Gaussian densities and stability for some Ricci solitons

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Perelman [Pe02] has discovered a remarkable variational structure for the Ricci flow: it can be viewed as the gradient flow of the entropy functional λ. There are also two monotonicity formulas of shrinking or localizing type: the shrinking entropy ν, and the reduced volume. Either of these can be seen as the analogue of Huisken's monotonicity formula for mean curvature flow [Hu90]. In various settings, they can be used to show that centered rescalings converge subsequentially to shrinking solitons, which function as idealized models for singularity formation.

In this note, we exhibit the second variation of the λ and ν functionals, and investigate the linear stability of examples. We also define the "central density" of a shrinking Ricci soliton (*shrinker*) and compute its value for certain examples in dimension 4. Using these tools, one can sometimes predict or limit the formation of singularities in the Ricci flow. In particular, we show that certain Einstein manifolds are unstable for the Ricci flow in the sense that generic perturbations acquire higher entropy and thus can never return near the original metric. A detailed version of the calculations summarized in this announcement will follow in [CHI].

In §1, we investigate the stability of Perelman's λ-functional. Its critical points are steady solitons (Ricci flat in the compact case). We compute the second variation $\mathcal{D}^2\lambda$; the corresponding Jacobi field operator L is a degenerate negative elliptic integro-differential operator. In fact, L equals half the Lichnerowicz Laplacian Δ_L on divergence-free symmetric tensors, and zero on Lie derivatives. This fact and further investigations of the second variation have been reported by Perelman [Pe03]. We call a steady soliton *linearly stable* if $L \leq 0$, otherwise *linearly unstable*. If g is linearly unstable, then g can be perturbed so that $\lambda(g) > 0$, which will destabilize it utterly: it will decay into a

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cacophony of shrinkers and disappear in finite time. One observes that $\lambda(g) \leq 0$ for any metric on the torus T^n, $n \leq 7$; in fact this is equivalent to the positive mass theorem. By Guenther, Isenberg and Knopf [GIK02] every $K3$ surface is linearly stable; more generally, by Dai, Wang and Wei [DWW04] any manifold with a parallel spinor is linearly stable. Other cases are open.

In §2, we investigate the stability of the ν-functional, whose critical points are shrinkers. The Jacobi field operator N of ν is like L but with lower order terms. We call a shrinker *linearly stable* if $N \leq 0$. Again, N is closely related to the Lichnerowicz Laplacian. We observe that $\mathbb{CP}^N$ (with the standard metric) is linearly stable, but all other compact complex surfaces with $c_1 > 0$ are linearly unstable. Using results of Gasqui and Goldschmidt [GG96, GG91] the complex hyperquadric Q^3 (a Hermitian symmetric space) is linearly unstable. This implies that Q^3 is irremediably unstable in the sense that a generic (non-Kähler!) perturbation of Q^3 will never approach the original geometry of Q^3 at any scale or time. On the other hand, the hyperquadric Q^4 is linearly stable. Other cases are open.

A notion of central density (or gaussian density) of a shrinker can be defined from either of Perelman's monotonicity formulas; we call these notions Θ and ν. On a shrinker, the two definitions are equivalent via $\Theta = e^\nu$. For a general solution, e^ν is a lower bound for the central density of any shrinker that arises later as a singularity model, which restricts the shrinkers that may occur later. This is presented in §3.

The central density of certain standard 4-dimensional examples are exhibited in a table in §4.

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1 Second Variation of the Entropy λ

The second variation of the Einstein functional is positive in the conformal direction but negative in all other directions. The glory of Perelman's entropy is that there is a preliminary minimization over scalar functions that absorbs nearly all the positive directions: the Jacobi field operator is a linear integro-differential operator with nonpositive symbol.[1]

Fix a compact manifold (M, g). Define [Pe02]

$$\mathcal{F}(g, f) := \int e^{-f}(|Df|^2 + R).$$

Define the *entropy*

$$\lambda(g) := \inf\{\mathcal{F}(g, f) : f \in C_c^\infty(M), \int e^{-f} = 1\}$$

[1] It is convenient that this happens without changing the metric via a conformal change. So the scalar variations do not affect the background geometry. Instead they satisfy a *linear* PDE.

