Asymptotic Lower Bounds for Optimal Tracking

A Linear Programming Approach

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Outline

Tracking Problem

- An agent observes a stochastic target X° in $\mathbb{R}^d$:

$$dX_t^\circ = b_t dt + \sqrt{a_t} dW_t, \quad X_0^\circ = 0.$$

- And adjusts her position $(\psi_t)_{t\geq 0}$ to minimize the deviation X :

$$X_t = -X_t^\circ + \psi_t.$$

- The objective of the agent is given by :

$$\inf_{(\psi_t)\in\mathcal{A}} J(\psi), \quad J(\psi) = H_0(X) + H(\psi),$$

where H_0 is the **deviation penalty** and H the **tracking effort.**

Tracking Problem

- Deviation penalty $H_0(X)$ is given by :

$$H_0(X) = \int_0^T r_t D(X_t) dt,$$

  where
  - (r_t) is a random weight process,
  - $D(\varepsilon x) = \varepsilon^{\zeta_D} D(x)$, e.g. $D(x) = \langle x, \Sigma^D x \rangle, \zeta_D = 2$.

- Tracking effort $H(\psi)$ depends on the cost structure.

Outline

Fixed Costs and Impulse Control

- The process (ψ_t) is given by

$$\psi_t = \sum_{0<\tau_j\leq t} \xi_j.$$

- The cumulated cost is then given by

$$H(\psi) = \sum_{0<\tau_j\leq T} k_{\tau_j} F(\xi_j) + h_{\tau_j} P(\xi_j),$$

  - (k_t) and (h_t) are random weight processes.
  - $F(\varepsilon\xi) = \varepsilon^{\zeta_F} F(\xi), \zeta_F = 0$, e.g. $F(\xi) = \sum_i \mathbb{1}_{\{\xi^i\neq 0\}}, \xi = (\xi^1, \cdots, \xi^d)^T$.
  - $P(\varepsilon\xi) = \varepsilon^{\zeta_P} P(\xi), \zeta_P = 1$, e.g. $P(\xi) = \langle p, |\xi|\rangle, p \in \mathbb{R}^d_+$.

- Examples :
  - Discretization of hedging strategies (Fukasawa 2011, 2014, Rosenbaum and Tankov 2012, Gobet and Landon 2014).
  - Indifference pricing for option with fixed costs (Wilmott and Whalley 1999).
  - Management of index fund (Korn 1999, Atkison and Wilmott 1995, Pliska and Suzuki 2004).
  - Utility maximization under fixed costs (Morton and Pliska 1995, Altorovici et al. 2013).

Proportional Costs and Singular Control

- The process (ψ_t) is given by

$$\psi_t = \int_0^t \gamma_s d\varphi_s,$$

with $\gamma_s \in \Delta = \{\gamma \in \mathbb{R}^d | \sum_{i=1}^d |\gamma^i| = 1\}$ and (φ_s) non-decreasing.

- The corresponding cost is usually given as

$$H(\psi) = \int_0^T h_t P(\gamma_t) d\varphi_t,$$

  - (h_t) is a random weight process.
  - $P(\varepsilon\gamma) = \varepsilon^{\zeta_P} P(\gamma), \zeta_P = 1$, e.g. $P(\gamma) = \langle p, |\gamma| \rangle$ with $p \in \mathbb{R}^d_+$.

- Examples :
  - Utility maximization under proportional costs (Jenecek and Shreve 2004, 2010, Bichuch and Shreve 2013, Soner and Touzi 2012, Possamai et al. 2013, Gerhold et al. 2013, Kallsen and Muhle-Karbe 2013).
  - Indifference pricing for option under proportional costs (Davis et al. 1993, Wilmott and Whalley 1997).
  - Trend following (Martin and Schöneborn 2011, Martin 2012).

