Publications - Tom Ilmanen

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Some of these papers are available online at http://www.math.ethz.ch/∼ilmanen.

1. T. Ilmanen, "Generalized Flow of Sets by Mean Curvature on a Manifold," Indiana J. Math. 41 (1992) 671-705.

   The flow of level sets by mean curvature of Evans-Spruck and Chen-Giga-Goto is generalized to a manifold, with discussion of the noncompact case and a new example of instantaneous "fattening". The evolving function is shown not to be C^2 in general.

2. T. Ilmanen, "The Level-Set Flow on a Manifold," *Diff. Geom.: Part. Diff. Equ. on Manifolds (Los Angeles, CA, 1990)*, Proc. Symp. Pure Math. #54, Part 1, Amer. Math. Soc., Providence, 1993, 193–204.

   We give a direct geometric characterization of the level-set flow, which yields an alternate (geometric) construction of the level-set flow of Evans-Spruck and Chen-Giga-Goto. We exhibit a proof of the maximum principle for viscosity solutions (in this case) via interposition of a $C^{1,1}$ barrier.

3. T. Ilmanen, *Elliptic Regularization and Partial Regularity for Motion by Mean Curvature*, Memoirs of the Amer. Math. Soc. #520, March, 1994.

   Elliptic regularization is a new approximation scheme for mean curvature motion, in which we minimize an increasingly degenerate elliptic parametric functional in space-time. It yields a new construction of Brakke's varifolds moving by mean curvature, linked to the motion of the reduced boundary of an open set. The relation to the level-set flow is clarified (codimension one). If the corresponding level-set does not fatten up, then the flow is unique, has unit density, and satisfies Brakke's almost-everywhere regularity theory.

4. T. Ilmanen, "Convergence of the Allen-Cahn Equation to Brakke's Motion by Mean Curvature," J. Diff. Geom 38 (1993) 417-461.

   We prove that the Allen-Cahn equation converges to Brakke's mean curvature flow of varifolds. In the process we develop a new method in weak convergence, combining the monotonicity formula with an equipartition of energy principle. For almost every time, the limit interface is supported on a closed, rectifiable set of finite $(n-1)$-dimensional Hausdorff measure. Off this set, the phase-field function converges to plus or minus one. In addition, the regularity theory of Brakke applies, provided the corresponding level-set motion does not fatten up.

5. T. Ilmanen, "Convergence to Brakke's Motion by Mean Curvature," Proc. of the Computational Crystal Growers Conf., J. Taylor, ed., Amer. Math. Soc., 1992.

   Announcement.

6. S. B. Angenent, D. Chopp, and T. Ilmanen, "A Computed Example of Nonuniqueness of Mean Curvature Flow in $\mathbf{R}^3$," Comm. Part. Diff. Eq. 20 (1995) 1937-1958.

Let C be a double cone in $\mathbf{R}^n$, that is, the union of some cone in $\{x_n > 0\} \cup \{0\}$ with its reflection across $x_n = 0$. We prove that if C has sufficiently large aperture, then C gives rise to two distinct self-expanding mean curvature evolutions, one of which one is connected, the other not. A computation shows that in $\mathbf{R}^3$, the critical aperture is roughly $66.04°$.
Using a numerical algorithm of Chopp, in $\mathbf{R}^3$ we construct a family of smooth, noncompact, genus three surfaces $\{M_t\}_{t<0}$ that shrink self-similarly by mean curvature, such that as $t \to 0$, M_t converges to a double cone of the above type, which then evolves nonuniquely for $T > 0$. This resolves the question of nonuniqueness, or "fattening" for motion by mean curvature, posed by DeGiorgi and by Evans-Spruck.

7. T. Ilmanen, "A Strong Maximum Principle for Singular Minimal Hypersurfaces," Calc. Var. 4 (1996) 443-467.

   If M is a stable minimal hypersurface of multiplicity one, then all the tangent cones of M have multiplicity one. The proof is a modification of a corresponding proof of L. Simon for area-minimizing hypersurfaces.
   Corollary: if V_1, V_2 are stationary integral n-varifolds in $\mathbf{R}^{n+1}$ such that $\mathcal{H}^{n-2}\,\mathrm{spt}\,V_1 \cap \mathrm{spt}\,V_2 = 0$, then actually $\mathrm{spt}\,V_1 \cap \mathrm{spt}\,V_2 = \emptyset$. Furthermore, if the supports are disjoint but approach closely in one place, and are each connected, then they approach closely everywhere (in the sense of Hausdorff distance). These results are proven by interposing stable minimal hypersurfaces.