The infimum is achieved by a function f solving

$$-2\Delta f + |Df|^2 - R = \lambda(g).$$

Now consider variations $g(s) = g + sh$. Following Perelman, the first variation $\mathcal{D}_g\lambda(h)$ of λ is given by

$$\left.\frac{d}{ds}\right|_{s=0} \lambda(g(s)) = \int e^{-f}(-Rc - D^2 f) : h,$$

where f is the minimizer. A stationary point satisfies

$$Rc + D^2 f = 0,$$

which implies that g is a *(gradient) steady soliton*, that is, the Ricci flow with initial condition g satisfies

$$g(t) = \phi_t^*(g)$$

where ϕ_t is a family of diffeomorphism generated by the gradient vector field Df. In fact, any compact steady is Ricci flat with $f = 0$, $\lambda = 0$.

Note by diffeomorphism invariance of λ that $\mathcal{D}_g\lambda$ vanishes on any Lie derivative $h = L_x g$. From this, by inserting $h = -2(Rc + D^2 f)$ one recovers Perelman's wonderful result that $\lambda(g(t))$ is nondecreasing on any Ricci a Ricci flow, and is constant if and only if $g(t)$ is a steady soliton.

We prove the following. Write $Rm(h,h) := R_{ijkl}h_{ik}h_{jl}$, $\operatorname{div}\omega := D_i\omega_i$, $(\operatorname{div} h)_i := D_j h_{ji}$, $(\operatorname{div}^*\omega)_{ij} = -(D_i\omega_j + D_j\omega_i)/2 = -(1/2)L_{\omega^\#}g_{ij}$.

1.1 Theorem *The second variation $\mathcal{D}_g^2\lambda(h,h)$ of λ on a compact Ricci flat manifold is given by*

$$\begin{aligned}\left.\frac{d^2}{ds^2}\right|_{s=0} \lambda(g(s)) &= \int -\frac{1}{2}|Dh|^2 + |\operatorname{div} h|^2 - \frac{1}{2}|Dv_h|^2 + Rm(h,h)\\ &= \int Lh : h,\end{aligned}$$

where

$$Lh := \frac{1}{2}\Delta h + \operatorname{div}^*\operatorname{div} h + \frac{1}{2}D^2 v_h + Rm(h,\cdot),$$

and v_h satisfies

$$\Delta v_h = \operatorname{div}\operatorname{div} h.$$

The symbol of L in the direction $\xi \in T_x^*M$ is

$$\sigma_\xi(h) = -\pi_{\xi^\perp}(h),$$

where $\pi_{\xi^\perp}(h)$ restricts h to the hyperplane $\xi^\perp$. So the operator L is degenerate negative elliptic, and has a discrete spectrum with at most a finite-dimensional space of positive eigenfunctions.

Decompose $C^\infty(\mathrm{Sym}^2(T^*M))$ as

$$\ker \mathrm{div} \oplus \mathrm{im}\, \mathrm{div}^* .$$

One verifies that L vanishes on $\mathrm{im}\, \mathrm{div}^*$, that is, on Lie derivatives. On $\ker \mathrm{div}$ one has

$$L = \frac{1}{2}\Delta_L$$

where

$$\Delta_L h := \Delta h + 2Rm(h, \cdot) - Rc \cdot h - h \cdot Rc$$

is the Lichnerowicz Laplacian on symmetric 2-tensors.

We call a critical point g of λ *linearly stable* if $L \leq 0$, and a *maximizer* if $\lambda(g_1) \leq \lambda(g)$ for all g_1. A compact Ricci flat metric is a maximizer if and only if it admits no metric of positive scalar curvature. (This follows from Schoen's solution of the Yamabe problem [S84].) Evidently a maximizer is stable. If g is not stable, then a slight perturbation will develop $\lambda > 0$ and $R > 0$ and (in principle) disappear in finite time as positive manifolds do. A good question is whether any Ricci-flat manifold is unstable. We call this the *positive mass problem for Ricci flat manifolds.*

1.2 Example T^n admits no metric of positive scalar curvature by the positive mass theorem, so $\lambda(g) \leq 0$ for all g on T^n.

