A Note on the Hamiltonian Area Conjecture (Draft)

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The aim of this note is to prove the following result.

Theorem. *Let T be a smooth, immersed torus in $\mathbf{C}^2$, Hamiltonian isotopic to the Clifford torus T_c. If T is stationary for area under Hamiltonian variations, then*

$$|T| \geq |T_c|,$$

where $|T|$ denotes the area of T.

Supplemented by the existence and regularity of an area minimizer in this class, our result would imply the the 2-dimensional case of the *Hamiltonian Area Conjecture* [O1], which states that the Clifford torus in C^n minimizes area in its Hamiltonian isotopy class. Progress has been made toward such minimization problems in [SW].

We build on a result of Minicozzi [Mi], who showed that the above Theorem is a consequence of the Willmore Conjecture [W,Mi,T] in this Hamiltonian isotopy class. Our proof has three easy steps. First, we express the mean curvature form in the conformally equivalent flat metric on T, and find its (constant) length using Minicozzi's observation. We apply the Isoperimetric Inequality to symplectic integrals to give a lower bound for the lengths of various curves in this flat metric. Finally we estimate the lattice in the universal cover to estimate the area of a fundamental domain, without, however, bounding the conformal type itself.

Preliminaries. We state a few definitions and facts about Lagrangian submanifolds and area. For more complete information see [O1,O2,Mo,Mi].

Let J denote the usual complex structure of $\mathbf{C}^2$ and ω the standard symplectic form, so $\omega(X, Y) = \langle JX, Y\rangle$. We identify forms with vectors using the metric whenever this is unambiguous. We use coordinates $p = (x^1 + iy^1, x^2 + iy^2)$.

An immersed 2-surface T in $\mathbf{C}^2$ is called *Lagrangian* if $\omega|T_pT$ vanishes for each $p \in T$, or equivalently, if $J(T_pT) = (T_pT)^\perp$, so that

$$J : T_pT \to (T_pT)^\perp$$

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defines an isometry.

A vector field of the form $X = J\eta$, where η is a closed (resp. exact) 1-form defined on some open set, is called *Lagrangian* (resp. *Hamiltonian.* An isotopy T_t of immersed surfaces is called *Lagrangian* or *Hamiltonian* if its velocity vector has the corresponding property in an immersed tubular neighborhood of T_t. Note that if T is Lagrangian, the normal component of the motion X is determined solely by η pulled back to T.

The property of being Lagrangian is invariant under Lagrangian isotopy. We write $T \sim T'$ if T is Hamiltonian isotopic to T'.

When T is Lagrangian, we define a 1-form h on T by rotating the mean curvature vector H into T_pT by the formula

$$h = -JH.$$

It follows from the Codazzi identity and the Lagrangian condition that

$$dh = 0,$$

so h represents a cohomology class $[h]$ in $H^1(T)$, called the Maslov class. This class is invariant under Lagrangian isotopy.

A surface T is Hamiltonian stationary for area if

$$\frac{d}{dt}\bigg|_{t=0} |T_t| = \int_T \operatorname{div}_T X = \int_T -\langle H, X\rangle = 0 \tag{1}$$

for any Hamiltonian vector field $X = Jdf$, $f : T \to \mathbf{R}$. Here T_t is a Hamiltonian isotopy starting from $T_0 = T$ with initial velocity X, and $\operatorname{div}_T X$ is the divergence of X along T. It follows by rotating into T_pT and integrating by parts that

$$d^*h = 0,$$

on a Hamiltonian stationary surface.

Define the action 1-form by

$$\tilde{\alpha} := \frac{1}{2}\sum_{i=1,2}(x^i dy^i - y^i dx^i), \qquad \alpha := i_T^*(\tilde{\alpha})$$

where i_T is the inclusion map of the Lagrangian surface T. Then $d\tilde{\alpha} = \omega$, and $d\alpha = 0$, so α represents a cohomology class on T. Note that

$$\tilde{\alpha}\big|_p = \frac{Jp}{2}, \quad p \in \mathbf{C}^2, \, \alpha\big|_p = \frac{Jp^{\perp}}{2}, \quad p \in T,$$

where $p^\perp$ is the projection of p to T_pT. The 1-form α records the symplectic area enclosed by a given curve γ, since by Stokes' Theorem

$$\int_\gamma \alpha = \int_D \omega,$$

where D is any surface in $\mathbf{C}^2$ with $\partial D = \gamma$. It follows that the class $[\alpha]$ is invariant under Hamiltonian variations of T.

Our Clifford torus is the surface

$$T_c := S^1(\sqrt{2}) \times S^1(\sqrt{2}) \subseteq \mathbf{C} \times \mathbf{C},$$

which lies in a sphere of radius 2 and has area $8\pi^2$. (The usual Clifford torus lies in the unit sphere and has area $2\pi^2$.) Write γ_1, γ_2 for curves homotopic to each of the two S^1 factors. Then

$$\int_{\gamma_1} h = \int_{\gamma_2} h = \int_{\gamma_1} \alpha = \int_{\gamma_2} \alpha = 2\pi, \tag{2}$$

so $h = \alpha$ on the Clifford torus T_c. For any $T \sim T_c$, by the invariance of $[h]$ and $[\alpha]$, we still have

$$[h] = [\alpha] \tag{3}$$

in $H^1(T)$.

