Semi-Static Completeness and Model-independent Pricing by Informed Investors

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LSE

joint work with Martin Larsson

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Outline

Model-independent framework

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  - $\mathcal{X}$: path-space, S: canonical process on $\mathcal{X}$
  - Ψ: set of claims ψ available for buy-and-hold trading
  - $\mathcal{M}$: martingale measures consistent w/ the market price of ψ's
  - Φ: a given derivative, robust pricing: $\sup_{\mathbf{Q}\in\mathcal{M}} \mathbb{E}_{\mathbf{Q}}[\Phi]$

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$$\sup_{\mathbf{Q}\in\mathcal{M}} \mathbb{E}_{\mathbf{Q}}[\Phi] = \inf\left\{c \in \mathbb{R} : \begin{array}{l}\Phi \text{ can be hedged pathwise} \\ \text{starting with initial capital } c\end{array}\right\}$$

Beiglböck, H.-Labordère, Penkner ‘13; Galichon, H.-Labordère, Touzi ‘14; Acciaio, Beiglböck, Penkner, Schachermayer ‘13; Bouchard, Nutz ‘13; Dolinsky, Soner ‘14a,‘14b; Beiglböck, Cox, Huesmann ‘14; Biagini, Bouchard, Kardaras, Nutz ‘14; Beiglböck, Nutz, Touzi ‘15; Guo, Tan, Touzi ‘15; Hou, Obłój ‘15; Beiglböck, Cox, Huesmann, Perkowski, Prömel ‘15, Beiglböck, Nutz, Touzi ’15,...

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- **Note**: $\mathcal{M}$ clearly depends on the underlying filtration, as does the set of available trading strategies.

- **Question**: What can be said about the relation between the super-hedging price and the choice of filtration? In particular, when passing from $\mathbb{F}$ to $\mathbb{G} \supseteq \mathbb{F}$?

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- **Question**: Which measures in $\mathcal{M}(\mathbb{F})$ are still relevant for pricing for the informed agent?

Setup

- $(\Omega, \mathbb{F}, \mathcal{F})$: Filtered measurable space with $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ right-continuous.

  $\hookrightarrow$ Later we will consider other filtrations.

- $S = (S_t)_{0 \le t \le T}$: càdlàg $\mathbb{F}$-adapted discounted price process of an asset available for dynamic trading. We assume $S_0 = 0$. (Everything works the same for multiple assets.)

- A risk-free asset with price $\equiv 1$ available for dynamic trading.

- $\Psi = \{\psi_1, \ldots, \psi_n\}$ a set of $\mathcal{F}_T$-measurable payoffs available for buy-and-hold trading. Today's price of ψ_i is zero for each i.

Martingale measures

Calibrated martingale measures:

$$\mathcal{M}(\mathbb{F}) = \left\{ \boldsymbol{Q} \in \mathcal{P} : \begin{array}{l} S \text{ is an } \mathbb{F}\text{-martingale}, \ \mathbb{E}_{\boldsymbol{Q}}[S_T^2] < \infty, \\ \mathbb{E}_{\boldsymbol{Q}}[\psi \mid \mathcal{F}_0] = 0, \ \mathbb{E}_{\boldsymbol{Q}}[\psi^2] < \infty \text{ for all } \psi \in \Psi \end{array} \right\}$$

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  $\hookrightarrow$ Can we reduce to the study of a special subset?

  $\hookrightarrow$ For example, if $\mathcal{P}$ is endowed with a topology s.t. $\mathcal{M}(\mathbb{F})$ is compact, then

$$\mathcal{M}(\mathbb{F}) = \overline{\mathrm{conv}(\mathrm{ext}\,\mathcal{M}(\mathbb{F}))},$$

  where $\mathrm{ext}\,\mathcal{M}(\mathbb{F})$ is the set of all extreme points in $\mathcal{M}(\mathbb{F})$.