(Absolutely Continuous) Stochastic Control

- The process (ψ_t) is given by

$$\psi_t = \int_0^t u_s ds,$$

- A typical cost structure is

$$H(\psi) = \int_0^T l_t Q(u_t) dt,$$

where
  - (l_t) is a random weight process.
  - $Q(\varepsilon u) = \varepsilon^{\zeta_Q} Q(u), \zeta_Q > 1$, e.g. $Q(u) = \langle u, \Sigma^Q u \rangle, \zeta_Q = 2$.

- Examples :
  - Trading with market impact/illiquidity (Almgren and Li, 2014, Rogers and Singh 2007, Guasoni and Weber 2012, 2015a, 2015b, Moreau et al. 2014).
  - Trading under proportional cost and market impact (Liu et al. 2014).

Combined Controls

- In the case of combined stochastic and impulse controls, we have

$$\psi_t = \sum_{0<\tau_j\leq t} \xi_j + \int_0^t u_s ds,$$

- The cost functional is given by

$$H(\psi) = \sum_{0<\tau_j\leq T} \left(k_{\tau_j} F(\xi_j) + h_{\tau_j} P(\xi_j)\right) + \int_0^T l_t Q(u_t) dt.$$

  Similarly, one can consider other combination of controls.
- Example :
  - Control of exchange rate (Mundaca and Oksendal 1997).

Outline

1. Formulation of Tracking Problem
   - Cost Structure and Control Types
   - Asymptotic Framework
2. Main Results
   - Asymptotic Lower Bounds
   - Closed-form Examples in Dimension One
   - Relation with Utility Maximization
3. Elements of Proof
   - Occupation Measures
   - Interpretation as Time-average Control of BM

Asymptotic Framework : Small Tracking Costs

Instead of the original tracking problem:

$$X_t = -X_t^\circ + \psi_t.$$

$$\inf_{(\psi_t)\in\mathcal{A}} J(\psi), \quad J(\psi) = H_0(X) + H(\psi),$$

we consider a sequence of tracking problems :

$$X_t^\varepsilon = -X_t^\circ + \psi_t^\varepsilon,$$

$$\inf_{(\psi_t^\epsilon)\in\mathcal{A}^\varepsilon} J^\varepsilon(\psi^\varepsilon), \quad J^\varepsilon(\psi^\varepsilon) = H_0(X^\varepsilon) + \varepsilon H(\psi^\varepsilon),$$

with $\varepsilon \to 0$.

Case Study : Combined Stochastic and Impulse Controls

In the presence of several controls, we consider

$$\inf_{(u^\varepsilon,\tau^\varepsilon,\xi^\varepsilon)\in\mathcal{A}^\varepsilon} J^\varepsilon(u^\varepsilon,\tau^\varepsilon,\xi^\varepsilon),$$

with

$$J^\varepsilon(u^\varepsilon,\tau^\varepsilon,\xi^\varepsilon) = \int_0^T (r_t D(X_t^\varepsilon) + \varepsilon^{\beta_Q} l_t Q(u_t^\varepsilon))dt + \sum_{j:0<\tau_j^\varepsilon\le T} (\varepsilon^{\beta_F} k_{\tau_j^\varepsilon} F(\xi_j^\varepsilon) + \varepsilon^{\beta_P} h_{\tau_j^\varepsilon} P(\xi_j^\varepsilon)),$$

and

$$X_t^\varepsilon = -X_t^\circ + \int_0^t u_s^\varepsilon ds + \sum_{j:0<\tau_j^\varepsilon\le t} \xi_j^\varepsilon.$$

The constants $\beta_Q, \beta_F, \beta_P$ are to be determined.

