Higher Regularity of the Inverse Mean Curvature Flow

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Abstract. We prove higher regularity properties of inverse mean curvature flow in Euclidean space: A sharp lower bound for the mean curvature is derived for star-shaped surfaces, independently of the initial mean curvature. It is also shown that solutions to the inverse mean curvature flow are smooth if the mean curvature is bounded from below. As a consequence we show that weak solutions of the inverse mean curvature flow are smooth for large times, beginning from the first time where a surface in the evolution is starhaped.

A classical solution of inverse mean curvature flow (IMCF) in Euclidean space is a smooth family $F : N^n \times [0,T] \to \mathbb{R}^{n+1}$ of regular and closed hypersurfaces satisfying

$$\frac{\partial}{\partial t} F(p,t) = \frac{1}{H}\,\nu(p,t), \qquad p \in N^n, \qquad 0 \leq t \leq T, \tag{0.1}$$

where $H(p,t) > 0$ and $\nu(p,t)$ are the mean curvature and exterior unit normal of the surface $N_t = F(\cdot,t)(N^n)$ at the point $F(p,t)$. It was shown by Gerhardt [3] that for smooth star-shaped initial data of strictly positive mean curvature equation (0.1) has a smooth solution for all times which approaches a homothetically expanding spherical solution as $t \to \infty$, see also Urbas [15]. For nonstar-shaped initial data it is well known that singularities may develop; in the case $n = 2$ Smoczyk [13] proved that such singularities can only occur if the speed becomes unbounded, or, equivalently, when the mean curvature tends to zero somewhere during the evolution.

In [6], [7], [8] the authors developed a new level set approach to weak solutions of the flow, allowing "jumps" of the surfaces and solutions of weakly positive mean curvature. Weak solutions of the flow can be used to derive energy estimates in General Relativity, see [7] and the references therein.

In section 1 we prove a lower bound for the mean curvature of star-shaped solutions to (0.1) which is independent of the curvature of the initial surface and only depends on gross geometric properties of the initial surface, see theorem 1.1. Our lower bound for H can be thought of as embodying a weak Harnack inequality, in which $\inf H$ is bounded below by some integral of H, which in turn is bounded below by the initial area in view of the Michael-Simon Sobolev inequality. The proof combines the evolution equation of the mean curvature with the evolution equation of the support function $w = \langle F, \nu\rangle$. In conjunction with the Sobolev inequality for hypersurfaces and an iteration scheme due to Stampacchia we derive as a central result of the

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paper an estimate interior in time stating that in the star-shaped setting the mean curvature has to grow at least of order $(t-t_o)^{1/2}$ when starting from a surface of nonnegative mean curvature at time t_o. Since time t is a dimensionless quantity in IMCF, the power $t^{1/2}$ is not explained by scaling but arises by some other mechanism unknown to us. It is sharp as can be seen from the one-dimensional case. From another point of view the result can be seen as a sharp Harnack inequality for the speed of the surface, capturing how the near infinite speed at some part of the surface is spread around a starshaped surface in time.
In section 2, theorem 2.1, we estimate the full second fundamental form A when the mean curvature H is bounded below by a positive constant, making use of a maximum principle for the tensor HA. This extends a result of Smoczyk [13] to all dimensions. As a direct consequence we obtain a sharp blowup criterion for IMCF: A solution can only become singular if the mean curvature goes to zero, see corollary 2.3.
Combining the results above with an approximation lemma we can construct smooth solutions to IMCF for star-shaped, weakly mean convex initial data of class C^1 in theorem 2.5. This result can then be applied to variational level set solutions of the flow as introduced and studied in [7]: Using the uniqueness of these weak solutions we conclude in theorem 2.7 that every level set solution to the flow is regular after the first instant t_o where a level set N_{t_o} is star-shaped. Since a blowdown argument shows that every weak solution eventually is star-shaped, this in particular implies that weak solutions will be regular outside some compact set.
We note that Heidusch [5] has derived local estimates for solutions of IMCF which are locally star-shaped, implying $C^{1,1}$-regularity of the level sets of weak solutions in low dimensions. Chow and Gulliver [1] obtain slope estimates for radial graphs, compare the remark at the end of section 2. Questions concerning large initial mean curvature for IMCF are discussed in [8].

In a forthcoming paper we will extend the results presented here to inverse mean curvature flow in asymptotically flat Riemannian manifolds, proving smoothness for large times and convergence to the center of mass.

1 A lower bound for the mean curvature

If $F : N^n \to \mathbb{R}^{n+1}$ is a smooth closed hypersurface, we say that $F(N^n)$ is star-shaped (with respect to the origin) if $\langle F, \nu \rangle > 0$ on M^n. We will prove the following theorem in this section.

Theorem 1.1 *Suppose $F : N^n \times [0,T] \to \mathbb{R}^{n+1}$ is a smooth star-shaped solution of (0.1) such that on N_0^n we have the estimates*

$$0 < R_1 \leq \langle F, \nu \rangle \leq R_2. \tag{1.1}$$

Then there is a constant $0 < c_n < \infty$ depending only on n such that the estimates

$$\frac{1}{H\langle F, \nu \rangle} \leq c_n \max\left(\frac{1}{t^{1/2}}, 1\right) R_1^{-1} \, |N_0^n|^{1/n}$$

and

$$H \geq c_n^{-1} \min(t^{1/2}, 1) \exp(-t/n) \, R_1 R_2^{-1} \, |N_0^n|^{-1/n}$$

hold everywhere on $N^n \times [0,T]$.

We recall the evolution equations for various geometric quantities under the inverse mean curvature flow. Let $g = \{g_{ij}\}_{1\le i,j\le n}$ and $A = \{h_{ij}\}_{1\le i,j\le n}$ be the first and second fundamental form of the evolving surfaces, let $H = g^{ij}h_{ij} = \mathrm{trace}_g A$ as before be the mean curvature, $w = \langle F, \nu\rangle$ the support function and $d\mu$ the induced measure on N_t. Since we already know from [3] that the smooth solution can be extended for all time, we will assume throughout this section that $T = \infty$.