1.3 Example A Calabi-Yau K3 surface and more generally, any manifold with a parallel spinor has $\Delta_L \leq 0$ [GIK02, DWW04]. So these manifolds are linearly stable in the sense presented here.

1.4 Example Let g be compact and Ricci flat. Following [B84, GIK02] we examine conformal variations. It is convenient to replace ug by

$$h = Su := (\Delta u)g - D^2 u$$

which differs from the conformal direction only by a Lie derivative and is divergence free. We have

$$\Delta_L Su = (S\Delta u)g,$$

so Δ_L has the same eigenvalues as Δ. In particular, $N \leq 0$ in the conformal direction. This contrasts with the Einstein functional.

2 Second Variation of the Shrinker Entropy ν

Fix a complete manifold (M, g). Define

$$\mathcal{W}(g, f, \tau) := \frac{1}{(4\pi\tau)^{n/2}} \int e^{-f} \left[\tau(|Df|^2 + R) + f - n\right] dV.$$

Define the *shrinker entropy* by

$$\nu(g) := \inf\{\mathcal{W}(g,f,\tau) : f \in C_c^\infty(M), \tau > 0, \frac{1}{(4\pi\tau)^{n/2}} \int e^{-f} = 1\}$$

Assume that M is compact or is asymptotic at infinity to a metric cone over a smooth, compact Riemannian manifold. One checks that $\nu(g)$ is realized by a pair (f,τ) that solve the equations

$$\tau(-2\Delta f + |Df|^2 - R) - f + n + \nu = 0, \qquad \frac{1}{(4\pi\tau)^{n/2}} \int f e^{-f} = \frac{n}{2} + \nu,$$

and f grows quadratically.

Consider variations $g(s) = g + sh$ where h is smooth of compact support. Following Perelman, one calculates the first variation $\mathcal{D}_g\nu(h)$ to be

$$\left.\frac{d}{ds}\right|_{s=0} \nu(g(s)) = \frac{1}{(4\pi\tau)^{n/2}} \int e^{-f}(\tau(-Rc - D^2 f) + g/2) : h.$$

A stationary point of ν satisfies

$$D^2 f + Rc - \frac{g}{2\tau} = 0 \tag{1}$$

which says that g is a (gradient) shrinker, that is, its Ricci flow $g(t)$ has the form

$$g(t) := (T-t)\psi_t^*(g)), \qquad t < T,$$

where ψ_t are the diffeomorphisms generated by $-Df$, and $\tau = T - t$.

As before, $\mathcal{D}_g\nu$ vanishes on Lie derivatives. By scale invariance it vanishes on multiples of the metric. Inserting $h = -2(Rc + D^2 f - g/2\tau)$, one recovers Perelman's brilliant formula that finds that $\nu(g(t))$ is monotone on a Ricci flow, and constant if and only if $g(t)$ is a gradient shrinker.

A positive Einstein manifold is a shrinker with $f \equiv n/2$, normalized by $Rc = g/2\tau$. We compute:

2.1 Theorem *Let (M,g) be a positive Einstein manifold. The second variation $\mathcal{D}_g^2\nu(h,h)$ is given by*

$$\begin{aligned}\left.\frac{d^2}{ds^2}\right|_{s=0} \nu(g(s)) &= \frac{\tau}{\mathrm{vol}(g)} \int -\frac{1}{2}|Dh|^2 + |\operatorname{div} \quad h|^2 - \frac{1}{2}|Dv_h|^2 + Rm(h,h) + \frac{v_h^2}{4\tau} \\ &\qquad -\frac{1}{2n}\left(\frac{1}{\mathrm{vol}(g)}\int \mathrm{tr}_g\, h\right)^2, \\ &= \frac{\tau}{\mathrm{vol}(g)} \int Nh : h,\end{aligned}$$

where

$$Nh := \frac{1}{2}\Delta h + \operatorname{div}^* \operatorname{div} h + \frac{1}{2}D^2 v_h + Rm(h,\cdot) - \frac{g}{2n\tau \operatorname{vol}(g)} \int \operatorname{tr}_g h.$$

and v_h is the unique solution of

$$\Delta v_h + \frac{v_h}{2\tau} = \operatorname{div}\operatorname{div} h, \qquad \int v_h = 0.$$

