \dots \\ \times \exp(s_k \operatorname{ad}_A) B ds_1 \dots ds_k,$$

where the adjoint action  $\operatorname{ad}$  is defined via

$$\exp(s \operatorname{ad}_A) B = \exp(sA) B \exp(-sA).$$

## Local volatility

A good approximation for implied volatility is given by local volatility, which can be calculated in many models by the following formula

$$\sigma(t, K)^2 = \frac{\frac{\partial}{\partial T} C(T, K)}{\frac{\partial^2}{\partial K^2} C(T, K)},$$

being the answer to the question which time-dependent volatility function  $\sigma$  to choose such that  $dS_t = \sigma(t, S_t)dW_t$  mimicks the given prices  $C(T, K)$ , i.e. for  $T, K \geq 0$  it holds

$$C(T, K) = E((S_T - K)_+).$$

The reasoning behind Dupire's formula for local volatility is that the transition distribution of a local volatility model satisfies Kolmogorov's forward equation in the forward variables

$$\frac{\partial^2}{\partial S^2} \sigma(t, S)^2 p(T, S, s) = \frac{\partial}{\partial T} p(T, S, s).$$

On the other hand it is well-known by Breeden-Litzenberger that

$$p(T, K, s) = \frac{\partial^2}{\partial K^2} C(T, K),$$

which leads after twofold integration of Kolmogorov's forward equation by parts to Dupire's formula.

In terms of the transition function of the SABR model Dupire's formula reads as

$$\sigma(T, K)^2 = \frac{C(K)^2 \int \Sigma^2 G(T, K, \Sigma, f, \sigma) d\Sigma}{\int G(T, K, \Sigma, f, \sigma) d\Sigma},$$

which can be evaluated by Laplace's principle as shown in [2] since one has at hand a sufficiently well-known expression for  $G$ .

## Lévy driven HJM models

Let  $L$  be a  $d$ -dimensional Lévy process with Lévy exponent  $\kappa$ , i.e.

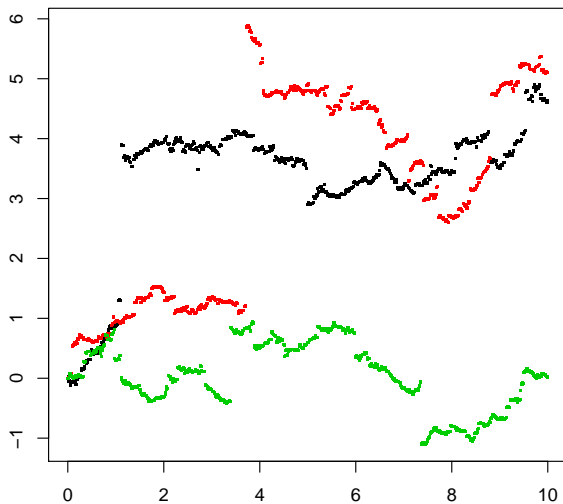
$$E(\exp(\langle u, L_t \rangle)) = \exp(\kappa(u)t)$$

for  $u \in \mathcal{U}$  an open strip in  $\mathbb{C}^d$  containing  $i\mathbb{R}^d$ , where  $\kappa$  is always defined. Then it is well-known that

$$\exp\left(-\int_0^t \kappa(\alpha_s) ds + \int_0^t \langle \alpha_s, dL_s \rangle\right)$$

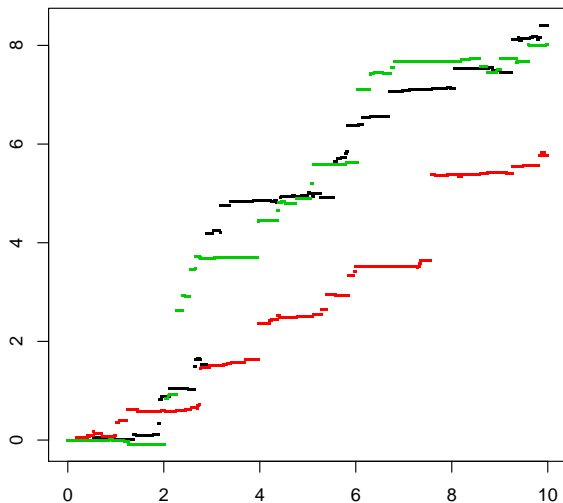
is a local martingale for predictable strategies  $\alpha$  such that both integrals are well-defined. Notice that the strategy  $\alpha$  is  $\mathbb{R}^d$ -valued and that  $\kappa$  has to be defined on  $\alpha_s$ .