Lemma 1.2 *Smooth solutions of (0.1) with $H > 0$ satisfy*

(i) $\dfrac{\partial}{\partial t} g_{ij} = 2H^{-1}h_{ij}$,

(ii) $\dfrac{\partial}{\partial t}(d\mu) = d\mu$,

(iii) $\dfrac{\partial}{\partial t}\nu = \dfrac{1}{H^2}\nabla H$,

(iv) $\dfrac{\partial}{\partial t} h_{ij} = -\nabla_i\nabla_j(\dfrac{1}{H}) + \dfrac{1}{H}h_{il}h^l_j = \dfrac{1}{H^2}\Delta H - \dfrac{2}{H^3}\nabla_i H\nabla_j H + \dfrac{|A|^2}{H^2}h_{ij}$,

(v) $\dfrac{\partial}{\partial t} H = \dfrac{1}{H^2}\Delta H - \dfrac{|A|^2}{H}$,

(vi) $\dfrac{\partial}{\partial t} w = \dfrac{1}{H^2}\Delta w + \dfrac{|A|^2}{H^2}\,w$.

Proof. The evolution equations for the metric and the second fundamental form have been established in [3], see also [9] for the evolution equations satisfied by general flows. The evolution equation for the support function w follows from (iii) and the identity

$$\Delta\langle F, \nu\rangle = -|A|^2\,\langle F, \nu\rangle + \langle\nabla H, F\rangle + \,H.$$

As a first consequence we conclude that star-shaped surfaces remain star-shaped, a fact first observed by Gerhardt [3].

Proposition 1.3 *If the initial surface satisfies*

$$0 < R_1 \,\le\, \langle F, \nu\rangle \,\le R_2,$$

then the solution of (0.1) satisfies

$$\exp{(t/n)}R_1 \,\le\, \langle F, \nu\rangle \,\le\, |F| \,\le\, R_2\exp{(t/n)}.$$

Proof. Since $|A|^2 \ge H^2/n$ we have for $w = \langle F, \nu\rangle$

$$\frac{\partial}{\partial t} w \ge \frac{1}{H^2}\Delta w + \frac{1}{n}w,$$

proving the first inequality in view of the maximum principle. The second inequality follows from the equation

$$\frac{\partial}{\partial t}|F|^2 \;=\; \frac{2}{H}\langle F, \nu\rangle$$

and the fact that in the point of the surface most distant from the origin we have $\langle F, \nu\rangle = |F|$ and $H \ge n|F|^{-1}$.

To prove the lower bound for the mean curvature we exploit the fact that the functions $1/H$ and $w = \langle F, \nu \rangle$ satisfy the same equation to get rid of the nonlinear zero order terms: We will derive an upper bound for the modified speed function

$$u = \frac{1}{Hw}.$$

Lemma 1.4 *The modified speed function $u = H^{-1}w^{-1}$ satisfies the evolution equation*

$$\begin{aligned} \frac{\partial}{\partial t}u &= \frac{1}{H^2}\Delta u - \frac{2}{H^2}u^{-1}|\nabla u|^2 - \frac{2}{H^3}\nabla_i H \nabla_i u \\ &= \nabla_i\left(\frac{1}{H^2}\nabla_i u\right) - \frac{2}{H^2}u^{-1}|\nabla u|^2. \end{aligned}$$

Proof. Combine the equations (v) and (vi) from lemma 1.2.
The strategy of proof aims for a Stampacchia iteration to estimate $\sup u$. The first step is an L^p–estimate for u that for each $2 < p < \infty$ behaves like $t^{-1/2}$ for small t and still depends on p at this stage. The $1/H^2$-term in front of the Laplacian is of crucial help, making the diffusion stronger when H is small.

Theorem 1.5 *Suppose the initial surface N_0 satisfies*

$$0 < R_1 \leq \langle F_0 \,, \nu \rangle$$

and N_t solves IMCF (0.1). Then there is a constant $c(n)$ depending only on n such that the modified speed function u satisfies for all $p > 2$ and $0 < t \leq T$ the estimate

$$||u||_{L^p(N_t)} \leq c(n)\, R_1^{-1}\, |N_0|^{\frac{p+n}{np}} \exp\left(\frac{2}{p}\,t\right)\left(\exp\left(\frac{2}{p}\,t\right) - 1\right)^{-1/2}.$$

Proof. From lemma 1.4 and lemma 1.2(ii) we compute for $p \geq 2$

$$\begin{aligned} \frac{d}{dt}\int u^p\, d\mu &= p\int u^{p-1}\nabla_i\left(\frac{1}{H^2}\nabla_i u\right) d\mu - 2p\int \frac{1}{H^2}u^{p-2}|\nabla u|^2\, d\mu + \int u^p\, d\mu \\ &= -p(p+1)\int \frac{1}{H^2}u^{p-2}|\nabla u|^2\, d\mu + \int u^p\, d\mu. \end{aligned}$$

Using now the first inequality in proposition 1.3 we conclude

$$\begin{aligned} \frac{d}{dt}\int u^p\, d\mu &= -p(p+1)\int \langle F, \nu\rangle^2 u^p |\nabla u|^2\, d\mu + \int u^p\, d\mu \\ &\leq -p(p+1)\exp\left(\frac{2}{n}\,t\right) R_1^2 \int u^p|\nabla u|^2\, d\mu + \int u^p\, d\mu \end{aligned}$$

and thus arrive for $g = u^{p/2+1}$ at the estimate

$$\frac{d}{dt}\int u^p\, d\mu \leq -\exp\left(\frac{2}{n}\,t\right) R_1^2 \int |\nabla g|^2\, d\mu + \int u^p\, d\mu. \qquad (1.2)$$

To proceed further we need the Sobolev inequality on hypersurfaces due to Michael and Simon, see [12].

Proposition 1.6 *There is a constant $c(n)$ depending only on $n \geq 2$ such that*

$$\Big(\int_{N^n} f^{\frac{n}{n-1}}\,d\mu\Big)^{\frac{n-1}{n}} \leq c(n)\int_{N^n} |\nabla f| + |H||f|\,d\mu$$

for any $f \in C^{0,1}_c(N^n)$.