Let $\{t_k^\varepsilon = k\delta^\varepsilon, k = 0, 1, \cdots, K^\varepsilon\}$ be a partition of $[0, T]$ with $\delta^\varepsilon \to 0$.

$$J^\varepsilon(u^\varepsilon, \tau^\varepsilon, \xi^\varepsilon) = \sum_{k=0}^{K^\varepsilon - 1} \Big(\int_{t_k^\varepsilon}^{t_k^\varepsilon + \delta^\varepsilon} \big(r_t D(X_t^\varepsilon) + \varepsilon^{\beta_Q} l_t Q(u_t^\varepsilon)\big) dt + \sum_{j: t_k^\varepsilon < \tau_j^\varepsilon \le t_k^\varepsilon + \delta^\varepsilon} \big(\varepsilon^{\beta_F} k_{\tau_j^\varepsilon} F(\xi_j^\varepsilon) + \varepsilon^{\beta_P} h_{\tau_j^\varepsilon} P(\xi_j^\varepsilon)\big)\Big)$$

$$= \sum_{k=0}^{K^\varepsilon - 1} j_{t_k^\varepsilon}^\varepsilon (t_{k+1}^\varepsilon - t_k^\varepsilon),$$

where

$$j_t^\varepsilon = \frac{1}{\delta^\varepsilon} \Big(\int_t^{t+\delta^\varepsilon} \big(r_s D(X_s^\varepsilon) + \varepsilon^{\beta_Q} l_s Q(u_s^\varepsilon)\big) ds + \sum_{j: t < \tau_j^\varepsilon \le t + \delta^\varepsilon} \big(\varepsilon^{\beta_F} k_{\tau_j^\varepsilon} F(\xi_j^\varepsilon) + \varepsilon^{\beta_P} h_{\tau_j^\varepsilon} P(\xi_j^\varepsilon)\big)\Big).$$

We have

$$j_t^\varepsilon = \frac{1}{\delta^\varepsilon}\Big(\int_t^{t+\delta^\varepsilon} (r_s D(X_s^\varepsilon) + \varepsilon^{\beta_Q} l_s Q(u_s^\varepsilon))ds + \sum_{j:t<\tau_j^\varepsilon\leq t+\delta^\varepsilon} (\varepsilon^{\beta_F} k_{\tau_j^\varepsilon} F(\xi_j^\varepsilon) + \varepsilon^{\beta_P} h_{\tau_j^\varepsilon} P(\xi_j^\varepsilon))\Big)$$

and as ε tends to zero, we approximately get

$$J^\varepsilon(u^\varepsilon, \tau^\varepsilon, \xi^\varepsilon) \simeq \int_0^T j_t^\varepsilon dt.$$

Then consider the following rescaling of X^ε over the horizon $(t, t+\delta^\varepsilon]$:

$$\widetilde{X}_s^{\varepsilon,t} = \frac{1}{\varepsilon^\beta} X^\varepsilon_{t+\varepsilon^{\alpha\beta}s}, \quad s \in (0, T^\varepsilon],$$

with $T^\varepsilon = \varepsilon^{-\alpha\beta}\delta^\varepsilon$ and $\alpha = 2$.

On the one hand, we have

$$d\widetilde{X}_s^{\varepsilon,t} = \widetilde{b}_s^{\varepsilon,t}ds + \sqrt{\widetilde{a}_s^{\varepsilon,t}}d\widetilde{W}_s^{\varepsilon,t} + \widetilde{u}_s^{\varepsilon,t}ds + d\Big(\sum_{0<\widetilde{\tau}_j^{\varepsilon,t}\leq s} \widetilde{\xi}_j^{\varepsilon}\Big),$$

with

$$\widetilde{b}_s^{\varepsilon,t} = -\varepsilon^{(\alpha-1)\beta} b_{t+\varepsilon^{\alpha\beta}s}, \quad \widetilde{a}_s^{\varepsilon,t} = a_{t+\varepsilon^{\alpha\beta}s}, \quad \widetilde{W}_s^{\varepsilon,t} = -\frac{1}{\varepsilon^{\beta}} W_{t+\varepsilon^{\alpha\beta}s},$$

$$\widetilde{u}_s^{\varepsilon,t} = \varepsilon^{(\alpha-1)\beta} u^{\varepsilon}_{t+\varepsilon^{\alpha\beta}s}, \quad \widetilde{\xi}_j^{\varepsilon} = \frac{1}{\varepsilon^{\beta}}\xi_j^{\varepsilon}, \quad \widetilde{\tau}_j^{\varepsilon,t} = \frac{1}{\varepsilon^{\alpha\beta}}(\tau_j^{\varepsilon} - t) \vee 0.$$