8. G. Huisken and T. Ilmanen, "The Riemannian Penrose Inequality," Int. Math. Res. Not. 20 (1997) 1045-1058.

   Announcement.

9. G. Huisken and T. Ilmanen, "A Note on the Inverse Mean Curvature Flow," Proceedings of Workshop at Saitama University (Sept. 1997), March, 1998.

   We summarize the analytic results concerning the weak formulation of the inverse mean curvature flow. In the second section, we present a family of rotationally symmetric "eternal" solutions starting from a half-line.

10. T. Ilmanen, P. Sternberg, and W. Ziemer, "Equilibria of the Level-Set Flow," J. Geom. Analysis (1998) 845-858.

    If u is a Lipschitz viscosity solution of the Evans-Spruck level-set mean curvature flow equation on a mean convex domain, then $u(\cdot,t)$ converges to a unique static solution v as $t \to \infty$. Any level-set of v with locally finite Hausdorff $(n-1)$-measure is a stable minimal surface with singularities of codimension at least 7.

11. T. Ilmanen, "Die Penrose-Ungleichung : Differentialgeometrie und Schwarze Loecher im Kosmos," Max-Planck Gesellschaft Jahrbuch, translated by W. Tuschmann, 1998. (English version: The Penrose Inequality : Differential Geometry and Black Holes.)

    For the scientific layman, a brief description of geometry, relativity, black holes and the Penrose inequality as a research highlight at the Max-Planck Institute for Mathematics in the Natural Sciences, Leipzig, focusing on the work of Huisken-Ilmanen.

12. G. Huisken and T. Ilmanen, "The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality," J. Diff. Geom. 59 (2001) 353-437.

    The Penrose conjecture of general relativity, in its purely Riemannian case, states the following: in an asymptotically flat 3-manifold of nonnegative scalar curvature, the ADM mass bounds the area of each outermost minimal surface. Using the inverse mean curvature flow put forward by Geroch and Jang-Wald, we succeed in proving this, even though the evolving surfaces become singular and jump around in the manifold. A corollary is the positive mass theorem of Schoen and Yau.

13. G. Huisken and T. Ilmanen, "Energy Inequalities for Isolated Systems and Hypersurfaces Moving by their Mean Curvature," *Proc. of the 16th Internat. Conf. on General Relativity (Durban, S. Africa, 2001),* World Scientific, 2002, 162–173.

    Conference proceedings.

14. T. Ilmanen and D. Knopf, "A Lower Bound for the Diameter of Solutions to the Ricci Flow with Nonzero $H^1(M^n, \mathbf{R})$," Math. Res. Letters 10 (2003) 161-168; also math.DG/0211230.

    We prove a diameter lower bound when $H^1(M^n, \mathbf{R}) \neq 0$. As a corollary, $S^1 \times S^2$ cannot occur as the blowup of a singularity of a compact 3-manifold evolving by the Ricci flow.

15. M. Feldman, T. Ilmanen and D. Knopf, "Rotationally Symmetric Shrinking and Expanding Gradient Kähler-Ricci Solitons," J. Diff. Geom. 65 (2003) 169-209.

    We construct three new families of Kähler-Ricci solitons. The first family is self-similarly expanding and lives on the holomorphic line bundle $L(\ell)$ over $\mathbf{CP}^{n-1}$, where $\ell < -n$. The second family is self-similarly shrinking and lives on $L(\ell)$, $-n < \ell < 0$. Both are asymptotic to a metric cone at infinity. The third family is self-similarly shrinking and lives on orbifold obtained by compactifying $L(\ell)$ be adding a point at infinity, $2 \leq \ell < k$. Putting the noncompact shrinking solution together with a self-similarly expanding solution found by H.-D. Cao, we obtain a singular solution of Ricci-Kähler in which a $\mathbf{CP}^{n-1}$ shrinks down to a point, disappears, and then the manifold smooths out again: a blowdown has occurred.