In applying the Sobolev inequality we have to distinguish the cases $n = 2$ and $n > 2$.
I. Case $n = 2$. Setting $f = |h|^q$ for $q > 1$ we conclude from proposition 1.6 that

$$\left(\int_{N^2} |h|^{2q}\,d\mu\right)^{1/q} \leq c\,q^2\,|N^2|^{1/q}\int_{N^2} |\nabla h|^2 + |H|^2|h|^2\,d\mu. \tag{1.3}$$

Hence we derive from (1.2) with $2 > q = \frac{2p}{p+2} > 1$ for $p > 2$ that

$$\frac{d}{dt}\int u^p\,d\mu \;\leq\; -c^{-1}\exp{(t)}\,R_1^2 q^{-2}\,|N_t|^{-1/q}\left(\int g^{2q}\,d\mu\right)^{1/q} +\;\exp{(t)}R_1^2\int H^2 g^2\,d\mu + \int u^p\,d\mu.$$

Using $u^p = g^q$ and Hölder's inequality we derive

$$\frac{d}{dt}\int g^q\,d\mu \;\leq\; -c^{-1}R_1^2 q^{-2}\,|N_t|^{-2/q}\exp{(t)}\left(\int g^q\,d\mu\right)^{2/q} +\;\exp{(t)}R_1^2\int H^2 g^2\,d\mu + \int u^p\,d\mu.$$

Now we estimate

$$\exp{(t)}R_1^2\int H^2 g^2\,d\mu \leq \int \langle F,\nu\rangle^2 H^2 g^2 d\mu = \int u^{-2}g^2\,d\mu = \int g^q\,d\mu,$$

such that finally in view of $|N_t| = |N_0|\exp{(t)}$ and $1 < q < 2$

$$\begin{aligned}\frac{d}{dt}\int g^q\,d\mu &\leq -c^{-1}q^{-2}R_1^2\,|N_t|^{-2/q}\exp{(t)}\left(\int g^q\,d\mu\right)^{1+2/p} + 2\int u^p\,d\mu\\ &= -c^{-1}q^{-2}R_1^2\,|N_0|^{-2/q}\exp{(-\frac{2}{p}\,t)}\left(\int g^q\,d\mu\right)^{1+2/p} + 2\int u^p\,d\mu.\end{aligned}$$

Setting $\varphi = \exp{(-2t)}\int g^q\,d\mu$ this is equivalent to

$$\frac{d}{dt}\varphi \leq -c^{-1}q^{-2}R_1^2\,\exp\Big(\frac{2t}{p}\Big)\,|N_0|^{-\frac{p+2}{p}}\varphi^{\frac{p+2}{p}}.$$

From the solution of the corresponding ODE we conclude

$$\varphi \leq |N_0|^{\frac{p+2}{2}}\left(c^{-1}R_1^2 q^{-2}\Big(\exp\Big(\frac{2t}{p}\Big) - 1\Big)\right)^{-p/2}$$

and derive the desired L^p–estimate for u:

$$||u||_{L^p} \;\leq\; c\,R_1^{-1}\,|N_0|^{\frac{p+2}{2p}}\,\exp\Big(\frac{2}{p}\,t\Big)\left(\exp\Big(\frac{2}{p}\,t\Big) - 1\right)^{-1/2}.$$

II. Case $n \geq 3$. Using the Sobolev inequality in the form

$$\left(\int_{N^n} |h|^{\frac{2n}{n-2}}\, d\mu\right)^{\frac{n-2}{n}} \leq c(n) \int_{N^n} |\nabla h|^2 + |H|^2 |h|^2\, d\mu. \tag{1.4}$$

we derive from (1.2) for $q = \frac{2p}{p+2}$, $p \geq 2$, as before

$$\frac{d}{dt}\int g^q\, d\mu \;\leq\; -c^{-1}(n) R_1^2 \exp\left(\frac{2t}{n}\right)\left(\int g^{\frac{2n}{n-2}}\, d\mu\right)^{\frac{n-2}{n}} + 2\int g^q\, d\mu.$$

The Hölder inequality and $|N_t| = |N_0| \exp(t)$ yield

$$\frac{d}{dt}\int g^q\, d\mu \;\leq\; -c^{-1}(n) R_1^2 |N_0|^{\frac{-2(p+n)}{np}} \exp\left(\frac{-2t}{p}\right)\left(\int g^q\, d\mu\right)^{\frac{p+2}{p}} + 2\int g^q\, d\mu$$

and the proof proceeds as before to the estimate

$$||u||_{L^p} \;\leq\; c(n)\, R_1^{-1}\, |N_0|^{\frac{p+n}{np}} \exp\left(\frac{2}{p}\,t\right)\left(\exp\left(\frac{2}{p}\,t\right) - 1\right)^{-1/2}$$

as required.

In the second step of the proof we derive an iteration inequality first employed by Stampacchia for the measure of the set where u is larger than a constant, weighted by t^β.

Proof of theorem 1.1. To obtain the supremum estimate for u from the L^p-estimate, let $t_0 > 0$ be arbitrary but fixed and set

$$v = (t - t_0)^\beta u = (t - t_0)^\beta \frac{1}{H\langle F, \nu\rangle}\,, \tag{1.5}$$

where $0 < \beta < 1$ will be chosen later. From lemma 1.4 we get the evolution equation

$$\frac{d}{dt} v \;=\; \nabla_i\left(\frac{1}{H^2}\nabla_i v\right) - \frac{2}{H^2} v^{-1} |\nabla v|^2 + \beta (t - t_0)^{-1} v.$$

Let $v_k = \max(v - k, 0)$ for $k \geq 0$ and let $A(k) = \{p \in N_t |\, v(p,t) > k\}$. We multiply the last equation with v_k and integrate to derive

$$\begin{aligned}
\frac{d}{dt}\int_{N_t} v_k^2\, d\mu \;&=\; 2\int_{A(k)} v_k \nabla_i\left(\frac{1}{H^2}\nabla_i v\right) d\mu + \int_{A(k)} v_k^2\, d\mu \\
&\quad -4\int_{A(k)} \frac{1}{H^2} v_k v^{-1} |\nabla v|^2\, d\mu + 2\beta (t - t_0)^{-1}\int_{A(k)} v_k v\, d\mu.
\end{aligned}$$

This implies

$$\begin{aligned}
\frac{d}{dt}\int_{N_t} v_k^2\, d\mu \;&\leq\; -2\int_{A(k)} \frac{1}{H^2} |\nabla v|^2\, d\mu + \int_{A(k)} v_k^2\, d\mu \\
&\quad +2\beta (t - t_0)^{-1}\int_{A(k)} v^2\, d\mu \\
&=\; -2(t - t_0)^{-2\beta}\int_{A(k)} \langle F, \nu\rangle^2 v^2 |\nabla v|^2\, d\mu + \int_{A(k)} v_k^2\, d\mu \\
&\quad +2\beta (t - t_0)^{-1}\int_{A(k)} v^2\, d\mu.
\end{aligned}$$