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Proof of the Main Theorem

Let g be the metric induced on T by the inclusion in $\mathbf{C}^2$, and let g_0 be the conformally equivalent flat metric of the same area. Quantities associated with g_0 will be given a subscript.

Lemma. *Let $T \sim T_c$ and suppose T is Hamiltonian stationary for area. Then*

$$g_0 := |h|^2 g.$$

Furthermore, h is parallel with respect to g_0, and $|h|_0 = 1$.

Proof. Recall that stationarity implies

$$dh = 0, \qquad d^*h = 0$$

in the metric g. These equations are conformally invariant, so they hold equally well in the flat metric g_0. Thus h is a harmonic 1-form, and therefore parallel, with respect to the flat metric g_0. In particular, $|h|_0$ is constant.

Let us deduce this constant following Minicozzi [M]. By (3), $h - \alpha$ is exact on T, so the vector field

$$X := J(h-\alpha) = H + \frac{p^\perp}{2}$$

is Hamiltonian. We insert X into (1) to obtain *Minicozzi's Lemma*, which avers

$$\int_T |H|^2 = \int_T -\left\langle H, \frac{p^\perp}{2} \right\rangle = \int_T \operatorname{div}_T \left(\frac{p}{2}\right) = |T|,$$

for a Hamiltonian stationary surface in the Hamiltonian isotopy class of T_c. Evidently, then

$$|T|_0 = |T| = \int_T |h|^2 \, d\mu = \int_T |h|_0^2 \, d\mu_0 = |h|_0^2 |T|_0,$$

which shows $|h|_0 = 1$. This implies that

$$\frac{g_0}{g} = \frac{|h|^2}{|h|_0^2} = |h|^2.$$

□

Next, we apply the Isoperimetric Inequality to get a lower bound for the length of curves in T. Since α records a minimum area spanned by a given curve, the curves can't get too short. Let γ_1 and γ_2 now denote isotoped versions of the original generating circles.

Lemma (Lower Bound of Curves). *Let γ be a curve in the homology class $k\gamma_1 + \ell\gamma_2$, $k, \ell \in \mathbf{Z}$. Then*

$$|\gamma|_0^2 \geq 8\pi^2 |k + \ell|. \tag{4}$$

Remark. Since inequality (4) is quadratic on one side and linear on the other, the best information is obtained for small values of k and ℓ. Of course, (4) can't see curves in the kernel of $[\alpha]$, but we will get around this later.

Proof. Let D be an area-minimizing disk in $\mathbf{C}^2$ spanned by γ [ref?]. By the Isoperimetric Inequality [ref?] and (2), we have

$$|\gamma|^2 \geq 4\pi |D| \geq 4\pi \left| \int_D \omega \right| = 4\pi \left| \int_\gamma \alpha \right| = 4\pi(2\pi|k+\ell|) = 8\pi^2 |k+\ell|.$$

The proof is completed by applying the next Lemma. □

Lemma. *Suppose that (T, g_0) is a flat 2-torus and (T, g) a conformally equivalent torus of the same area. Then in each homotopy class β, we have*

$$\inf_{\gamma\in\beta} |\gamma| \leq \inf_{\gamma\in\beta} |\gamma|_0.$$

Proof. This Lemma should be well known. [ref?] There are two convenient proofs.

(1) Under the Ricci flow

$$\frac{\partial g}{\partial t} = (-R + \frac{1}{|T|}\int_T R\, d\mu_t)$$

the area is preserved and g converges smoothly to g_0 as $t \to \infty$ [Ha1]. Furthermore, the length of the shortest geodesic in each homotopy class is nondecreasing [Ha2], so we obtain the Lemma.

(2) Alternately, we can prove it directly by Fubini's theorem and the Schwarz inequality. Let γ_0 be a shortest g_0-geodesic in the class β, with length $a := |\gamma_0|_0$, and let let γ_t, $0 \leq t \leq b$ be parallel copies of γ_0 that foliate P. We can choose the parameter t so that $d\mu_0 = ds\, dt$ where ds is g_0-arclength along γ_t. Then $|T|_0 = ab$, and writing $g = \lambda^2 g_0$, we have

$$\begin{aligned}
\int_0^b |\gamma_t|\, dt &= \int_0^b \int_0^a \lambda\, ds\, dt \\
&\leq \left(\int_0^b\int_0^a \lambda^2\, ds\, dt\right)^{1/2} \left(\int_0^b\int_0^a ds\, dt\right)^{1/2} \\
&= |T|^{1/2}|T|_0^{1/2} = |T|_0 = ab,
\end{aligned}$$

so

$$\inf_{\gamma\in\beta} |\gamma| \leq \frac{1}{b}\int_0^b |\gamma_t|\, dt = \frac{ab}{b} = |\gamma_0|_0,$$

which proves the Lemma. □

Proof of Main Theorem. Let $T \sim T_c$ be Hamiltonian stationary for area. We must prove $|T| \geq 8\pi^2$.