By continuity of a_t, we have

$$d\widetilde{X}_s^{\varepsilon,t} \simeq \sqrt{a_t}d\widetilde{W}_s^{\varepsilon,t} + \widetilde{u}_s^{\varepsilon,t}ds + d\Big(\sum_{0<\widetilde{\tau}_j^{\varepsilon,t}\leq s} \widetilde{\xi}_j^{\varepsilon}\Big).$$

On the other hand, we have (by continuity of r_t, l_t, k_t and h_t),

$$j_t^\varepsilon \simeq \frac{1}{T^\varepsilon}\Big(\int_0^{T^\varepsilon}(\varepsilon^{\beta\zeta_D} r_t D(\widetilde{X}_s^{\varepsilon,t}) + \varepsilon^{\beta_Q-(\alpha-1)\zeta_Q\beta} l_t Q(\widetilde{u}_s^{\varepsilon,t}))ds + \sum_{0<\widetilde{\tau}_j^{\varepsilon,t}\le T^\varepsilon}(\varepsilon^{\beta_F-(\alpha-\zeta_F)\beta} k_t F(\widetilde{\xi}_j^\varepsilon) + \varepsilon^{\beta_P-(\alpha-\zeta_P)\beta} h_t P(\widetilde{\xi}_j^\varepsilon))\Big).$$

Assume that

$$\beta\zeta_D = \beta_Q - (\alpha-1)\zeta_Q\beta = \beta_F - (\alpha-\zeta_F)\beta = \beta_P - (\alpha-\zeta_P)\beta,$$

then

$$j_t^\varepsilon \simeq \varepsilon^{\beta\zeta_D} I_t^\varepsilon,$$

with

$$\boxed{I_t^\varepsilon = \frac{1}{T^\varepsilon}\Big(\int_0^{T^\varepsilon}(r_t D(\widetilde{X}_s^{\varepsilon,t}) + l_t Q(\widetilde{u}_s^{\varepsilon,t}))ds + \sum_{0<\widetilde{\tau}_j^{\varepsilon,t}\le T^\varepsilon}(k_t F(\widetilde{\xi}_j^\varepsilon) + h_t P(\widetilde{\xi}_j^\varepsilon))\Big).}$$

Lower Bounds and the Time-average Control of BM (TACBM)

In summary, we expect to have

$$\varepsilon^{-\beta\zeta_D} J^\varepsilon(u^\varepsilon, \tau^\varepsilon, \xi^\varepsilon) \simeq \int_0^T I_t^\varepsilon dt \gtrsim \int_0^T I(a_t, r_t, l_t, k_t, h_t)dt,$$

where $I = I(a, r, l, k, h)$ is the optimal cost of time-average control of BM :

$$I = \inf_{(u,\tau,\xi)} \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\Big[\int_0^T \big(rD(X_s) + lQ(u_s)\big)ds + \sum_{0<\tau_j\leq S} \big(kF(\xi_j) + hP(\xi_j)\big)\Big],$$

with

$$dX_s = \sqrt{a}dW_s + u_s ds + d\Big(\sum_{0<\tau_j\leq s} \xi_j\Big).$$

Outline

Theorem: Combined Stochastic and Impulse Controls

Consider

$$J^\varepsilon(u^\varepsilon,\tau^\varepsilon,\xi^\varepsilon) = \int_0^T (r_t D(X_t^\varepsilon) + \varepsilon^{\beta_Q} l_t Q(u_t^\varepsilon))dt$$
$$+ \sum_{j:0<\tau_j^\varepsilon\leq T} (\varepsilon^{\beta_F} k_{\tau_j^\varepsilon} F(\xi_j^\varepsilon) + \varepsilon^{\beta_P} h_{\tau_j^\varepsilon} P(\xi_j^\varepsilon)),$$

$$X_t^\varepsilon = -X_t^\circ + \int_0^t u_s^\varepsilon ds + \sum_{j:0<\tau_j^\varepsilon\leq t} \xi_j^\varepsilon.$$