Extreme points

- **Extreme points:** $\boldsymbol{Q} \in \mathcal{M}(\mathbb{F})$ is called an extreme point if

$$\begin{array}{l} \boldsymbol{Q} = \lambda \boldsymbol{Q}^1 + (1-\lambda)\boldsymbol{Q}^2 \\ \text{for } \boldsymbol{Q}^i \in \mathcal{M}(\mathbb{F}),\ \lambda \in (0,1) \end{array} \quad \Longrightarrow \quad \boldsymbol{Q}^1 = \boldsymbol{Q}^2 = \boldsymbol{Q}$$

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- **Note:** The notion of extreme point is purely algebraic, independent of any topology we may put on the space of probability measures.

Examples

Example (Discrete time and bounded prices)

- $\Omega = [a, b]^T$, S is the coordinate process,
- each $\omega \mapsto \psi_i(\omega)$ is continuous,
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Example (Continuous time and bounded volatility)

- $\Omega = C_0[0,T]$, S is the coordinate process,
- $\omega \mapsto \psi_i(\omega)$ bounded and continuous, $\mathbb{F}$ generated by S
- $\mathcal{P} = \left\{ \mathbf{Q} : \mathbb{E}_{\mathbf{Q}}\left[X \sup_{s \le u \le t} |S_u - S_s|^p \right] \le C_p \, \overline{\sigma}^p \, (t-s)^{p/2} \, \mathbb{E}_{\mathbf{Q}}[X] \right\}$, for all $0 \le s < t \le T$, $X \ge 0$ $\mathcal{F}_s$-measurable, $p \ge 1$.

Then $\mathcal{M}(\mathbb{F})$ is weakly compact.

Examples

Example (Jakubowski topology)

- $\Omega = D_0([0,T],[-1,1])$ with Jakubowski's S-topology,
- S is the coordinate process, ψ_i suitable continuity conditions,
- $\mathbb{F}$ is generated by S

Then $\mathcal{M}(\mathbb{F})$ is sequentially S-compact. Cf. Jakubowski (1997) and Guo, Tan, Touzi (2015).

Semi-static completeness and the Jacod-Yor theorem

The classical Jacod-Yor theorem

- Suppose $\Psi = \emptyset$ (no static claims).
- For $\boldsymbol{Q} \in \mathcal{M}(\mathbb{F})$, by the classical **Jacod-Yor (1977) theorem**:

$$\boldsymbol{Q} \in \operatorname{ext} \mathcal{M}(\mathbb{F}) \quad \Longleftrightarrow \quad \underbrace{L^2(\mathcal{F}_T) = \{x + (H \cdot S)_T : H \in L^2(S)\}}_{\text{classical completeness (in } L^2\text{)}}$$

- This result can be generalized to the semi-static case.

Generalization of the Jacod-Yor theorem

Definition

For $\boldsymbol{Q} \in \mathcal{M}(\mathbb{F})$, we say that **semi-static completeness** holds if any $X \in L^2(\mathcal{F}_T)$ can be represented as

$$X = x + a_1\psi_1 + \cdots + a_n\psi_n + (H \cdot S)_T$$

for some $x, a_1, \ldots, a_n \in \mathbb{R}$ and $H \in L^2(S)$.

Notation:

$$\boldsymbol{SSC}(\mathbb{F}) = \{\boldsymbol{Q} \in \mathcal{M}(\mathbb{F}) : \text{semi-static completeness holds}\}$$

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Theorem (semi-static Jacod-Yor theorem)

The extreme martingale measures are exactly the semi-statically complete models, i.e.

$$\text{ext}\,\mathcal{M}(\mathbb{F}) = \boldsymbol{SSC}(\mathbb{F}).$$

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About the proof.