16. M. Feldman, L. Ni, and T. Ilmanen, "Entropy and Reduced Distance for Ricci Expanders," J. Geom. Analysis 15 (2005) no. 1 ... ; also math.DG/0405036.

    Perelman has discovered two integral quantities, the shrinker entropy $\mathcal{W}$ and the (backward) reduced volume, that are monotone under the Ricci flow $\partial g_{ij}/\partial t = -2R_{ij}$ and constant on shrinking solitons. Tweaking some signs, we find similar formulae corresponding to the expanding case. The *expanding entropy* $\mathcal{W}_+$ is monotone on any compact Ricci flow and constant precisely on expanders; as in Perelman, it follows from a differential inequality for a Harnack-like quantity for the conjugate heat equation, and leads to functionals μ_+ and ν_+. The *forward reduced volume* θ_+ is monotone in general and constant exactly on expanders.
    A natural conjecture asserts that $g(t)/t$ converges as $t \to \infty$ to a negative Einstein manifold in some weak sense (in particular ignoring collapsing parts). If the limit is known a-priori to be smooth and compact, this statement follows easily from any monotone quantity that

is constant on expanders; these include $\text{vol}(g)/t^{n/2}$ (Hamilton) and $\bar{\lambda}$ (Perelman), as well as our new quantities. In general, we show that if $\text{vol}(g)$ grows like $t^{n/2}$ (maximal volume growth) then $\mathcal{W}_+$, θ_+ and $\bar{\lambda}$ remain bounded (in their appropriate ways) for all time. We attempt a sharp formulation of the conjecture.

The following papers are available online at http://www.math.ethz.ch/~ilmanen:

17. T. Ilmanen, "Singularities of Mean Curvature Flow of Surfaces," preprint, 1995.

    Let $\{M_t\}_{0\leq t<T}$ be a family of 2-dimensional surfaces in R^n moving by mean curvature. Using a new local estimate of curvature based on the Gauss-Bonnet formula, we prove that there is a sequence $t_j \to T$ such that the rescaled surfaces $(T-t_j)^{-1/2} \cdot M_{t_j}$ converge to a "blowup" surface that is smooth except for a discrete set of singularities. If $n=3$ and M_t is embedded, then the blowup is smooth.

18. T. Ilmanen, *Lectures on Mean Curvature Flow and Related Equations,* lecture notes, ICTP, Trieste, 1995.

    Lecture 1 - Self-shrinking surfaces; Examples of self-shrinkers in $\mathbf{R}^3$; Nonuniqueness in $\mathbf{R}^3$.
    Lecture 2 - Level-set flow; Topological changes past a singularity; Evolution of cones in $\mathbf{R}^3$.
    Lecture 3 - Monotonicity formula; Blowup theorem in $\mathbf{R}^3$.
    Lecture 4 - Nonuniqueness in geometric heat flows

19. G. Huisken and T. Ilmanen, "Higher Regularity of the Inverse Mean Curvature Flow," submitted to JDG, 2002.

    Let N_t be a inverse mean curvature flow in $\mathbf{R}^n$. We prove
    (a) A weak Harnack inequality for the mean curvature H when N_t is uniformly star-shaped.
    (b) Estimate for the second fundamental form of N_t when H is bounded below by a positive constant.
    Consequences:
    (c) sharp criterion criterion for the formation of a singularity, namely H going to zero.
    (d) Existence starting from uniformly star-shaped, weakly mean-convex initial data.
    (e) Eventual smoothness of the weak inverse mean curvature flow starting from any data.

20. T. Ilmanen, "A Note on the Hamiltonian Area Conjecture," 1999.

    Let T be a 2-torus Hamiltonian isotopic to the Clifford torus in $\mathbf{C}^2$. If T is stationary for area under Hamiltonian perturbations, then the area of T is bounded below by that of the Clifford torus.

21. T. Ilmanen and R. Schätzle, "The Singular Set of Minimal Hypersurfaces with Second Fundamental Form in L^2," announcement, 2000.

    Announcement.

22. H.-D. Cao, R. Hamilton, and T. Ilmanen, "Gaussian Densities and Stability for some Ricci Solitons," math.DG/0404165, 2004.