There is a strictly more complicated formula in the case of non-Einstein shrinkers.

As in the previous case, N is degenerate negative elliptic and vanishes on $\operatorname{im}\operatorname{div}^*$. Write

$$\ker \operatorname{div} = (\ker \operatorname{div})_0 \oplus \mathbb{R}g$$

where $(\ker \operatorname{div})_0$ is defined by $\int \operatorname{tr}_g h = 0$. Then on $(\ker \operatorname{div})_0$ we have

$$N = \frac{1}{2}\left(\Delta_L - \frac{1}{\tau}\right)$$

where Δ_L is the Lichnerowicz Laplacian. So the linear stability of a shrinker comes down to the (divergence free) eigenvalues of the Lichnerowicz Laplacian. Let us write μ_L for the maximum eigenvalue of Δ_L on symmetric 2-tensors and μ_N for the maximum eigenvalue of N on $(\ker \operatorname{div})_0$,

2.2 Example The round sphere is *geometrically stable* (i.e. nearby metrics are attracted to it up to scale and gauge) by the results of Hamilton [Ha82, Ha86, Ha88] and Huisken [Hu88]. In particular it is linearly stable: $\mu_N = -2/(n-1)\tau < 0$.

2.3 Example For $\mathbb{CP}^N$, the maximum eigenvalue of Δ_L on $(\ker \operatorname{div})_0$ is $\mu_L = 1/\tau$ by work of Goldschmidt [G04], so $\mathbb{CP}^N$ is neutrally linearly stable, i.e. the maximum eigenvalue of N on $(\ker \operatorname{div})_0$ is $\mu_N = 0$.

Any product of two nonflat shrinkers $N_1^{n_1} \times N_2^{n_2}$ is linearly unstable, with $\mu_N = 1/2\tau$. The destabilizing direction $h = g_1/n_1 - g_2/n_2$ corresponds to a growing discrepancy in the size of the factors.

More generally, any compact Kähler shrinker with $\dim H^{1,1}(M) \geq 2$ is linearly unstable. Again, this can be seen directly: a small perturbation into a non-canonical Kähler class will move in a straight line nearly toward the vertex of the Kähler cone, hence away from the canonical class (in a scale invariant sense). If M is Kähler-Einstein, we compute μ_N as follows. Let σ be a harmonic 2-form and h be the corresponding metric perturbation; then $\Delta_L h = 0$, and if σ is chosen perpendicular to the Kähler form, then as above we obtain $\mu_N = 1/2\tau$.

A complete list of compact complex surfaces with $c_1 > 0$ is $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, and $\mathbb{CP}^2 \# k(-\mathbb{CP}^2)$, $k = 1, \ldots 8$. Each of these has a unique Kähler shrinker metric (Kähler-Einstein unless $k = 1, 2$). By the above, all are linearly unstable except $\mathbb{CP}^2$.

Let Q^N denote the complex hyperquadric in $\mathbb{CP}^{N+1}$ defined by

$$\sum_{i=0}^{N+1} z_i^2 = 0,$$

a Hermitian symmetric space of compact type, hence a positive Kähler-Einstein manifold. Then Q^2 is isometric to $\mathbb{CP}^1 \times \mathbb{CP}^1$, the simplest example of the above instability phenomenon.