- The proof is very close to the classical case . . .
- . . . but uses duality for random variables ($L^1 - L^\infty$) instead of processes ($\mathcal{H}^1 - BMO$):

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1. Fix $\boldsymbol{Q} \in \text{ext}\,\mathcal{M}(\mathbb{F})$ and show that this set is dense in $L^1(\mathcal{F}_T)$

$$\left\{x + \textstyle\sum_i a_i \psi_i + (H \cdot S)_T : \; x,\, a_i \in \mathbb{R},\; H \in L^2(S)\right\}.$$

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2. Prove it is dense and closed in $L^2(\mathcal{F}_T)$ using Hahn-Banach and a result by Yor (see also Delbaen/Schachermayer, 1999):

Theorem (Yor (1978))

Let $H^n \in L(S)$ be such that $H^n \cdot S$ is a martingale for each n, and suppose $\lim_n (H^n \cdot S)_T = X$ in L^1 for some r.v. X. Then there is $H \in L(S)$ such that $H \cdot S$ is a martingale with $(H \cdot S)_T = X$.

Generalization of the Jacod-Yor theorem

Remarks.

- Infinitely many ψ_i's would allow to treat the case of a fixed (by the market) marginal law $S_T \sim \mu$
- But the arguments we use in the above proof break down in this case – for the moment we are only able to deal with finitely many ψ_i's

Generalization of the Jacod-Yor theorem

Can we say more?

- Already in the classical case ($\Psi = \emptyset$), completeness is a strong property, but yet we do not have “control” on the complete models. For instance, completeness holds if $\mathbb{F} = \mathbb{F}^S$, and S is a strong solution to an SDE of the form

$$dS_t = \sigma(t; S_u : u \leq t)dW_t, \qquad (W_t)_t \ BM, \ \sigma > 0.$$

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Notation: For any martingale N, denote
$$\mathcal{S}(N) = \left\{H \cdot N \colon H \in L^2(N)\right\}.$$

This is a closed subspace of $\mathcal{H}^2$ (stable subspace generated by N).

A curious consequence of semi-static completeness

- For simplicity let $\Psi = \{\psi\}$, and fix $\boldsymbol{Q} \in SSC(\mathbb{F})$
- Let $K \cdot S$ be the orthogonal projection of $\mathbb{E}_{\boldsymbol{Q}}[\psi \mid \mathcal{F}_t]$ onto $\mathcal{S}(S)$, and define

$$M_t = \mathbb{E}_{\boldsymbol{Q}}[\psi \mid \mathcal{F}_t] - (K \cdot S)_t$$

Note: M_T is the part of ψ which is not replicable by trading on S

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- Consequently,

$$\mathcal{S}(M) = \text{span}\{M\},$$

which is **one-dimensional!**

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We will use the following result on ψ:

Lemma

Let N be a square-integrable martingale null at zero. The following are equivalent:

(i) $\mathcal{S}(N) = \text{span}\{N\}$

(ii) $N = N_T \mathbf{1}_{B\times[t^*,T]}$ *for some* $t^* \in (0, T]$ *and some atom B of* $\mathcal{F}_{t^*-}$

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And the following one on S, when S **is continuous**:

Lemma

Let N be a continuous local martingale, and let B be an atom of $\mathcal{F}_{t^*-}$ *for some* $t^* \in (0, T]$*. Then* $N_t = N_0$ *on B for all* $t < t^*$*.*

A curious consequence of semi-static completeness

Recall: $\Psi = \{\psi\}$, $\boldsymbol{Q} \in SSC(\mathbb{F})$. Now, for S continuous we have

$$M = M_T \mathbf{1}_{B\times[t^*,T]} \qquad \text{and} \qquad S_t = S_0 \text{ on } B \text{ for } t \leq t^*$$

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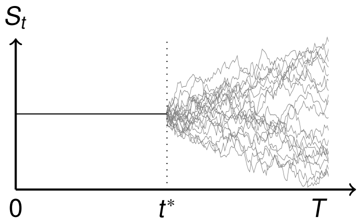
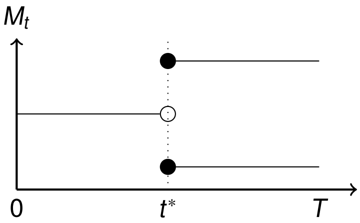