Under mild conditions, we have for any sequence $(u^\varepsilon,\tau^\varepsilon,\xi^\varepsilon)\in\mathcal{A}^\varepsilon$,

$$\liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^{\beta\zeta_D}} J^\varepsilon(u^\varepsilon,\tau^\varepsilon,\xi^\varepsilon) \geq_p \int_0^T I(a_t,r_t,l_t,k_t,h_t)dt.$$

See below for an exact definition of I.

Theorem: Singular Control Only

Consider

$$J^\varepsilon(u^\varepsilon,\gamma^\varepsilon,\varphi^\varepsilon)=\int_0^T r_t D(X_t^\varepsilon)dt+\int_0^T \varepsilon^{\beta_P} h_t P(\gamma_{t-}^\varepsilon)d\varphi_t^\varepsilon,$$

$$X_t^\varepsilon=-X_t^\circ+\int_0^t \gamma_{s-}^\varepsilon d\varphi_s^\varepsilon.$$

Under mild conditions, we have for any sequence $(\gamma^\varepsilon,\varphi^\varepsilon)\in\mathcal{A}^\varepsilon$,

$$\liminf_{\varepsilon\to 0}\frac{1}{\varepsilon^{\beta\zeta_D}}J^\varepsilon(\gamma^\varepsilon,\varphi^\varepsilon)\geq_p\int_0^T I(a_t,r_t,h_t)dt.$$

Here, $I=I(a,r,h)$ can be related to

$$I=\inf_{(u,\gamma,\varphi)}\overline{\lim_{T\to\infty}}\frac{1}{T}\mathbb{E}\Big[\int_0^T rD(X_s)ds+\int_0^T hP(\gamma_s)d\varphi_s\Big],\quad dX_s=\sqrt{a}dW_s+\gamma_s d\varphi_s.$$

Outline

Explicit expressions

Explicit expressions in dimension one

We obtain explicit expressions (several of them already known) for the lower bounds and optimal controls in the local and global cases in the following situations in dimension one:

- Stochastic control.
- Impulse control.
- Singular control.
- Combined Stochastic and Impulse controls.
- Combined Stochastic and Singular controls.

Explicit expressions: Combined Stochastic and Impulse controls

Local problem

$$\overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\Big[\int_0^T (rX_t^2 + lu_t^2)dt + \sum_{0<\tau_j\le T}(k + h|\xi_j|)\Big]$$

- Optimal cost: $\iota(a^2 rl)^{1/2}$, $\iota(a, r, l, k, h) \in (0, 1)$.
- $u^*(x) = -\frac{1}{2l}w'(x; a, r, l, k, h)$.
- Impulse part: hitting times of domain $[-x^*, x^*]$ with $x^* = x^*(a, r, l, k, h)$.
- $\xi^*(\pm x^*) = \pm\widetilde{x}^*(a, r, l, k, h)$.
- Optimally controlled process:

$$dX_t^* = \sqrt{a}dW_t - \frac{w'(X_t^*)}{2l}dt + d\Big(\sum_{\tau_j\le t}\big(1_{X^*_{\tau_j}=-x^*}\xi^* - 1_{X^*_{\tau_j}=x^*}\xi^*\big)\Big).$$

Explicit expressions: Combined Stochastic and Impulse controls

Global problem

$$\int_0^T (r_t X_t^2 + \varepsilon^{\beta_Q} l u_t^2)dt + \sum_{0<\tau_j\leq T} (\varepsilon^{\beta_F} k_{\tau_j} + \varepsilon^{\beta_P} h_{\tau_j}|\xi_j|).$$

- Optimal cost: $\int_0^T \iota(a_t, r_t, l_t, k_t, h_t)(a_t^2 r_t l_t)^{1/2} dt$.
- $u_t^*(x) = -\frac{1}{2l_t} w'(x; a_t, r_t, l_t, k_t, h_t)$,
- Impulse part: hitting times of domain $[-x_t^*, x_t^*]$ with $x_t^* = x^*(a_t, r_t, l_t, k_t, h_t)$.
- $\xi_t^*(\pm x_t^*) = \pm\widetilde{x}^*(a_t, r_t, l_t, k_t, h_t)$.