We now use $v > k$ on $A(k)$ as well as $\langle F, \nu\rangle \geq R_1 \exp(t/n)$ from proposition 1.3 to estimate

$$\begin{aligned}\frac{d}{dt}\int_{N_t} v_k^2\,d\mu \;&+\; 2k^2R_1^2(t-t_0)^{-2\beta}\exp\Big(\frac{2t}{n}\Big)\int_{N_t}|\nabla v_k|^2 + H^2v_k^2\,d\mu \\ &\leq\; 2\beta(t-t_0)^{-1}\int_{A(k)} v^2\,d\mu + 3\int_{N_t} v_k^2\,d\mu. \end{aligned}\tag{1.6}$$

Here we also used the fact that on $A(k)$

$$\exp\Big(\frac{2t}{n}\Big)(t-t_0)^{-2\beta}H^2R_1^2 \leq (t-t_0)^{-2\beta}H^2\langle F,\nu\rangle^2 = v^{-2} < k^{-2}.$$

Again we have to distinguish the cases $n = 2$ and $n \geq 3$.
I. Case $n = 2$. The Sobolev inequality (1.3) for some fixed $1 < q < \infty$ yields the estimate

$$\begin{aligned}\frac{d}{dt}\int_{N_t} v_k^2\,d\mu \;&+\; c^{-1}k^2R_1^2\,q^{-2}(t-t_0)^{-2\beta}\exp(t)|N_t|^{-\frac{1}{q}}\Big(\int_{N_t} v_k^{2q}\,d\mu\Big)^{\frac{1}{q}} \\ &\leq\; 2\beta(t-t_0)^{-1}\int_{A(k)} v^2\,d\mu + 3\int_{N_t} v_k^2\,d\mu.\end{aligned}\tag{1.7}$$

In view of the Hölder inequality

$$\int_{N_t} v_k^2\,d\mu \leq |N_t|^{1-\frac{1}{q}}\Big(\int_{N_t} v_k^{2q}\,d\mu\Big)^{\frac{1}{q}}$$

the second term on the RHS can be absorbed if $|N_t| \leq \frac{1}{3}c^{-1}k^2R_1^2q^{-2}(t-t_0)^{-2\beta}\exp(t)$. As $|N_t| = |N_0|\exp(t)$ this inequality will hold in the interval $[t_0, t_1]$ for $k \geq k_0 > 0$ if we make sure that

$$k_0^2 \geq 3\,c\,(t_1-t_0)^{2\beta}R_1^{-2}\,q^2\,|N_0|.\tag{1.8}$$

Hence for $n = 2$ we derive the inequality

$$\begin{aligned}\frac{d}{dt}\int_{N_t} v_k^2\,d\mu \;&+\; c^{-1}k^2\,R_1^2\,q^{-2}\,(t-t_0)^{-2\beta}\exp(t)|N_t|^{-\frac{1}{q}}\Big(\int_{N_t} v_k^{2q}\,d\mu\Big)^{\frac{1}{q}} \\ &\leq\; 2\beta(t-t_0)^{-1}\int_{A(k)} v^2\,d\mu\end{aligned}\tag{1.9}$$

provided (1.8) holds.
II. Case $n \geq 3$. Again starting from (1.6) we use the Sobolev inequality in the form (1.4), set $q = \frac{n}{n-2}$ and infer with the help of the Hölder inequality

$$\int_{N_t} v_k^2\,d\mu \leq |N_t|^{\frac{2}{n}}\Big(\int_{N_t} v_k^{2q}\,d\mu\Big)^{\frac{1}{q}}$$

the estimate

$$\begin{aligned}\frac{d}{dt}\int_{N_t} v_k^2\,d\mu \;&+\; c^{-1}(n)k^2R_1^2(t-t_0)^{-2\beta}\exp\Big(\frac{2t}{n}\Big)\Big(\int_{N_t} v_k^{2q}\,d\mu\Big)^{\frac{1}{q}} \\ &\leq\; 2\beta(t-t_0)^{-1}\int_{A(k)} v^2\,d\mu,\end{aligned}\tag{1.10}$$

provided $t \in [t_0, t_1], k \geq k_0 > 0$ and

$$k_0^2 \geq 3\,c(n)\,(t-t_0)^{2\beta}R_1^{-2}\,|N_0|^{\frac{2}{n}}.\tag{1.11}$$

Now define a constant $B(k)$ by setting

$$B(k) := \begin{cases} c^{-1}\, q^{-2}\, k^2\, R_1^2\, |N_{t_1}|^{-\frac{1}{q}} \exp(t_0), & n = 2, \\ c^{-1}(n)\, k^2\, R_1^2 \exp(2t_0/n), & n \geq 3, \end{cases}$$

such that for all $n \geq 2$ we have from (1.9) and (1.10) the estimate

$$\frac{d}{dt}\int_{N_t} v_k^2\, d\mu + B(k)(t-t_0)^{-2\beta}\Big(\int_{N_t} v_k^{2q}\, d\mu\Big)^{\frac{1}{q}} \leq 2\beta(t-t_0)^{-1}\int_{A(k)} v^2\, d\mu.$$

Integrating from t_0 to any $t \in [t_0, t_1]$ and having in mind that v vanishes at $t = t_0$ we deduce

$$\sup_{[t_0,t_1]}\int_{N_t} v_k^2\, d\mu + B(k)\int_{t_0}^{t_1}(t-t_0)^{-2\beta}\Big(\int_{A(k)} v_k^{2q}\, d\mu\Big)^{\frac{1}{q}} dt \leq 2\beta\int_{t_0}^{t_1}(t-t_0)^{-1}\int_{A(k)} v^2\, d\mu\, dt. \quad (1.12)$$

To proceed further we define an interpolating exponent $1 < q_0 < q$ by

$$\frac{1}{q_0} = \frac{a}{q} + (1-a), \quad a = \frac{1}{q_0}.$$

With this choice we have $1 - a = (1/q_0)(1 - 1/q)$ and

$$\begin{aligned} 1 < q < \infty, &\qquad q_0 = 2 - \frac{1}{q}, &&\text{if } n = 2, \\ q = \frac{n}{n-2}, &\qquad q_0 = 2 - \frac{1}{q} = \frac{n+2}{n}, &&\text{if } n = 3. \end{aligned} \quad (1.13)$$

