The filtering Problem:

$(S_t)_{t\in[0,1]}$ $\mathbb{R}^d$-valued semimartingale, locally bounded

$$S = A + M \qquad (S \text{ is special})$$

predictable, loc. martingale

Two filtrations:

$$\mathcal{F}_t = \overline{\sigma}(S_s,\ s \le t)$$

$$\mathcal{G}_t = \overline{\sigma}(A_s, M_s,\ s \le t)$$

for $t \in [0,1]$. $\overline{\sigma}(A_s, S_s, s\le t)$

S satisfies (NFLVR) w.r.t. $(\mathcal{F}_t)_{t \in [0,1]}$

$\iff \exists\, Q \sim P$ on $\mathcal{F}$, s.t. S Q-loc. mart.

No drift estimator with vanishing error

$\iff$ (NDEVE) w.r.t. $(\mathcal{F}_t)$, i.e.

there is no sequence of drift estimators

$(g^n \cdot S)$ bounded from below by $-\varepsilon_n \nearrow 0$

s.t. $(g^n \cdot S)_1 \xrightarrow[n\to\infty]{} X \gneq 0$ in probability

Remark: $(\varphi^{u} \cdot S)$ is called a $\boxed{\text{drift estimator}}$ since it is a predictably weighted sum of increments, i.e. $\varphi^{u} \cdot S \approx \sum_i \varphi^{u}_i \cdot \Delta S_i$, which reminds of $\frac{1}{N} \sum_{i=1}^{N} \Delta S_i$ being a standard drift estimator.

__Definition:__ A can be detected $\Leftrightarrow$

$\exists\ (D \in \mathcal{V}R) \Longleftrightarrow \exists\ (\varphi^n)$ predictable

$$(\varphi^n \cdot A) + (\varphi^n \cdot M) \geq -\varepsilon_n \quad \&$$

$$(\varphi^n \cdot A)_1 + (\varphi^n \cdot M)_1 \xrightarrow[\text{in prob}]{n\to\infty} X \not\geq 0$$

X is the arbitrage result of A.

__Question:__ Can we always achieve that

$$(\varphi^n \cdot M)_1 \xrightarrow[n\to\infty]{} 0 \quad ?$$

In the fundamental theorem of Statistics (FTS) it might be better to replace L^∞ by L^p, $1 \leq p < \infty$, and maybe also to relax admissibility, since a lower bound seems unnatural for an estimator of drifts.

What is filtering?

Calculate $E[f(A_t) \mid \mathcal{F}_t]$ w.r.t. P for test functions $f : \mathbb{R}^d \to \mathbb{R}$!

The martingale measure $Q \sim P$ might help if we can extend Q from $\mathcal{F}_1$ to $\mathcal{G}_1$.

Assumption: Assume there exists an extension $\overline{Q} \sim P$ to $\mathcal{G}_1$ and that $(A_s)_{s \le t}$ $\perp_{\overline{Q}}$ $(S_{s \le t})$.

$$E_P[f(A_t) \mid \mathcal{F}_t] = \frac{1}{E_{\overline{Q}}\left[\frac{dP}{d\overline{Q}} \mid \mathcal{F}_t\right]} E_{\overline{Q}}\left[\frac{dP}{d\overline{Q}} f(A_t) \mid \mathcal{F}_t\right].$$

(Kushner - Stratonovich equation, if we clear away the denominator we obtain the Zakai equation)

Of course S is a martingale w.r.t. to $\tilde{Q}$ on $\mathcal{G}$, so even on $\mathcal{G}$ we do not care a LDE of the form $\{(g^n \cdot S)\}_{n \geq 0}$

<u>Related theories</u>:
* enlargement of filtrations.
* insider trading

What can finance contribute?

(?) $\mathbb{E}_P[f(A_t) \mid \mathcal{F}_t]$ appears as mean variance hedge $\Rightarrow$ use more general criteria

u utility function;

search x, φ s.t.

$$E_P\left[u\left(f(A_t) - x - (\varphi \cdot S) \,\middle|\, \mathcal{F}_t \right) \right] \to \min$$

i.e. utility optimization with random endowment.

What does duality tell in filtering?

② What happens if we only assume (NUPBR) instead of (NFLVR)?

③ What does incompleteness mean?