2.4 Example Consider Q^3. It has $\dim H^{1,1}(Q^3) = 1$, so the above discussion does not apply. But the maximum eigenvalue of Δ_L on $(\ker \operatorname{div})_0$ is $\mu_L = -2/3\tau$ by work of Gasqui and Goldschmidt [GG96] (or see [GG04]). The proximate cause is a representation that appears in the sections of the symmetric tensors but not in scalars or vectors. Therefore, Q^3 is linearly unstable with

$$\mu_N = \frac{1}{6\tau}.$$

Since ν increases along the Ricci flow, this implies that a generic small perturbation of the Einstein metric $\bar{g}$ will grow and $g(t)$ will never return near $\bar{g}$ at any scale or time. We say that $\bar{g}$ is *geometrically unstable*. Now imagine that we start with a random metric on Q^3 and propose to use the Ricci flow to find an Einstein metric or other canonical geometry for Q^3. Assuming there are no other critical points, we find that the flow combusts in singularities of more elementary type and the topology of the underlying manifold simplifies drastically, unless it happens to get hung up at the Einstein metric ν. So the Ricci flow has fundamentally more complicated behavior than in dimension three, as one expects. Further exploration of this example will appear in [BGIM].

2.5 Example Let Q^4 be the 4-dimensional hyperquadric. The maximum eigenvalue of Δ_L on symmetric tensors is $\mu_L = -1/\tau$ by work of Gasqui and Goldschmidt [GG91] (or see [GG04]). So Q^4 is neutrally linearly stable: $\mu_N = 0$.

Let g be a positive Einstein metric, and let us examine conformal variations. As before, without loss replace ug by the divergence-free variation

$$h = Su := (\Delta u)g - D^2 u + \frac{ug}{2\tau}.$$

As before, $\Delta_L Su = (S\Delta u)g$. Thus Δ_L has the same eigenvalues as $\Delta_{\mathrm{fns}}|(\ker S)^{\perp}$. But $\ker S$ is empty except on round S^n, which is linearly stable. Note that $\mu_{\mathrm{fns}} \le -n/2(n-1)\tau$ with equality only on round S^n. So we have:

2.6 Proposition *A positive Einstein metric is linearly unstable for conformal variations if and only if the maximum eigenvalue of Δ on functions satisfies*

$$-\frac{1}{\tau} < \mu_{\mathit{fns}} < -\frac{n}{2(n-1)\tau}.$$

We do not know whether this inequality can ever be satisfied on a positive Einstein manifold.

3 The Central Density of a Shrinker

Our aim in this section is to define the central density of a gradient shrinker. First we define a suitable class of gradient shrinkers, then we review the two Perelman monotonicity formulas of shrinking type and apply them by taking the center point to be the parabolic vertex of the shrinker. Our principal result is that the two notions of density coincide.

A gradient shrinker solves

$$\partial g/\partial t = -2Rc, \qquad g(t) := -t\psi_t^*(g(-1)), \qquad t < 0,$$

where ψ_t are the diffeomorphisms generated by the gradient of a function $F(x,t)$. Differentiating the above expression yields

$$D^2F + Rc - \frac{g}{2\tau} = 0, \tag{2}$$

where $\tau = -t$. Normalizing F by adding a time-dependent constant, we obtain

$$\frac{\partial F}{\partial \tau} + |DF|^2 = 0. \tag{3}$$

Differentiating (2), taking the trace two ways, and applying Bianchi II and commutation rules yields $D(|DF|^2 + R - F/\tau) = 0$. Adding a further global constant to F leads to the classical *auxiliary equation*

$$|DF|^2 + R - \frac{F}{\tau} = 0. \tag{4}$$

Equations (2)-(4) are the fundamental equations for a gradient shrinker. Combining (2) and (3) yields the backward heat equation

$$\frac{\partial F}{\partial \tau} = \Delta F - |DF|^2 + R - \frac{n}{2\tau}. \tag{5}$$

In order to prove our results we need some analytic hypotheses on the metric of M. We assume that M is complete, connected, and κ-noncollapsed at all scales. We also assume that the curvature decays quadratically as $x \to \infty$. (This is satisfied, for example, by the blowdown shrinker $L(N,-1)$ [FIK04].) However, many of our results hold under the weaker hypothesis of bounded curvature. Under the quadratic decay hypothesis, $g(t)$ converges in the Gromov-Hausdorff sense as $t \nearrow 0$ to a metric cone C which is smooth except at the vertex, which we call 0. The convergence is smooth except on a compact set, which falls into the vertex, which we call 0. For a proof and further details, see [I].