Semi-static completeness and filtration structure

Atomic tree

- Fix $\boldsymbol{Q} \in \mathcal{M}(\mathbb{F})$
- For $A \in \mathcal{F}_T$, denote by $t(A)$ the first time A becomes measurable,
$$t(A) = \inf\{t \in [0, T] : A \in \mathcal{F}_t\}.$$

Definition

An **atomic tree** is a finite collection $\boldsymbol{T}$ of events in $\mathcal{F}_T$ s.t.:

(i) every $A \in \boldsymbol{T}$ is a non-null atom of $\mathcal{F}_{t(A)}$;

(ii) $\forall\, A, A' \in \boldsymbol{T}$ s.t. $t(A) < t(A')$, either $A \supseteq A'$ or $A \cap A' = \emptyset$;

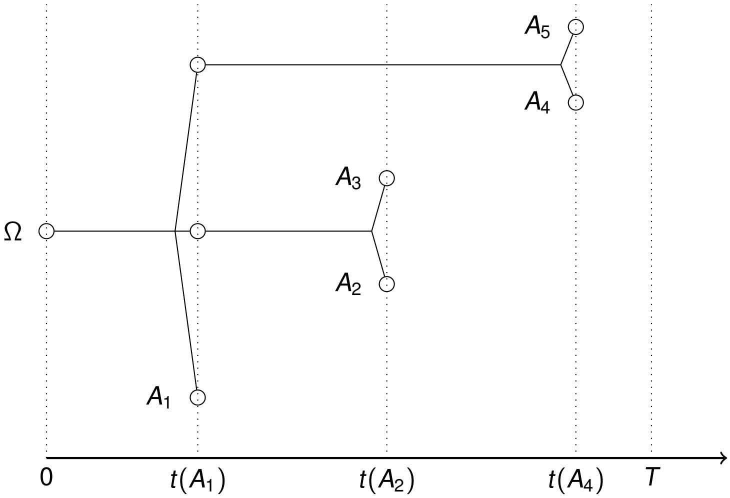
(iii) $\forall\, A, A' \in \boldsymbol{T}$ such that $A \supsetneq A'$, $\boldsymbol{Q}(A \setminus A') > 0$;

(iv) the leaves form a partition of Ω (up to nullsets), and A is an atom of $\mathcal{F}_{t(A')-}$ whenever A' is a child of A.

leaf: $A \in \boldsymbol{T}$ s.t. there is no $A' \in \boldsymbol{T}$ s.t. $A' \subsetneq A$; **dim T:** $\#$ leaves

child: A' is a child of A if $A, A' \in \boldsymbol{T}$ satisfy $A' \subsetneq A$ and there is no $A'' \in \boldsymbol{T}$ such that $A' \subsetneq A'' \subsetneq A$

Atomic tree



Atomic tree

Remarks.

- $\sigma(\boldsymbol{T})$ is well-defined. It can be described as $\sigma(\boldsymbol{T}) = \mathcal{F}_{\zeta(\boldsymbol{T})}$, where the stopping time $\zeta(\boldsymbol{T})$ is the "end" of the tree:

$$\zeta(\boldsymbol{T}) = \sum_{A \in \boldsymbol{T} \text{ is a leaf}} t(A) \mathbf{1}_A .$$

- Note that $\dim \boldsymbol{T} = \dim L^2(\sigma(\boldsymbol{T}))$.

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Definition

We say that S is **complete on** $A \times [t, T]$ for given $t \in [0, T]$ and $A \in \mathcal{F}_t$ if any $X \in L^2(\mathcal{F}_T)$ can be dynamically replicated there:

$$X = x + (H \cdot S)_T \quad \text{on} \quad A$$

for some $x \in \mathbb{R}$ and some $H \in L^2(S)$ with $H = 0$ on $[\![0, t]\!]$.

