The singular set of minimal hypersurfaces with second fundamental form in L^2

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Abstract: We consider smooth, embedded, minimal n-hypersurfaces M_j converging to a stationary varifold Σ. Under the assumption that the second fundamental A_{M_j} of M_j is bounded in L^2, that is

$$\int_{M_j} |A_{M_j}|^2 \, \mathrm{d}\mathcal{H}^n \leq \Lambda_0,$$

we prove that the Hausdorff dimension of the singular set is estimated by

$$dim_{\mathcal{H}} \; sing \; \Sigma \leq n-2.$$

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1 Introduction

The dimension of the singular set of area-minimizing integral n-currents Σ in $I\!R^{n+1}$ is estimated by

$$dim_{\mathcal{H}} \; sing \; \Sigma \leq n-7, \tag{1.1}$$

where $dim_{\mathcal{H}}$ denotes the Hausdorff dimension, see [**?**] Theorem 37.7.

For stationary, stable n-hypersurfaces M in $I\!R^{n+1}$ with small singular set $\mathcal{H}^{n-2}(sing \; M) = 0$, estimates on the second fundamental form were derived in [**?**], when M is squeezed between two planes, and the dimension estimate of the singular set of M was improved to (**??**).

For varifolds that are merely stationary, Allard's Regularity Theorem shows that points which have a tangent plane of density one are regular, see [**?**] Section 8 or [**?**] Theorem 23.1, 24.2. However, in the presence of higher density, it is not even known whether stationary varifolds are regular almost everywhere.

In this paper, we consider smooth, embedded hypersurfaces M_j of dimension n in an open set $\Omega \subseteq I\!R^{n+1},\ n \geq 1$, minimal in the sense that the mean curvature $H_{M_j} = 0$ vanishes. Let A_{M_j} denote the second fundamental form of M_j . We assume that

$$\mathcal{H}^n(M_j), \int\limits_{M_j} |A_{M_j}|^2 \, \mathrm{d}\mathcal{H}^n \leq \Lambda_0 \tag{1.2}$$

for some $\Lambda_0 < \infty$. The bound on the area yields a subsequence such that the corresponding unit-density n-varifolds converge weakly to a stationary integral n-varifold

$$\mu_{M_j} := \mathcal{H}^n \lfloor M_j \to \mu_V \tag{1.3}$$

in Ω . We put $\Sigma := spt\ \mu_V$ and call $x \in \Sigma$ a regular point of the varifold μ_V if μ_V equals a smooth graph of constant integer density in neighbourhood of x . We denote

$$reg\ \Sigma := \{\ x \in \Sigma \mid x \text{ regular point of } \mu_V\ \}$$

and

$$sing\ \Sigma := \Sigma - reg\ \Sigma. \tag{1.4}$$

We show the following theorem which states that the singular set is of codimension 2 in the support of μ_V .

Theorem 1.1 *Under the above assumptions,*

$$dim_{\mathcal{H}}\ sing\ \Sigma \leq n - 2, \tag{1.5}$$

where $dim_{\mathcal{H}}$ *denotes the Hausdorff dimension.*

1.1 Notation

We fix some notation.

We define the d-dimensional density by

$$\theta^d(\mu, x) := \lim_{\varrho \to 0} \frac{\mu(B_\varrho^{n+1}(x))}{\omega_d \varrho^d}$$

if this limit exists. Likewise, we define the upper and lower densities $\theta^{*,d}$ and θ_*^d , respectively, and abbreviate the density-ratio by

$$\theta^d(\mu, x, \varrho) := \frac{\mu(B_\varrho^{n+1}(x))}{\omega_d \varrho^d}$$

The density of a set $A \subseteq I\!R^{n+1}$ in $x \in I\!R^{n+1}$ with respect to a Radon measure μ on $I\!R^{n+1}$ is defined by

$$\theta(\mu, A, x) := \lim_{\varrho \to 0} \frac{\mu(B_\varrho^{n+1}(x) \cap A)}{\mu(B_\varrho^{n+1}(x))}$$

if this limit exists.

2 Graphical decomposition

Following [?], [?], [?] and [?], we prove a graphical decomposition theorem.

3 Non-decay of the mean value

We prove in a purely PDE-setting that the mean value of a nonnegative function which is superharmonic outside a set of small 2-capacity decays only by a factor arbitrarily close to one. Actually, we have to truncate the function by a median. We start with the following definition.