3 trajectories of a stable process

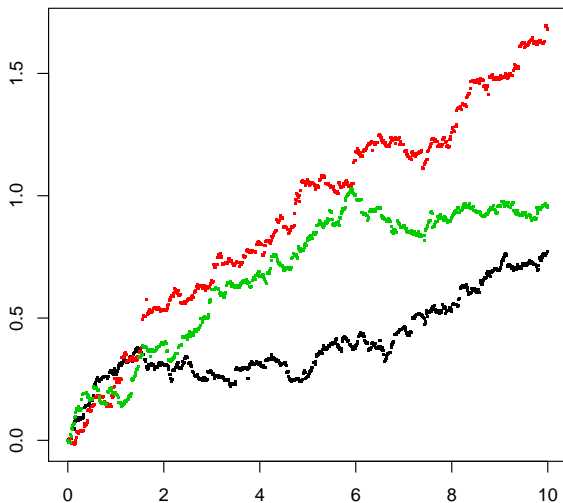




3 trajectories of VG process



3 trajectories of NIG process



We can formulate a small generalization of the previous result by considering a parameter-dependence in the strategies  $\alpha^S$ . There is one Fubini argument necessary to prove the result: we assume continuous dependence of  $\alpha^S$  on  $S$ , then

$$N_t^u = \exp\left(\int_0^t \int_u^T \frac{d}{dS} \kappa\left(\int_u^S \alpha_s^U dU\right) ds + \int_0^t \int_u^T \langle \alpha_s^S dS, dL_s \rangle\right)$$

is a local martingale. Notice here that  $N_t^t$  is not a local martingale, but

$$\int_0^t (dN_t^u)_{u=t}$$

is one, since – loosely speaking – it is the sum of local martingale increments.

## The general HJM-drift condition for Lévy-driven term structures

If

$$f(t, T) = f(0, T) + \int_0^t \frac{d}{dT} \kappa \left( - \int_t^T \alpha_t^T \right) dt + \int_0^t \langle \alpha_t, dL_t \rangle$$

defines a stochastic process of forward rates, where continuous dependence in  $T$  of all quantities is assumed, such that

$$\begin{aligned} M(t, T) &= P(t, T) \exp\left(- \int_0^t f(s, s) ds\right) \\ &= \exp\left(\int_t^T f(t, S) dS - \int_0^t f(s, s) ds\right) \end{aligned}$$

is a local martingale, since

its differential equals

$$-f(t, t)M(t, T)dt + f(t, t)M(t, T)dt + \exp\left(-\int_0^t f(s, s)ds\right)(dN_t^u)|_{u=t},$$

where the first two terms cancel and the third one is the increment of a local martingale as was shown before.

## HJM-drift condition in case of driving Brownian motion

When the HJM equation is driven by Brownian motions, we speak of an Itô process model, in particular  $\kappa(u) = \frac{\|u\|^2}{2}$ .

If we assume an Itô process model with the HJM equation reads as

$$\begin{aligned}df(t, T) &= \sum_{i=1}^d \alpha^i(t, T) \int_0^x \alpha^i(t, y) dy dt + \\ &+ \sum_{i=1}^d \alpha^i(t, T) dB_t^i,\end{aligned}$$

where the volatilities  $\alpha^i(t, T)_{0 \leq t \leq T}$  are predictable stochastic integrands.

## Musiela parameterization

The forward rates  $(f(t, T))_{0 \leq t \leq T}$  are best parametrized through

$$r(t, x) := f(t, t + x)$$

for  $t, x \geq 0$  (Musiela parametrization). This allows to consider spaces of forward rate curves, otherwise the domain of definition of the forward rate changes along running time as it equals  $[t, \infty[$ .

## Forward Rates as states

This equation is best analysed as stochastic evolution on a Hilbert space  $H$  of forward curves making it thereon into a Markov process

$$\tilde{\sigma}^i(t, \cdot) = \sigma^i(r_t), \sigma^i : H \rightarrow H$$

for some initial value  $r_0 \in H$ . We require:

- ▶  $H$  is a separable Hilbert space of continuous functions.
- ▶ point evaluations are continuous with respect to the topology of a Hilbert space.
- ▶ The shift semigroup  $(S_t r)(x) = r(t + x)$  is a strongly continuous semigroup on  $H$  with generator  $\frac{d}{dx}$ .
- ▶ The map  $h \mapsto \mathcal{S}(h)$  with  $\mathcal{S}(h)(x) := h(x) \int_0^x h(y) dy$  satisfies

$$\|\mathcal{S}(h)\| \leq K \|h\|^2$$

for all  $h \in H$  with  $\mathcal{S}(h) \in H$ .



## An example

Let  $w : \mathbb{R}_{\geq 0} \rightarrow [1, \infty[$  be a non-decreasing  $C^1$ -function with

$$\frac{1}{w^{\frac{1}{3}}} \in L^1(\mathbb{R}_{\geq 0}),$$

then we define

$$\|h\|_w := |h(0)| + \int_{\mathbb{R}_{\geq 0}} |h'(x)|w(x)dx$$

for all  $h \in L^1_{loc}$  with  $h' \in L^1_{loc}$  (where  $h'$  denotes the weak derivative). We define  $H_w$  to be the space of all functions  $h \in L^1_{loc}$  with  $h' \in L^1_{loc}$  such that  $\|h\|_w < \infty$ .