By interpolation and Young's inequality we obtain

$$\begin{aligned} &\Big(\int_{t_0}^{t_1} B(k)(t-t_0)^{-2\beta}\int_{A(k)} v_k^{2q_o}\, d\mu\, dt\Big)^{1/q_o} \\ \leq\ &\Big[\int_{t_0}^{t_1} B(k)(t-t_0)^{-2\beta}\Big(\int_{A(k)} v_k^{2q}\, d\mu\Big)^{aq_o/q}\Big(\int_{A(k)} v_k^2\, d\mu\Big)^{q_o(1-a)} dt\Big]^{1/q_o} \\ \leq\ &\Big(\sup_{[t_0,t_1]}\int_{A(k)} v_k^2\, d\mu\Big)^{1-a}\Big[\int_{t_0}^{t_1} B(k)(t-t_0)^{-2\beta}\Big(\int_{A(k)} v_k^{2q}\, d\mu\Big)^{1/q} dt\Big]^{a} \\ \leq\ &\sup_{[t_0,t_1]}\int_{A(k)} v_k^2\, d\mu + c(n)\int_{t_0}^{t_1} B(k)(t-t_0)^{-2\beta}\Big(\int_{A(k)} v_k^{2q}\, d\mu\Big)^{1/q} dt, \end{aligned}$$

since q, q_0 are fixed depending only on n. Combining this estimate with (1.12) we derive

$$\Big(\int_{t_0}^{t_1} B(k)(t-t_0)^{-2\beta}\int_{A(k)} v_k^{2q_0}\, d\mu\, dt\Big)^{1/q_0} \leq c(n)\beta\int_{t_0}^{t_1}(t-t_0)^{-1}\int_{A(k)} v^2\, d\mu\, dt.$$

Now let $d\sigma := (t-t_0)^{-2\beta} d\mu\, dt$, $||A(k)|| := \int_{t_0}^{t_1}\int_{A(k)} d\sigma$ and apply Hölder's inequality with respect to $d\sigma$ on the LHS to conclude

$$B(k)^{\frac{1}{q_0}}\int_{t_0}^{t_1}\int_{A(k)} v_k^2\, d\sigma \leq c(n)\beta||A(k)||^{1-\frac{1}{q_0}}\int_{t_0}^{t_1}(t-t_0)^{-1}\int_{A(k)} v^2\, d\mu\, dt.$$

To properly match the powers of $(t-t_0)$ on both sides of the inequality, a good choice for β is $\beta = 1/4$. Then

$$\int_{t_0}^{t_1}\int_{A(k)} v_k^2\,d\sigma \le c(n)B(k)^{-\frac{1}{q_0}}||A(k)||^{1-\frac{1}{q_0}}\int_{t_0}^{t_1}(t-t_0)^{-1}|A(k)|^{1-\frac{1}{r}}\Big(\int_{N_t} v^{2r}\,d\mu\Big)^{\frac{1}{r}}dt \tag{1.14}$$

for some $r>1$ to be chosen. We use the L^p–estimate in theorem 1.5 for $u = H^{-1}w^{-1} = v(t-t_0)^{-1/4}$,

$$\begin{aligned} ||u||_{L^p(N_t)} &\le c(n)\exp\Big(\frac{2}{p}t\Big)|N_0|^{\frac{p+n}{np}}\Big(R_1^2\Big(\exp\Big(\frac{2}{p}t\Big)-1\Big)\Big)^{-1/2} \\ &\le c(n)\,R_1^{-1}\,|N_0|^{\frac{p+n}{np}}\,\max\Big(1,\,\frac{p}{t}\Big)^{1/2}\exp\Big(\frac{t}{p}\Big), \end{aligned}$$

and thus from (1.14)

$$\begin{aligned} \int_{t_0}^{t_1}\int_{A(k)} v_k^2\,d\sigma &\le c(n)R_1^{-2}B(k)^{-\frac{1}{q_0}}||A(k)||^{1-\frac{1}{q_0}}\,|N_0|^{\frac{2r+n}{nr}} \\ &\qquad\times\max\Big(1,\,\frac{r}{t_0}\Big)\,\exp\Big(\frac{t_1}{r}\Big)\int_{t_0}^{t_1}(t-t_0)^{-1/2}|A(k)|^{1-\frac{1}{r}}dt. \end{aligned}$$

The integral on the RHS can be estimated by

$$\begin{aligned} \int_{t_0}^{t_1}(t-t_0)^{-1/2}|A(k)|^{1-\frac{1}{r}}dt &\le \Big(\int_{t_0}^{t_1}(t-t_0)^{-1/2}|A(k)|\,dt\Big)^{1-\frac{1}{r}}\Big(\int_{t_0}^{t_1}(t-t_0)^{-1/2}dt\Big)^{\frac{1}{r}} \\ &= 2||A(k)||^{1-\frac{1}{r}}(t_1-t_0)^{\frac{1}{2r}}. \end{aligned}$$

So we finally arrive at

$$\begin{aligned} \int_{t_0}^{t_1}\int_{A(k)} v_k^2\,d\sigma &\le c(n)\,R_1^{-2}B(k)^{-\frac{1}{q_0}}|N_0|^{\frac{2r+n}{nr}} \\ &\qquad\times\max\Big(1,\,\frac{r}{t_0}\Big)\,\exp\Big(\frac{t_1}{r}\Big)(t_1-t_0)^{\frac{1}{2r}}||A(k)||^{2-\frac{1}{q_0}-\frac{1}{r}}. \end{aligned}$$

Choosing now r depending only on $q_0 = q_0(n)$ large enough that $\gamma = 2-\frac{1}{q_0}-\frac{1}{r} > 1$ we get the iteration inequality

$$|h-k|^2||A(h)|| \le c(n)\,R_1^{-2}\,B(k_0)^{-\frac{1}{q_0}}\,|N_0|^{\frac{2r+n}{nr}}\,\max\Big(1,\,\frac{r}{t_0}\Big)\,\exp\Big(\frac{t_1}{r}\Big)(t_1-t_0)^{\frac{1}{2r}}||A(k)||^{\gamma}$$

for $h > k \ge k_0 > 0$. A well known lemma due to Stampacchia ([14], Lemma 4.1) yields

$$||A(k_0+d)|| = 0, \quad d^2 = c(n)\,R_1^{-2}B(k_0)^{-\frac{1}{q_0}}|N_0|^{\frac{2r+n}{nr}}\,\max\Big(1,\,\frac{r}{t_0}\Big)\,\exp\big(\frac{t_1}{r}\big)(t_1-t_0)^{\frac{1}{2r}}||A(k_0)||^{\gamma-1}.$$