Let $\tilde{T}$ be the universal cover of T, equipped with the lift of the flat metric g_0 by the covering map π. Pick an origin 0 in $\tilde{T}$ and let Γ be the lattice $\pi^{-1}(\pi(0))$. Then $\tilde{T}$ gets a vector space structure and $T = \tilde{T}/\Gamma$.

Let γ_1, γ_2 be the generators of Γ isotopic to the original generators, so that a typical element of Γ has the form

$$\gamma = k\gamma_1 + \ell\gamma_2$$

using vector space addition in $\tilde{T}$.

Let $\theta : \tilde{T} \to \mathbf{R}$ be the potential with $\theta(0) = 0$, $d\theta = h = \alpha$. Let L_a be the straight line $\{\theta = a\}$ in $\tilde{T}$. Since $|d\theta|_0 = 1$, the lines

$$\ldots, L_0, L_{2\pi}, L_{4\pi}, \ldots \tag{5}$$

ara parallel lines at distance 2π. According to (2),

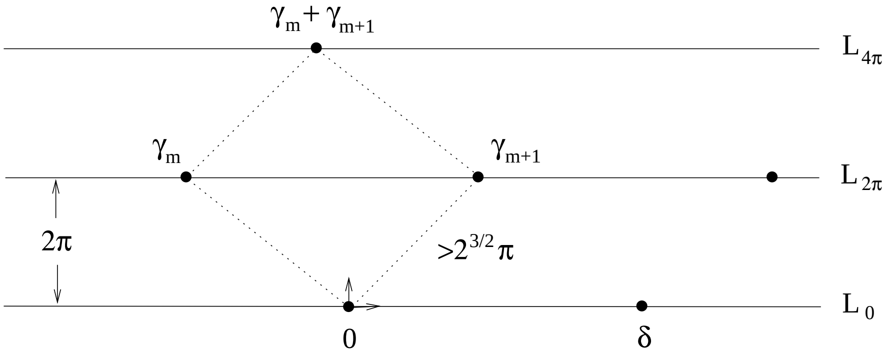
$$\theta(k\gamma_1 + \ell\gamma_2) = \int_0^{k\gamma_1+\ell\gamma_2} h = 2\pi(k + l),$$

so $\Gamma \cap L_{2\pi}$ consists of the points

$$\ldots, \gamma_1 - \delta, \gamma_1, \gamma_1 + \delta, \gamma_1 + 2\delta, \ldots$$

with $\delta := \gamma_2 - \gamma_1$, and constant spacing equal to $|\delta_|0$.

Let δ_1, δ_2 be the orthonormal basis of $\tilde{T}$ with $\delta_1 := \delta/|\delta_0 \in L_0$, $\langle \delta_2, \cdot \rangle_0 = h$.



Fundamental Domain.

By (4), each point $\gamma_m := \gamma_1 + (m-1)\delta$ in $\Gamma \cap L_{2\pi}$ has length at least

$$|\gamma_m|_0 \geq 2\sqrt{2}\pi,$$

whereas

$$\langle \gamma_m, \delta_2 \rangle = 2\pi,$$

so by the Pythagorean theorem, $|\langle \gamma_m, \delta_1 \rangle_0| \geq 2\pi$. Then there exists a unique m such that

$$\langle \gamma_m, \delta_1 \rangle_0 \leq -2\pi, \qquad \langle \gamma_{m+1}, \delta_1 \rangle_0 \geq 2\pi,$$

which implies

$$|\delta|_0 = |\gamma_{m+1} - \gamma_m|_0 \geq 4\pi,$$

and by (5),

$$\mathrm{dist}_0(\gamma_{m+1} + \gamma_m, L_0) = 4\pi,$$

so using the parallelogram with corners 0, γ_m, γ_{m+1}, $\gamma_m + \gamma_{m+1}$ as a fundamental domain, we find that

$$|T| = |T|_0 \geq \frac{1}{2}(4\pi)(4\pi) = 8\pi^2,$$

as required. □

Questions. 1. Is the Clifford torus the unique Hamiltonian stationary torus in its Hamiltonian isotopy class?

2. Is there a Hamiltonian isotopy of $\mathbf{C}^n$ that takes T_c to T_c, while taking γ_m to γ_{m+k}, for some $k \neq 0$?

If not, then the value of m in the proof gives an integer invariant on the class of Hamiltonian stationary tori Hamiltonian isotopic to the Clifford torus. If so, then it is likely that there is a saddle point somewhere along an optimized isotopy. Of course, both alternatives will be moot if the Clifford torus is unique.

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