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- Denote the frictionless optimal wealth/strategy by (w_t^*) and (φ_t^*), and

$$dS_t = b_t^S dt + \sqrt{a_t^S} dW_t.$$

- The indirect risk tolerance process is defined by

$$R_t = -u'(t, w_t^*)/u''(t, w_t^*).$$

- Denote the dual martingale measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \frac{u'(T, w_T^*)}{u'(0, w_0^*)}.$$

- In a market with proportional costs, the portfolio dynamics is given by

$$w_s^{t,w_t,\varepsilon} = w_t^\varepsilon + \int_t^s \varphi_u^\varepsilon dS_u - \int_t^s \varepsilon^{\beta_P} h_u d\|\varphi^\varepsilon\|_u,$$

The problem of utility maximization is given by

$$u^\varepsilon(t, w_t) = \sup_{\varphi^\varepsilon} \mathbb{E}[U(w_T^{t,w_t,\varepsilon})].$$

- As $\varepsilon \to 0$, we expect that φ^ε_t is close to φ^*_t and obtain heuristically

$$u^\varepsilon(0,w) - u(0,w) \simeq -u'(w_0)\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T \frac{a^S_t}{2R_t}(\varphi^\varepsilon_t - \varphi^*_t)^2 dt + \varepsilon^{\beta_P}\int_0^T h_t d\|\varphi^\varepsilon\|_t\Big].$$

- In general, the problem of utility maximization with small market frictions can be formally approximated by the problem of tracking if we take

$$\boxed{r_t D(x) = \frac{1}{2R_t} x^T a^S_t x.}$$

- It follows that

$$\boxed{\frac{1}{\varepsilon^{\beta\zeta_D}}\big(u^\varepsilon(0,w) - u(0,w)\big) \simeq -u'(w_0)\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T I_t dt\Big],}$$

cf. Soner and Touzi 2012, Kallsen and Muhle-Karbe 2013.

Outline

Occupation Measures

Key quantities

Define

$$\mu_t^\varepsilon = \frac{1}{T^\varepsilon}\int_0^{T^\varepsilon} \delta_{\{(\widetilde{X}_s^{\varepsilon,t},\widetilde{u}_s^{\varepsilon,t})\}} ds,$$

$$\rho_t^\varepsilon = \frac{1}{T^\varepsilon}\sum_{0<\widetilde{\tau}_j^{\varepsilon,t}\leq T^\varepsilon} \delta_{\{(\widetilde{X}^{\varepsilon,t}_{\widetilde{\tau}_j^{\varepsilon,t}-},\widetilde{\xi}_j^\varepsilon)\}}.$$

Cost Functional

On the one hand,

$$I_t^\varepsilon = \frac{1}{T^\varepsilon}\Big(\int_0^{T^\varepsilon}(r_t D(\widetilde{X}_s^{\varepsilon,t}) + l_t Q(\widetilde{u}_s^{\varepsilon,t}))ds + \sum_{0<\widetilde{\tau}_j^{\varepsilon,t}\leq T^\varepsilon}(k_t F(\widetilde{\xi}_j^\varepsilon) + h_t P(\widetilde{\xi}_j^\varepsilon))\Big)$$

can be written as

$$\boxed{I_t^\varepsilon = \int C_t^A(x,u)d\mu_t^\varepsilon(dx,du) + \int C_t^B(x,\xi)d\rho_t^\varepsilon(dx,d\xi),}$$

where

$$C_t^A(x,u) = r_t D(x) + l_t Q(u), \quad C_t^B(x,\xi) = k_t F(\xi) + h_t P(\xi).$$