Semi-static completeness for continuous price processes

Recall: $\boldsymbol{Q} \in \mathcal{M}(\mathbb{F})$ is fixed.

Theorem

Let S be continuous. Then $\boldsymbol{Q} \in SSC(\mathbb{F})$ IFF $\exists$ an **atomic tree** $\boldsymbol{T}$ *s.t.*

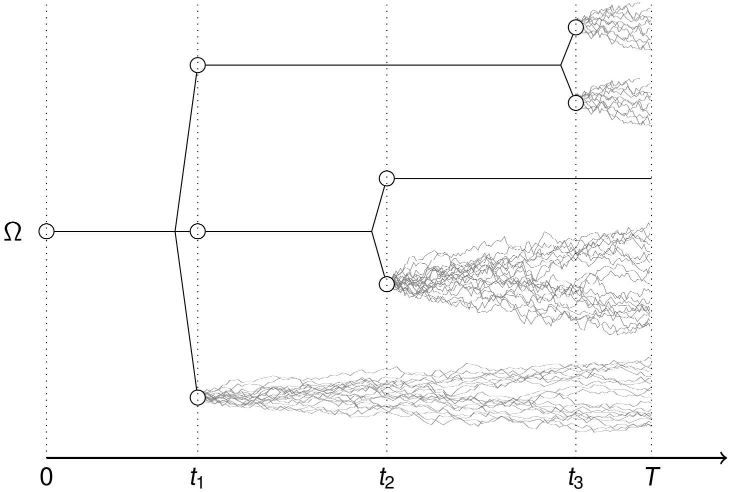
1. $\left\{ \mathbb{E}_{\boldsymbol{Q}}[\psi_i \mid \sigma(\boldsymbol{T})] : i = 1, ..., n \right\}$ *has* $\dim \boldsymbol{T} - 1$ *lin. indep. elements,*
2. S *is* **complete** *on* $A \times [t(A), T]$ *for each leaf* $A \in \boldsymbol{T}$*.*

In this case, S *is constant on* $[\![0, \zeta(\boldsymbol{T})]\!]$ *and*

$$L^2(\mathcal{F}_T) = \operatorname{span}\{1, \Psi\} + \mathcal{S}(S) = L^2(\sigma(\boldsymbol{T})) \oplus \mathcal{S}(S).$$

Remark: $\psi_i = \mathbb{E}_{\boldsymbol{Q}}[\psi_i \mid \sigma(\boldsymbol{T})] + \underbrace{(H^i \cdot S)_T}_{\text{orthog. proj.}}, \quad i = 1, \ldots, n\,.$

Semi-static completeness for continuous price processes



The filtration $\mathbb{F}$ under $\boldsymbol{Q} \in SSC(\mathbb{F})$. Each set of lines emanating from the leaves of $\boldsymbol{T}$ corresponds to a dynamically complete stock price model.

Semi-static completeness for continuous price processes

Example (Semi-statically complete continuous model)

One static claim $\psi = \langle S, S\rangle_T - K$ with zero value at $t = 0$.

- Pick $t^* \in (0, T)$, $\sigma_1, \sigma_2 > 0$ with $\sigma_1 \neq \sigma_2$.
- Set $\boldsymbol{Q} = \lambda \boldsymbol{Q}^1 + (1-\lambda)\boldsymbol{Q}^2$ where

$$S_t = \sigma_i W_{t-t^*} \mathbf{1}_{\{t \geq t^*\}} \quad \text{under} \quad \boldsymbol{Q}^i,$$

  where W is Brownian motion, and λ is determined by calibration:

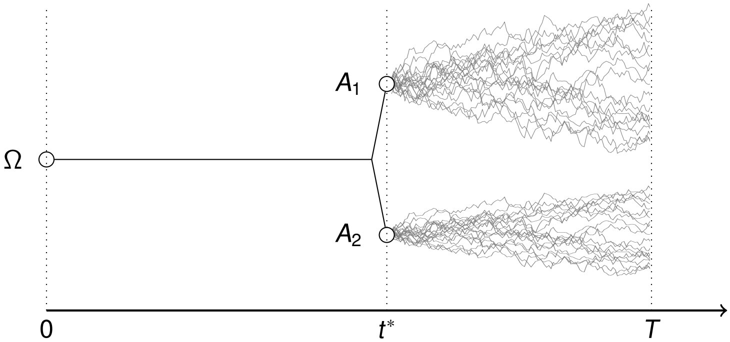
$$0 = \mathbb{E}_{\boldsymbol{Q}}[\psi \mid \mathcal{F}_0] = \lambda\sigma_1^2(T - t^*) + (1-\lambda)\sigma_2^2(T - t^*) - K.$$

- Define $A_i = \{\partial^+ \langle S, S\rangle_{t^*} = \sigma_i^2\}$ and set $\boldsymbol{T} = \{\Omega, A_1, A_2\}$.
- $\boldsymbol{T}$ is an atomic tree with dim $\boldsymbol{T} = 2$ and

$$\mathbb{E}_{\boldsymbol{Q}}[\psi \mid \sigma(\boldsymbol{T})] = \sigma_1^2(T - t^*)\mathbf{1}_{A_1} + \sigma_2^2(T - t^*)\mathbf{1}_{A_2} - K \not\equiv 0.$$

- By the theorem, $\boldsymbol{Q} \in SSC(\mathbb{F})$.

Semi-static completeness for continuous price processes



The leaves A_1, A_2 correspond to Bachelier models with volatilities $\sigma_1 > \sigma_2$. Thus the "variance swap" $\psi = \langle S \rangle_T$ is priced differently under the two models, and can be used to hedge against A_1 or A_2.

Semi-static completeness for continuous price processes

Example (Semi-statically complete jump model, but no atomic tree)

- $\psi = [S,S]_T - K$
- $S_t = \begin{cases} -t & t < \theta \wedge t^* \\ 1-\theta+f(\theta)W_{t-\theta} & t \geq \theta,\ \theta < t^* \\ -t^* + \mathbf{1}_{A_1}\sigma_1 W_{t-t^*} + \mathbf{1}_{A_2}\sigma_2 W_{t-t^*} & t \geq t^*,\ t^* \leq \theta \end{cases}$

with $\theta \sim \mathrm{Exp}(1)$, $W, t^*, \sigma_1, \sigma_2 > 0$ as above, $f(t) : [0, t^*) \to \mathbb{R}_+$.

Conclusion: When the asset is allowed to jump, we do not have anymore the tree structure.

Pricing by informed investors

Setup

- $\mathbb{G} = (\mathcal{G}_t)_{0 \leq t \leq T}$: right-continuous filtration (of the informed agent) with

$$\mathcal{F}_t \subseteq \mathcal{G}_t, \qquad 0 \leq t \leq T.$$

- Access to the same trading instruments: risk-free asset, S, Ψ

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Question: How are $SSC(\mathbb{G})$ and $SSC(\mathbb{F})$ related?

Progressive filtration enlargement

Specification of $\mathbb{G}$: Progressive enlargement of $\mathbb{F}$ with $\mathbb{H}$

$$\mathcal{G}_t = \bigcap_{u>t} \mathcal{F}_u \vee \mathcal{H}_u.$$

Smallest right-continuous filtration that contains both $\mathbb{F}$ and $\mathbb{H}$.

- $\mathbb{H}$ generated by a collection of single-jump processes $X\mathbf{1}_{[\![\tau,T]\!]}$, where X is a non-negative bounded random variable and τ is a random time (that is, $[0,T] \cup \{\infty\}$-valued random variable). (W.l.g., suppose $\tau = \infty$ on $\{X = 0\}$.)
- Remark: **special cases** are the classical progressive enlargement with a random time and initial enlargement with a random variable.

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- Remark: **special cases** are the classical progressive enlargement with a random time and initial enlargement with a random variable.
- For this kind of filtration enlargement there are clear-cut results between $SSC(\mathbb{G})$ and $SSC(\mathbb{F})$.