We now wish to define a gaussian density centered at the parabolic vertex $(y,s) = (0,0)$ of the spacetime $\mathcal{M} := (M \times (-\infty, 0)) \cup (C \times \{0\})$. We may do this in two ways: via the shrinking entropy or the reduced volume, both due to Perelman [Pe02].

The reduced volume generalizes Bishop volume monotonicity to the space-time setting. For a smooth point (y,s) in any Ricci flow, define the *reduced distance* $\ell = \ell_{y,s}$ by

$$\ell(x,t) := \frac{1}{2\sqrt{\tau}} \inf_{\gamma} \int_0^{\tau} \sqrt{\sigma} \left(\left| \frac{d\gamma}{du} \right|^2 + R \right) d\sigma, \qquad t < s, \quad x \in M, \tag{6}$$

where the infimum is taken over all paths $(\gamma(u), u)$, $t \leq u \leq s$ that connect (x,t) to (y,s). The *reduced volume centered at* (y,s) is defined by

$$\theta_{y,s}(t) := \frac{1}{(4\pi\tau)^{n/2}} \int_M e^{-\ell(x,t)}\, dV_t(x).$$

Perelman wonderfully shows that $\theta_{y,s}(t)$ is increasing in t and is constant precisely on a gradient shrinker. Now define $\ell_{0,0}$ by passing smooth points (y_i, s_i) to $(0,0)$. We have:

3.1 Proposition *$\ell_{0,0}$ is well-defined and is locally Lipschitz on $\mathcal{M}$. For $t < 0$, $\theta_{0,0}(t)$ is well defined, constant, and contained in (0,1].*

This constant value we call the *central density of (M,g(t))* and denote

$$\Theta(M) = \Theta(M, g(\cdot)) := \theta_{0,0}(t), \qquad t < 0.$$

Next we turn to the shrinking entropy ν. Let $(M, g(t))$ be a smooth Ricci flow existing up to $t = s$ and set $\tau := s - t$. Let f solve the heat equation

$$\frac{\partial f}{\partial t} = \Delta f - |Df|^2 + R - \frac{n}{2\tau},$$

that came up for the soliton potential of a shrinker (5). Define u by

$$u := \frac{e^{-f}}{(4\pi\tau)^{n/2}}.$$

Remarkably, u solves the adjoint heat equation

$$\frac{\partial u}{\partial t} = \Delta u - Ru. \tag{7}$$

This leads to the conservation law

$$\frac{1}{(4\pi\tau)^{n/2}} \int e^{-f} = \int u = 1 \qquad \text{for } t < s.$$

Perelman has shown [Pe02]

$$\frac{\partial}{\partial t} \mathcal{W}(s-t, f(t), g(t)) = \frac{1}{(4\pi\tau)^{n/2}} \int 2e^{-f} \left| D^2 f + Rc - \frac{g}{2\tau} \right|^2 dV_t \geq 0.$$

and the right hand side vanishes precisely when $g(t)$ is a gradient shrinker and f is its soliton potential. (This shows, as mentioned above, that ν increases in general and is constant on a shrinker.)