Linear Constraint

On the other hand, by Ito's formula,

$$
\begin{aligned}
f(\widetilde{X}^{\varepsilon,t}_{T^\varepsilon}) - f(\widetilde{X}^{\varepsilon,t}_{0+}) &= \int_0^{T^\varepsilon} f'(\widetilde{X}^{\varepsilon,t}_s)\sqrt{\widetilde{a}^{\varepsilon,t}_s}\, d\widetilde{W}^{\varepsilon,t}_s \\
&+ \int_0^{T^\varepsilon} \frac{1}{2}\sum_{ij} \widetilde{a}^{\varepsilon,t}_{ij,s}\partial^2_{ij} f(\widetilde{X}^{\varepsilon,t}_s)ds + \int_0^{T^\varepsilon} \sum_i \widetilde{u}^{\varepsilon,t}_{i,s}\partial_i f(\widetilde{X}^{\varepsilon,t}_s)ds \\
&+ \sum_{0<\widetilde{\tau}^{\varepsilon,t}_j \le T^\varepsilon} \left(f(\widetilde{X}^{\varepsilon,t}_{\widetilde{\tau}^{\varepsilon,t}_j-} + \widetilde{\xi}^{\varepsilon}_j) - f(\widetilde{X}^{\varepsilon,t}_{\widetilde{\tau}^{\varepsilon,t}_j-})\right).
\end{aligned}
$$

Linear Constraint

Hence,

$$\frac{1}{T^\varepsilon}\Big(\int_0^{T^\varepsilon}\frac{1}{2}\sum_{ij}\widetilde{a}^{\varepsilon,t}_{ij,s}\partial^2_{ij}f(\widetilde{X}^{\varepsilon,t}_s)ds+\int_0^{T^\varepsilon}\sum_i\widetilde{u}^{\varepsilon,t}_{i,s}\partial_i f(\widetilde{X}^{\varepsilon,t}_s)ds+\sum_{0<\widetilde{\tau}^{\varepsilon,t}_j\le T^\varepsilon}\big(f(\widetilde{X}^{\varepsilon,t}_{\widetilde{\tau}^{\varepsilon,t}_j-}+\widetilde{\xi}^\varepsilon_j)-f(\widetilde{X}^{\varepsilon,t}_{\widetilde{\tau}^{\varepsilon,t}_j-})\big)\Big)$$

$$=\frac{1}{T^\varepsilon}\Big(f(\widetilde{X}^{\varepsilon,t}_{T^\varepsilon})-f(\widetilde{X}^{\varepsilon,t}_{0+})-\int_0^{T^\varepsilon}f'(\widetilde{X}^{\varepsilon,t}_s)\sqrt{\widetilde{a}^{\varepsilon,t}_s}\,d\widetilde{W}^{\varepsilon,t}_s\Big).$$

We deduce that

$$\boxed{\int A^{a_t}f(x,u)d\mu^\varepsilon_t(x,u)+\int Bf(x,\xi)d\rho^\varepsilon_t(x,\xi)\simeq 0,\quad \forall f\in C^2_0(\mathbb{R}^d),}$$

where

$$A^a f(x,u)=\frac{1}{2}\sum_{i,j}a_{ij}\partial^2_{ij}f(x)+\langle u,\nabla f(x)\rangle,\qquad Bf(x,\xi)=f(x+\xi)-f(x).$$