Progressive filtration enlargement

Let σ be the first time S starts to move: $\sigma = \inf\{t \in [0, T] : S_t \neq 0\}$.

Theorem

Let S be continuous and $\mathbb{H}$ generated by $X_k \mathbf{1}_{[\![\tau_k, T]\!]}$, $k = 1, \dots, p$. Assume $\tau_k > \sigma$ on $\{0 < \tau_k < \infty\}$ for all k. Then

$$\boldsymbol{SSC}(\mathbb{G}) = \{\, \boldsymbol{Q} \in \boldsymbol{SSC}(\mathbb{F}) \;:\; \mathbb{F} = \mathbb{G} \text{ under } \boldsymbol{Q} \,\}$$

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In the proof we use an extension of the classical Jeulin-Yor theorem.

- Fix $\boldsymbol{Q} \in SSC(\mathbb{G})$
- Let Z be the Azéma supermartingale: $Z_t = \boldsymbol{Q}(\tau > t \mid \mathcal{F}_t)$
- Let A be is the dual predictable projection of $X\mathbf{1}_{[\![\tau,\infty[\![}$

Theorem (Jeulin-Yor (1978))

The following process is a $\mathbb{G}$-martingale w.r.to $\boldsymbol{Q}$:

$$M_t = X\mathbf{1}_{\{\tau \leq t\}} - \int_0^{t\wedge\tau} \frac{1}{Z_{s-}} dA_s.$$

Progressive filtration enlargement

Sketch of the proof of "$\subseteq$" (for $p = 1$, $X \equiv 1$)

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- By semi-static completeness,

$$M = M_0 + V + H \cdot S, \qquad (2)$$

for some $H \in L(S)$ and martingale V with $V_T \in L^2(\sigma(\boldsymbol{T}))$

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- By assumption, $\tau > \sigma = \inf\{t > 0 : S_t \neq S_0\}$
- And V is constant on $]\!]\sigma, \infty[\![$ by our characterization Theorem
- Therefore $\tau = \inf\left\{t \in [0,T] : \frac{1}{Z_{t-}}\Delta A_t = 1\right\}$ $\mathbb{F}$-stopping time.

Progressive filtration enlargement

Remarks.

- From the proof it is clear that the set equivalence still holds true without any assumption on S when $\Psi = \emptyset$.

Progressive filtration enlargement

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- From the proof it is clear that the set equivalence still holds true without any assumption on S when $\Psi = \emptyset$.
- We can generalize the theorem for filtration enlargements with countably many single-jump processes.

Theorem

Let S be continuous and $\mathbb{H}$ generated by $X_k \mathbf{1}_{[\![\tau_k, T]\!]}$, $k \in \mathbb{N}$. Assume $\tau_k > \sigma$ on $\{0 < \tau_k < \infty\}$ for all k, and $|\{k : \tau_k(\omega) \le T\}| < \infty \ \forall\ \omega$. Then

$$\boldsymbol{SSC}(\mathbb{G}) = \{\, \boldsymbol{Q} \in \boldsymbol{SSC}(\mathbb{F}) \ : \ \mathbb{F} = \mathbb{G} \text{ under } \boldsymbol{Q} \,\}$$

Conclusions

- Motivated by robust super-hedging price computation, we study extreme calibrated martingale measures
- We obtain:
  - Semi-static version of the Jacod-Yor theorem.
  - Description of semi-statically complete models in terms of dynamically complete models glued together by means of an atomic tree.
  - Application to robust pricing by informed agents: under structural assumptions, informed agents price using only those models that render the additional information uninformative.
- Lots of things remain to be done and appear to be within reach:
  - Infinitely many static claims ($\rightarrow$ case $S_T \sim \mu$)
  - Better understanding of price processes with jumps
  - More general filtration enlargements
  - ...

Thank you for your attention!

@ Walter: have a great year in Zurich!