Definition 3.1 *For a measurable function* $u : I\!R^n \to I\!R$ *, we define the lower* $\sigma-$*median for* $0 < \sigma < 1$ *on* $B_\varrho^n(0)$ *by*

$$m_{\sigma,\varrho}(u) := \inf\{ t \mid \mathcal{L}^n(B_\varrho^n(0) \cap [u \geq t]) \leq \sigma \mathcal{L}^n(B_\varrho^n(0)) \ \}$$

We abbreviate

$$u^{\sigma,\varrho} := \min(u, m_{\sigma,\varrho}(u)).$$

Lemma 3.2 *Let* $0 < \sigma < 1/2$ *and* $u \geq 0$ *be a nonnegative, measurable function on* $B_1^n(0)$ *which is a supersolution of an elliptic equation with measurable coefficients* a_{kl} *outside a closed set* $X \subseteq I\!R^n$ *, more precisely* $u \in C^{0,1}_{loc}(B_1^n(0) - X)$ *,*

$$-\partial_k(a_{kl}\partial_l u) \geq 0 \quad \textit{weakly locally in } B_1^n(0) - X, \tag{3.1}$$

and we assume the bounds

$$\| \, a_{kl} - \delta_{kl} \, \|_{L^\infty(B_1^n(0))}, \ cap_2(X, I\!R^n) \leq \delta \tag{3.2}$$

for some small $\delta = \delta(n, \sigma)$ *depending on* n *and* σ *.*

Then

$$\fint_{B_{1/2}^n(0)} u^{2\sigma,1/2} \geq (1 - C\sigma^\beta) \fint_{B_1^n(0)} u^{\sigma,1} \tag{3.3}$$

where $C < \infty$ *and* $\beta > 0$ *depend only on* n *.*

4 Self-density of the density sets

We consider the Radon-measures

$$\alpha_j := |A_{M_j}|^2 \ \mathcal{H}^n \lfloor M_j. \tag{4.1}$$

By (**??**), these measures are bounded, hence converge weakly for a subsequence

$$\alpha_j \to \alpha. \tag{4.2}$$

For $\theta_0 \in I\!N$, we put $\Sigma_{\theta_0} := \{ x \in \Sigma \mid \theta^n(\mu_V, x) = \theta_0 \}$.

Lemma 4.4 *If*

$$\theta^{*,n-2+\varepsilon}(\alpha,0)<\infty, \quad \theta_0:=\theta^n(\mu_V,0)\in I\!N,$$

then

$$\theta(\mu_V,\Sigma-\Sigma_{\theta_0},0)=0.$$

Proposition 4.5 *For any* $\sigma>0$ *, if*

$$(\varrho_0)^{2-n}\alpha(B^{n+1}_{\varrho_0}(0))\leq\varepsilon, \tag{4.3}$$

$$0\in\Sigma, \quad \theta^n(\mu_V,0,\varrho_0)\leq\theta_0+\varepsilon, \tag{4.4}$$

and $\varepsilon=\varepsilon(n,\theta_0,\sigma)$ *and* $\gamma=\gamma(n,\theta_0,\sigma)$ *are small enough, then*

$$\frac{\mu_V(B^{n+1}_{\gamma\varrho_0}(0)\cap\Sigma_{\theta_0})}{\mu_V(B^{n+1}_{\gamma\varrho_0}(0))}\in[0,\sigma)\cup(1-\sigma,1]. \tag{4.5}$$

5 Separation of the density sets

For $\theta_0\in I\!N$, we put

$$\mu_{V_{\theta_0}}:=\mu_V\lfloor\Sigma_{\theta_0}=\theta_0\mathcal{H}^n\lfloor\Sigma_{\theta_0}.$$

Lemma 5.1 $\mu_{V_{\theta_0}}$ *is stationary.*

References

[An86] Anderson, M.T., (1986) Remarks on Curvature Integrals and Minimal Varieties, Contemporary Mathematics, **49**, pp. 11-18.

[All72] Allard, W.K., (1972) On the first variation of a varifold, Annals of Mathematics, **95**, pp. 417-491.

[Hu86a] Hutchinson, J., (1986) Second fundamental form for varifolds and the existence of surface minimizing curvature, Indiana University Mathematical Journal, **35**, pp. 45-71.

[Hu86b] Hutchinson, J., (1986) $C^{1,\alpha}-$Multiple Function Regularity and Tangent Cone Behaviour for Varifolds with Second Fundamental Form in L^p , Proceedings of Symposia in Pure Mathematics, **44**, pp. 281-306.

[SchoSim81] Schoen, R., Simon, L., (1981) Regulartiy of Stable Minimal Hypersurfaces, Communications on Pure and Applied Mathematics, **34**, No. 6, pp. 741-797.

[Sim] Simon, L., (1983) Lectures on Geometric Measure Theory, Proceedings of the Centre for Mathematical Analysis Australian National University, Volume 3.

[Sim93] Simon, L., (1993) Existence of surfaces minimizing the Willmore functional, Communications in Analysis and Geometry, **Vol. 1**, No. 2, pp. 281-326.