In this announcement, we exhibit the second variation of Perelman's λ and ν functionals for the Ricci flow, and investigate the linear stability of examples. We also define the "central density" of a shrinking Ricci soliton and compute its values for certain examples in dimension 4. Using these tools, one can sometimes predict or limit the formation of singularities in the Ricci flow. In particular, we show that certain Einstein manifolds are unstable for the Ricci flow in the sense that generic perturbations acquire higher entropy and thus can never return near the original metric. We include a chart of some known 4-dimensional shrinking solitons ordered by their density value ν.

In preparation:

23. T. Ilmanen, Dynamics of Stationary Cones.

    Let C be a hypercone in $\mathbf{R}^{n+1}$ of locally finite $\mathcal{H}^n$-measure. Then the boundary of the level-set flow starting from C is self-similarly expanding by mean curvature, and is smooth except for singularities of codimension at least 7.
    Suppose C is a stationary hypercone. *Fattening:* if C does not minimize area, then C evolves nonuniquely by mean curvature (and at least one of the evolutions is a smooth radial graph). *Nonfattening:* if C minimizes area and $C \setminus \{0\}$ is smooth, then C has a unique evolution by mean curvature (equal to C for all time).

24. T. Ilmanen and B. White, "Optimal Lower Density Bounds for Area-Minimizing Cones."

    Let C be an area-minimizing hypercone with an isolated singularity at the vertex, and assume that the complement of C has a non-contractible component. Then the density of C is greater than or equal to the Gaussiam density b_{n-1} of a shrinking $(n-1)$-sphere. In particular, the density of C is greater than $\sqrt{2}$. The technique is to use mean curvature flows to connect various stationary hypercones and hypersurfaces, sweeping away potential competitors via the maximum principle.

25. S. B. Angenent, T. Ilmanen, and J. J. L. Velazquez, "Fattening from Smooth Initial Data in Mean Curvature Flow." [title to be changed...]

    Let $C(p,q)$ be the stationary hypercone in $\mathbf{R}^{p+q}$ that is rotationally invariant under $SO(p) \times SO(q)$.
    (1) The cone $C(6,2)$ is stable, and minimizing on exactly one side. Using the asymptotic pasting technique developed by Velazquez, we construct a mean curvature evolution near $C(6,2)$ with the same rotational symmetry. It starts out smooth, develops a point singularity, and then evolves nonuniquely.
    (2) For $4 \leq p+q \leq 7$, the cone $C(p,q)$ is unstable under small perturbations of area. We construct a mean curvature flow near $C(p,q)$ with the same symmetries, exactly one singular point $(0,0)$ in spacetime, and nonunique evolution past the singularity. It is smooth and self-similarly shrinking for $t<0$, a smooth nonstationary cone for $t=0$, and smooth and self-expanding for $t>0$.
    More generally: this nonuniqueness phenomenon is common to mean curvature flow, harmonic map heat flow, the equation $u_t = \Delta u + u^p$, and the Yang-Mills heat flow (Gastel).

For each of these equations, let a *cone* denote a rotationally and homothety invariant object. Then we have the following general principle:

> If C is an unstable, static cone, then C has an infinite number of self-similarly expanding forward evolutions. If C' is a cone close to an unstable, static cone, then C' has a large but finite number of such forward evolutions.

For harmonic map flow: the cones are the maps $x/|x| : \mathbf{R}^n \to S^n$, $3 \leq n \leq 6$. For the u^p equation: the cones are $C|x|^{-2/(p-1)}$ for a range of p above the critical value $(n+2)/(n-2)$.

26. T. Ilmanen and R. Schätzle, "The Singular Set of Minimal Hypersurfaces with Second Fundamental Form in L^2."

    Consider smooth, embedded, minimal hypersurfaces M_j in $\mathbf{R}^{n+1}$ converging to a stationary varifold Σ. Under the assumption that the second fundamental form A of M_j is bounded in L^2, that is,

$$\sup_j \int_{M_j} |A|^2 \, d\mathcal{H}^n < \infty,$$

    we prove that the Hausdorff dimension of the singular set is at most $n-2$.

27. J. Hättenschweiler and T. Ilmanen, "Shrinking Networks."

28. T. Ilmanen and F. Schulze, "Estimates for Curvature Flow of Planar Networks."