Now observe that

$$||A(k_0)|| = \int_{t_0}^{t_1}(t-t_0)^{-1/2}\int_{A(k)}d\mu\,dt \le 2|N_0|\exp(t_1)(t_1-t_0)^{1/2},$$

so

$$d^2 \le c(n)\,R_1^{-2}B(k_0)^{-\frac{1}{q_0}}|N_0|^{1+\frac{2}{n}-\frac{1}{q_0}}\max\Big(1,\,\frac{r}{t_0}\Big)(t_1-t_0)^{\frac{1}{2}(1-\frac{1}{q_0})}\exp\Big((1-\frac{1}{q_0})t_1\Big). \tag{1.15}$$

Now we have to distinguish small and large times:
I. Case $t_0 \le 1$: We choose $k_0 = c(n)\, t_0^{-1/4}\, R_1^{-1}\, |N_0|^{1/n}$, $t_1 = 2t_0$, where $c(n)$ is chosen such that inequalities (1.8) and (1.11) are satisfied. Using the relation between q, q_0 and n as in (1.13) we note that $1 - 2/q_0 - 1/qq_0 = 0$ if $n = 2$ and $1 - 1/q_0 - 2/nq_0 = 0$ if $n \ge 3$ and derive from (1.15), the definition of $B(k_0)$ and routine calculation that

$$d^2 \;\le\; c(n)\, R_1^{-2}\, |N_0|^{2/n}\, t_0^{-1/2}. \tag{1.16}$$

Here we also used the fact that the exponential function is bounded since $t_0 \le 1$.
II. Case $t_0 \ge 1$: We set $k_0 = c(n)\, R_1^{-1}\, |N_0|^{1/n}$, $t_1 = t_0 + 1$, where $c(n)$ is again chosen large enough to guarantee inequalities (1.8) and (1.11). Exploiting the relations between q, q_0 and n as in the first case and noting that now the coefficient of t_0 in the exponential function vanishes we infer from (1.15)

$$d^2 \;\le\; c(n)\, R_1^{-2}\, |N_0|^{2/n}. \tag{1.17}$$

In view of the definition of v in (1.5),

$$v = (t - t_0)^{1/4} u = (t - t_0)^{1/4} \frac{1}{H\langle F,\, \nu\rangle},$$

and in view of our choice $t_1 = \min(2t_0,\, t_0 + 1)$, this shows in both cases that

$$\sup_{N_{t_1}} \frac{1}{H\langle F,\, \nu\rangle} \;\le c_n \max\left(1,\, t_1^{-1/2}\right) R_1^{-1}\, |N_0|^{1/n},$$

proving the first estimate of theorem 1.1. The second estimate of the theorem,

$$H \ge c_n^{-1}\, \min(1,\, t^{-1/2})\, \exp(-t/n)\, R_1\, R_2^{-1}\, |N_0|^{-1/n}$$

is then a direct consequence of proposition 1.3.

2 Regularity for positive mean curvature

The classical formulation of inverse mean curvature flow is parabolic if the mean curvature is positive. We show in this section that a solution of the flow remains smooth as long as the mean curvature is bounded away from zero. In the two-dimensional case this was shown by Smoczyk [13]. Combining with the Harnack inequality of section 1, we show that star-shaped, weakly mean convex initial data have a unique, smooth solution of strictly positive mean curvature for all times $t > 0$ and also show that general weak solutions of IMCF constructed in [6] are smooth after some finite time.
We begin with a curvature estimate which is interior in time.

Theorem 2.1 *Let $F : N^n \times [0, T) \to \mathbb{R}^{n+1}$ be a smooth solution of IMCF (0.1) satisfying uniform bounds $0 < H_0 \le H \le H_1$. Then the largest eigenvalue κ_n of the tensor M,*

$$M_{ij} = H\, h_{ij},$$

and the largest eigenvalue λ_n of the second fundamental form A satisfy the estimates

$$\kappa_n \le \frac{H_1^2}{2t}, \qquad \lambda_n \le \frac{H_1^2}{2H_0 t},$$

everywhere on $N^n \times [0, T)$.

Corollary 2.2 *If the mean curvature H satisfies $0 < H_0 \leq H \leq H_1$ on $N^n \times [0,T)$, the full second fundamental form satisfies an estimate of the form*

$$|A| \leq c_n \frac{H_1^2}{H_0} \frac{1}{t}$$

on $N^n \times [0,T)$.

Proof. We combine the evolution equations of H and h_{ij} to get rid of the zero order terms: From lemma 1.2 we compute the evolution equation

$$\begin{aligned}
\frac{\partial}{\partial t} M_{ij} &= h_{ij}\left(\frac{1}{H^2}\Delta H - \frac{2}{H^3}|\nabla H|^2 - \frac{|A|^2}{H}\right) + H\left(\frac{1}{H^2}\Delta h_{ij} - \frac{2}{H^3}\nabla_i H \nabla_j H + \frac{|A|^2}{H^2} h_{ij}\right) \\
&= \frac{1}{H^2}\Delta M_{ij} - \frac{2}{H^2}\nabla_k H \nabla_k h_{ij} - \frac{2h_{ij}}{H^3}|\nabla H|^2 - \frac{2}{H^2}\nabla_i H \nabla_j H \\
&= \frac{1}{H^2}\Delta M_{ij} - \frac{2}{H^3}\nabla_k H \nabla_k M_{ij} - \frac{2}{H^2}\nabla_i H \nabla_j H.
\end{aligned}$$

To obtain the interior estimate of the theorem for the eigenvalues of M, we compute

$$\frac{\partial}{\partial t} M^i_j = \frac{1}{H^2}\Delta M^i_j - \frac{2}{H^3}\nabla_k H \nabla_k M^i_j - \frac{2}{H^2}\nabla^i H \nabla_j H - \frac{2}{H^2} M^{ik} M_{kj}.$$