LP Characterization for Lower Bounds

Theorem: LP version of the lower bound

The lower bound is given by

$$I^P = \inf_{(\mu,\rho)} \int_{\mathbb{R}^d_x \times \mathbb{R}^d_u} C^A(x,u)\mu(dx \times du) + \int_{\mathbb{R}^d_x \times \mathbb{R}^d_\xi \setminus \{0_\xi\}} C^B(x,\xi)\rho(dx \times d\xi),$$

with $(\mu, \rho) \in \mathcal{P}(\mathbb{R}^d_x \times \mathbb{R}^d_u) \times \mathcal{M}(\mathbb{R}^d_x \times \mathbb{R}^d_\xi \setminus \{0_\xi\})$ verifying

$$\int_{\mathbb{R}^d_x \times \mathbb{R}^d_u} A^a f(x,u)\mu(dx \times du) + \int_{\mathbb{R}^d_x \times \mathbb{R}^d_\xi \setminus \{0_\xi\}} Bf(x,\xi)\rho(dx \times d\xi) = 0, \ \ \forall f \in C^2_0(\mathbb{R}^d_x).$$

- For the previous examples in dimension one, I^P is equal to

$$I = \inf_{(u,\tau,\xi)} \overline{\lim_{T\to\infty}} \frac{1}{T}\mathbb{E}\Big[\int_0^T \big(rD(X_s) + lQ(u_s)\big)\,ds + \sum_{0<\tau_j\leq T} \big(kF(\xi_j) + hP(\xi_j)\big)\Big],$$

with

$$dX_s = \sqrt{a}dW_s + u_s ds + d\Big(\sum_{0<\tau_j\leq s} \xi_j\Big).$$

- But a relaxed version of controlled BM is needed for general case.

Outline

Time-average control of BM via Martingale Problem

Definition (Kurtz and Stockbridge 1998, 2001)

A triplet (X, Λ, Γ) with (X, Λ) an $\mathbb{R}^d_x \times \mathcal{P}(\mathbb{R}^d_u)$-valued process and Γ an $\mathcal{L}(\mathbb{R}^d_x \times \mathbb{R}^d_\xi)$-valued random variable is a solution of the controlled martingale problem for (A^a, B) with initial distribution $\nu_0 \in \mathcal{P}(\mathbb{R}^d_x)$ if there exists a filtration $(\mathcal{F}_t)$ such that the process (X, Λ, Γ_t) is $\mathcal{F}_t$-progressive, X_0 has distribution ν_0 and for every $f \in C^2_0(\mathbb{R}^d_x)$,

$$f(X_t) - \int_0^t \int_{\mathbb{R}^d_u} A^a f(X_s, u)\Lambda_s(du)ds - \int_{\mathbb{R}^d_x \times \mathbb{R}^d_\xi \times [0,t]} Bf(x, \xi)\Gamma(dx, d\xi, ds)$$

is an $\mathcal{F}_t$-martingale.

Definition (MP formulation of time-average control problem)

The time-average control problem under the martingale formulation is given by

$$I^M = \inf_{(X,\Lambda,\Gamma)} \limsup_{t\to\infty} \frac{1}{t}\mathbb{E}\Big[\int_0^t \int_{\mathbb{R}^d_u} C_A(X_s, u)\Lambda_s(du)ds + \int_{\mathbb{R}^d_x\times\mathbb{R}^d_\xi\times[0,t]} C_B(x,\xi)\Gamma(dx, d\xi, ds)\Big].$$

over all solutions of the martingale problem (A^a, B) with any initial distribution $\nu_0 \in \mathcal{P}(\mathbb{R}^d_x)$.

Theorem: Equivalence between I^P and I^M

We have $I^M = I^P$, if the following conditions holds.

1. A and B satisfy Condition 1.2 in Kurtz/Stockbridge 2001. In particular,

$$|Af(x,u)| \leq a_f \psi_A(x,u), \quad |Bf(x,\xi)| \leq b_f \psi_B(x,\xi).$$

2. C_A is non-negative and inf-compact.
3. C_B is non-negative and lower semi-continuous, and

$$\inf_{(x,\xi)\in\mathbb{R}^d_x\times\mathbb{R}^d_\xi} C_B(x,\xi) > 0.$$

4. There exist constants θ and $0 < \beta < 1$ such that

$$\psi_A(x,u)^{1/\beta} \leq \theta(1 + C_A(x,u)), \quad \psi_B(x,\xi)^{1/\beta} \leq \theta C_B(x,\xi).$$