## Finite Factor models

Given an initial forward rate  $T \mapsto f(0, T)$  or  $T \mapsto P(0, T)$ , respectively. A finite factor model at initial value  $r^*$  is a mapping

$$G : \{0 \leq t \leq T\} \times \mathbb{R}^n \subset \mathbb{R}_{\geq 0}^2 \times \mathbb{R}^n \rightarrow \mathbb{R}$$

together with an Markov process  $(X_t)_{t \geq 0}$  such that

$$f(t, T) = G(t, T, X_t^1, \dots, X_t^n)$$

for  $0 \leq t \leq T$  and  $T \geq 0$  is an arbitrage-free evolution of forward rates. The process  $(X_t)_{t \geq 0}$  is called factor process, its dimension  $n$  is the dimension of the factor model.

In many cases the map  $G$  is chosen to have a particularly simple structure

$$G(t, T, z^1, \dots, z^n) = A_0(t, T) + \sum_{i=1}^n A_i(t, T)z^i.$$

In these cases we speak of affine term structure models, the factor processes are also affine processes. Remark that  $G$  must reproduce the initial value

$$G(0, T, z_0^1, \dots, z_0^n) =: r^*(T)$$

for  $T \geq 0$ . The famous short rate models appear as 1- or 2-dimensional cases ( $n = 1, 2$  – time is counted as additional factor).

## The consistency problem

It is an interesting and far reaching question if – for a given function  $G(t, T, z)$  – there is a Markov process such that they constitute a finite factor model together.

## Svensson family

An interesting example for a map  $G$  is given by the Svensson family

$$G(t, T, z_1, \dots, z_6) = z_1 + z_2 \exp(-z_3(T - t)) + \\ + (z_4 + z_5(T - t)) \exp(-z_6(T - t)),$$

since it is often applied by national banks. The wishful thought to find an underlying Itô-Markov process such that  $G$  is consistent with an arbitrage free evolution of interest rates is realized by a one factor Gaussian process.

- [1] Damir Filipovic.  
Term structure models: A Graduate Course.  
see <http://sfi.epfl.ch/op/edit/page-12795.html>
- [2] Patrick Hagan, Andrew Lesniewski, and Diana Woodward.  
Probability Distribution in the SABR Model of Stochastic Volatility.  
see <http://lesniewski.us/working.html>
- [3] Martin Keller-Ressel, Antonis Papapantoleon, Josef Teichmann.  
A new approach to LIBOR modeling.  
see <http://www.math.ethz.ch/~jteichma/index.php?content=publications>

[4] Damir Filipovic, Stefan Tappe.

Existence of Lévy term structure models.

see

<http://www.math.ethz.ch/~tappes/publications.php>

## Catalogue of possible questions for the oral exam

- ▶ What are ZCBs, yield curves, forward curves, short rates, caplets, floorlets, swaps, swap rates, roll-over-portfolios?
- ▶ What is a LIBOR rate (simple forward rate) on nominal one received at terminal date and what is its fair value before? Some relations of cap, floors, swaptions like in exercise 2 of Sheet 2.
- ▶ Black's formula for caps and floors – derivation and assumptions?
- ▶ Black's formula for swaptions – derivation and assumptions?



## Catalogue of possible questions for the oral exam

- ▶ Change of numeraire theorem: exercise 1,2 of Sheet 7.
- ▶ What is the LIBOR market model (Exercise 3 of Sheet 7)?
- ▶ What is the forward measure model (Exercise 2 of Sheet 8).
- ▶ What is an affine LIBOR model and what are its main characteristics (Axiom 1 – 3 can be satisfied)?
- ▶ What is the Fourier method of derivative pricing (Exercise 1 of Sheet 9)?

## Catalogue of possible questions for the oral exam

- ▶ Lévy processes and their cumulant generating function (Exercise 1 of Sheet 11).
- ▶ What is the HJM-drift condition for Brownian motion and which Hilbert spaces of forward rates can be considered?
- ▶ Derivation of the HJM-drift condition for Lévy driven interest rate models.
- ▶ Why are models for the whole term structure attractive? What lie their difficulties?

## Catalogue of possible questions for the oral exam

- ▶ What is the general SABR model and how is it related to the Poincare halfplane?
- ▶ What is the geodesic distance in the Poincare halfplane (definition) and how can we calculate it (eikonal equation), Exercise 5 on Sheet 9? What is the natural stochastic process on the Poincare halfplane – can we calculate its heat kernel?
- ▶ What is local volatility and how can we calculate it (Dupire's formula)?
- ▶ Implied volatility in interest rate markets: which tractability do we need from models and what do market models, forward measure models, affine models, the SABR model, HJM models provide?

## Catalogue of possible questions for the oral exam

- ▶ Short rate models: the Vasicek model (Exercise 1 of Sheet 4).
- ▶ Short rate models: the CIR model (Exercise 2 of Sheet 4, not every detail).
- ▶ Short rate models: Hull-White extension of the Vasicek model (Exercise 2 of Sheet 6).
- ▶ Is short rate easy to model from an econometric point of view?