It follows immediately by the maximum principle for such parabolic systems, see e.g. ([4], section 4), that the largest eigenvalue of M^i_j remains bounded above by its initial data. In fact, due to the negative forcing term on the RHS, if on the time interval considered we have an upper bound $H \leq H_1$ for the mean curvature, it follows by comparison with the ODE

$$\frac{d}{dt}\varphi = -\frac{2}{H_1^2}\varphi^2$$

that the largest eigenvalue κ_n of M satisfies

$$\kappa_n \leq \frac{H_1^2}{2t}.$$

For the largest principal curvature $\lambda_n = \kappa_n / H$ this implies the estimate

$$\lambda_n = \frac{\kappa_n}{H} \leq \frac{H_1^2}{2H} \frac{1}{t} \leq \frac{H_1^2}{2H_0} \frac{1}{t},$$

completing the proof of theorem 2.1. Since $H \geq 0$, it follows that the full second fundamental form is bounded by

$$|A| \leq c(n) \frac{H_1^2}{H_0} \frac{1}{t},$$

as claimed in corollary 2.2.
In view of this curvature estimate we can now characterize the maximal time interval of smooth existence by

Corollary 2.3 *Let $F : N^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth solution of IMCF (0.1) with $H > 0$, $0 < T < \infty$. If the mean curvature H remains bounded from below by a constant $H_0 > 0$ for all $t \in (0,T)$, then the solution can be extended beyond T. In particular, if $[0,T), T < \infty$, is the maximal time interval of existence for a smooth solution of IMCF (0.1), then the speed $1/H$ is unbounded for $t \to T$.*

Remark 2.4 *The maximal smooth solution constructed here may not coincide with the weak solution of [7] on the entire interval of existence* $[0, T)$.

Proof. In view of the evolution equation for H in lemma 1.2(v) the mean curvature is uniformly bounded above by its initial value $H_1 = \sup_{N_0} H$ on $N^n \times [0, T)$. Given an additional uniform lower bound $H_0 > 0$ for H, theorem 2.1 and corollary 2.2 imply that the second fundamental form is bounded by $|A| \leq c(n) H_1^2 / H_0 t$, which is bounded for $t \to T$. The regularity results of Krylov [10], see also [3] and [15], then guarantee higher regularity of the solution and convergence to a smooth limit surface N_T as $t \to T$, satisfying $H \geq H_0 > 0$. The short-time existence of solutions to (0.1) in case of smooth initial data with positive mean curvature then yields the desired extension.

We are now ready to prove global existence and regularity of solutions to IMCF for star-shaped initial data with weakly positive mean curvature.

Theorem 2.5 *Let* $F_0 : N^n \to \mathbb{R}^{n+1}$ *be a closed embedded hypersurface of class* C^1 *with measurable, bounded, nonnegative weak mean curvature* $H \geq 0$. *Assume that* $F_0(N^n) = N_0^n$ *is strictly star-shaped, i.e.*

$$0 < R_1 \leq \langle F, \nu \rangle \leq R_2$$

holds everywhere on N_0^n *with positive constants* R_1, R_2. *Then IMCF (0.1) has a global smooth solution* $F : N^n \times (0, \infty) \to \mathbb{R}^{n+1}$ *satisfying the estimates established in theorem 1.1, theorem 2.1 and corollary 2.2. As* $t \to 0$, N_t *converges to* N_0^n *uniformly in* C^0.

To apply our a priori estimates and prove theorem 2.5 we need the following approximation Lemma.

Lemma 2.6 *Let* $F_0 : N^n \to \mathbb{R}^{n+1}$ *be a closed, oriented hypersurface immersion of class* C^1, *with measurable, bounded, nonnegative weak mean curvature. Then* N_0^n *is of class* $C^{1,\beta} \cap W^{2,p}$ *for all* $0 < \beta < 1$, $1 \leq p < \infty$, *and can be approximated locally uniformly in* $C^{1,\beta} \cap W^{2,p}$ *by a family of smooth surfaces* $\tilde{N}_\epsilon^n, 0 < \epsilon < \epsilon_0$, *satisfying* $H > 0$.

Proof. The weak mean curvature H is defined as in ([6], section 1) by the first variation formula

$$\int_{N^n} \mathrm{div}_N X \, d\mu = \int_{N^n} H \langle X, \nu \rangle \, d\mu$$

for vectorfields $X \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Since N_0^n is C^1 and H is bounded, standard regularity results of Allard and Calderon-Zygmund imply that N_0^n is of class $C^{1,\beta} \cap W^{2,p}$ for all $0 < \beta < 1$, $1 \leq p < \infty$. By mollification we can pick a sequence of surfaces $\hat{N}_i^n$ converging locally uniformly to N_0^n in $C^{1,\beta} \cap W^{2,p}$. Now consider standard mean curvature flow starting from the smooth approximating surfaces $\hat{F}_i : N^n \to \mathbb{R}^{n+1}$, $\hat{F}_i(N^n) = \hat{N}_i^n$:

$$\begin{aligned} \frac{\partial}{\partial \epsilon} F(p, \epsilon) &= -H \, \nu(p, \epsilon), \qquad p \in N^n, \qquad 0 < \epsilon < \epsilon_0, \\ F_i(p, 0) &= \hat{F}_i(p). \end{aligned}$$

In view of the local gradient estimates for mean curvature flow in [2] the surfaces $\hat{N}_{i,\epsilon} = F_i(\cdot, \epsilon)(N^n)$ exist on some fixed time interval $[0, \epsilon_0)$ independent of i and remain graphs over N_0^n in Gaussian adapted coordinates. These graphs therefore satisfy uniformly parabolic quasilinear equations with initial data in $C^{1,\beta} \cap W^{2,p}$. By interior parabolic Schauder regularity theory (see e.g. [11], section IV) the curvature of the surfaces $\hat{N}_{i,\epsilon}$ satisfies

$$|A_{i,\epsilon}| \leq \frac{c}{\epsilon^{1/2-\beta/2}},$$

where c is a constant uniform in i, depending on the $C^{1,\beta}$ regularity of N_0^n. Furthermore, for each i and $1 \leq p < \infty$ we can compute from the evolution equation for the second fundamental form,

$$\frac{\partial}{\partial \epsilon}|A_{i,\epsilon}|^2 = \Delta |A_{i,\epsilon}|^2 - 2|\nabla A_{i,\epsilon}|^2 + 2|A_{i,\epsilon}|^4,$$

that

$$\begin{aligned} \frac{d}{d\epsilon}\int_{\hat{N}_{i,\epsilon}} |A_{i,\epsilon}|^p \, d\mu &\leq -p(p-1)\int_{\hat{N}_{i,\epsilon}} |A_{i,\epsilon}|^{p-2}|\nabla A|^2 \, d\mu + p\int_{\hat{N}_{i,\epsilon}} |A_{i,\epsilon}|^2 |A_{i,\epsilon}|^p \, d\mu \\ &\leq \frac{c^2\, p}{\epsilon^{1-\beta}}\int_{\hat{N}_{i,\epsilon}} |A_{i,\epsilon}|^p \, d\mu. \end{aligned}$$

It follows by Gronwall's Lemma that hence

$$\int_{\hat{N}_{i,\epsilon}} |A_{i,\epsilon}|^p \, d\mu \leq \exp\left(\frac{c^2\, p}{\beta}\epsilon_0^\beta\right)\int_{\hat{N}_i} |A_{i,\epsilon}|^p \, d\mu \leq c(p)$$

uniformly in i for $0 < \epsilon < \epsilon_0$. In view of this uniform $W^{2,p}$-estimate we obtain for $i \to \infty$ a solution $\tilde{F} : N^n \times (0, \epsilon_0) \to \mathbb{R}^{n+1}$ of mean curvature flow still satisfying

$$|A_\epsilon| \leq \frac{c}{\epsilon^{1/2-\beta/2}}$$

on $(0, \epsilon)$ and converging to N_0^n in $C^{1,\beta} \cap W^{2,p}$ for all $0 < \beta < 1$, $1 \leq p < \infty$ as $\epsilon \to 0$. Note that the solution of mean curvature flow is unique in this class. It follows in particular that $H_\epsilon \to H$ strongly in L^p as $\epsilon \to 0$, $1 \leq p < \infty$, and similarly, $H_{\epsilon -} = \min(H_\epsilon, 0) \to H_- = \min(H, 0)$ strongly in L^p, $1 \leq p < \infty$. We may then use the evolution equation for the mean curvature $\frac{\partial}{\partial \epsilon}H_{i,\epsilon} = \Delta H_{i,\epsilon} + |A_{i,\epsilon}|^2 H_{i,\epsilon}$ and Gronwalls lemma for the L^2-norm of H_- to conclude that

$$\int_{N_\epsilon^n} |H_{\epsilon -}|^2 \, d\mu \leq \exp(c\,\epsilon_0^\beta)\int_{N_0^n} |H_-|^2 \, d\mu = 0,$$

proving that $H_\epsilon \geq 0$ for all $0 < \epsilon < \epsilon_0$. By the strong maximum principle and the compactness of $N_\epsilon^n \subset \mathbb{R}^{n+1}$ it follows that $H_\epsilon > 0$ for all $0 < \epsilon < \epsilon_0$ as required.

Proof of theorem 2.5. Given N_0^n, let $\tilde{N}_\epsilon^n$ be the family of approximating surfaces of positive mean curvature constructed in lemma 2.6. For each $0 < \epsilon < \epsilon_0$ IMCF (0.1) has a global smooth solution in view of theorem 1.1 and corollary 2.3, see also [3]. As the estimates in theorem 1.1 and all resulting higher regularity estimates are uniform in ϵ for each positive fixed $t > 0$, we may let $\epsilon \to 0$ and obtain the desired global solution to IMCF. It approaches the initial data uniformly for $t \to 0$ in view of the estimate on the speed in theorem 1.1 and in view of the fact that N_t can be written as a graph of bounded gradient over N_0.

Finally we apply theorem 2.5 to weak solutions of IMCF in $\mathbb{R}^{n+1}$ to show that weak solutions are smooth outside some compact region:

Theorem 2.7 *Let $N_t^n = \partial E_t$, $E_t = \{x \in \mathbb{R}^{n+1} | \, u(x) < t\}$ for some function $u : \mathbb{R}^{n+1} \to \mathbb{R}$, $u \in C^{0,1}_{loc}$, $u|_{\partial E_0} = 0$, be a weak (level set) solution of IMCF as in [6] with compact initial data $E_0 \subset \mathbb{R}^{n+1}$. Then the following is true:*
a) If there is $t_0 \geq 0$ such that N_0^n is C^1 and strictly star-shaped as in (1.1), then $\{N_t^n\}_{t>t_0}$ is a smooth solution of IMCF satisfying all estimates of theorem 1.1, theorem 2.1 and corollary 2.3.
b) Given E_0 there indeed exists some $t_0 \geq 0$, such that $N_{t_0}^n$ is strictly star-shaped and C^1. In particular, there is a compact set $K \subset \mathbb{R}^{n+1}$ depending only on the initial data E_0, such that the solution $\{N_t^n\}$ is smooth in $\mathbb{R}^{n+1} \backslash K$.

Proof. a) All level sets of a weak solution have non-negative, bounded, measurable mean curvature, see [7]. Since $N_{t_0}^n$ is assumed to be C^1 and strictly star-shaped, we may use $N_{t_0}^n$ as initial data in theorem 2.5 to obtain a smooth solution $\{\tilde{N}_t^n\}_{t>t_0}$ of IMCF with strictly positive mean curvature outside $N_{t_0}^n$. By [5], lemma 2.3 (see also lemma 1.1) $\{\tilde{N}_t^n\}_{t\geq t_0}$ is a weak solution in the set $R^n \backslash E_{t_0}$. By restriction, $\{N_t^n\}_{t\geq t_0}$ is also a weak solution in $\bar{R}^n \backslash E_{t_0}$. Since the level sets are compact, the uniqueness theorem for weak solutions [5, lemma 2.2] then shows that $\tilde{N}_t = N_t$ for all $t \geq t_0$, proving a).
b) In [5], theorem 7.1, it was shown that the blowdown of a weak solution to IMCF with compact level sets converges to the standard expanding sphere solution. In particular, the scaled-down level sets converge to spheres in C^1, so N_t^n is star-shaped for large t. Therefore a) applies to conclude the proof of the theorem.

Remark 2.8 a) Theorem 2.1 carries over directly to analogous results for IMCF in general smooth Riemannian manifolds, since the additional curvature terms in the evolution equations for H and h_{ij} are of lower order.
b) Chow-Gulliver [1] have shown that a smooth family of surfaces that obeys an outward parabolic flow must become star-shaped with respect to x by the time it has left the smallest ball $B_R(x)$ that contains N_0. We believe that their method can be made to work in Euclidean space for variational weak solutions as well by employing the comparison principle implied by lemma 2.3 in [7]. Their estimates of the slope as a radial graph lead to area estimates that imply that IMCF becomes smooth before the time

$$t_* = C + \log\left(\frac{\mathrm{diam}(N_0)^n}{|N_0|}\right).$$

where C is an explicit constant, in fact $C = \log(2 \cdot 4^n n\, \omega_n)$.

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