If u emerges from a dirac source at a smooth point (y,s), we write $u = u_{y,s}$, $f = f_{y,s}$ and define the *shrinker entropy centered at* (y,s) by

$$\phi_{y,s}(t) := \mathcal{W}(s-t, f_{y,s}(t), g(t))$$

Passing smooth points (y_i, s_i) to $(0,0)$, we prove:

3.2 Proposition *$u_{0,0}$ is well-defined, smooth, and positive on $M \times (-\infty, 0)$ and solves equation (7). It satisfies*

$$\int_M u_{0,0} \equiv \frac{1}{(4\pi\tau)^{n/2}} \int e^{-f_{0,0}} = 1, \qquad t < 0. \tag{8}$$

Also, $\phi_{0,0}(t)$ is well-defined and lies in $(-\infty, 0]$ for all $t < 0$. In fact, it is constant, with

$$\phi_{0,0}(t) = \nu(M), \qquad t < 0.$$

Since $\phi_{0,0}(t)$ is constant, $f_{0,0}$ is a soliton potential and so

$$f_{0,0} = F + C \tag{9}$$

for some constant C depending only on $\mathcal{M}$.

We now wish to relate $\Theta(M)$ and $\nu(M)$ via F. In the process we determine the value of C, and sharpen on a shrinker the general Perelman relation $\ell_{0,0} \leq f_{0,0}$ [Pe02].

We begin with $\Theta(M)$. Using the symmetry of the shrinker and a simple comparison argument, one checks:

3.3 Proposition *The integral curves of F are minimizing $\mathcal{L}$-geodesics emanating from $(0,0)$.*

Then using the homothetic time-symmetry of the shrinker, one obtains after a straightforward computation:

$$\ell_{0,0} = \tau(|DF|^2 + R) = F = f_{0,0} - C,$$

and thus by the definition and (8), one gets:

$$\Theta(M) = e^C.$$

Next, we evaluate ν. Compute

$$\begin{aligned}
\nu(M) &= \phi_{0,0}(t) \\
&= \frac{1}{(4\pi\tau)^{n/2}} \int e^{-f_{0,0}} \left[\tau(|Df_{0,0}|^2 + R) + f_{0,0} - n\right] dV_t \\
&= \frac{1}{(4\pi\tau)^{n/2}} \int e^{-f_{0,0}} \left[\tau(\Delta f_{0,0} + R) + f_{0,0} - n\right] dV_t \\
&= \frac{1}{(4\pi\tau)^{n/2}} \int e^{-f_{0,0}} \left[f_{0,0} - n/2\right] dV_t
\end{aligned}$$

by integrating by parts and the trace of (2). On the other hand, by (4) and (9), the first integral expression also equals

$$\nu(M) = \frac{1}{(4\pi\tau)^{n/2}} \int e^{-f_{0,0}} \left[f_{0,0} - C + f_{0,0} - n\right] dV_t$$

We conclude that $\nu(M) = 2\nu(M) - C$, so $\nu(M) = C$. We summarize these results in a theorem.

3.4 Theorem *On a shrinker satisfying the above assumptions, we have*

$$\ell_{0,0} = F = f_{0,0} - \nu(M)$$

and

$$\Theta(M) = e^{\nu(M)}.$$

Details will appear in [CHI].

4 Table of Values

In this section we calculate Θ for some standard shrinkers. The computations are simplified by several observations. Normalize positive Einstein manifolds by $Rc = g/2\tau$, $\tau = 1/2(n-1)$, so that S^n has radius 1.

(1) For any shrinker M, $\Theta(M) \leq 1$ with equality if and only if $M = \mathbb{R}^n$.

(2) Let M be a Ricci flat cone with $g = dr^2 + r^2 g_\Sigma$ where Σ is positive Einstein. Then M is a shrinker (with interior singularity), and

$$\Theta(M) = \frac{\mathrm{vol}(\Sigma)}{\mathrm{vol}(S^n)}.$$

(3) If M is a positive Einstein manifold, then (for any τ)

$$\Theta(M) = \left(\frac{1}{4\pi\tau e}\right)^{n/2} \mathrm{vol}_\tau(M) \leq \Theta(S^n),$$

with equality if and only if $M = S^n$.

(4) $\Theta(S^n) = \left(\frac{n-1}{2\pi e}\right)^{n/2} \mathrm{vol}(S^n)$. By way of comparison, note that for mean curvature flow, $\Theta_{MCF}(S^n) = \left(\frac{n}{2\pi e}\right)^{n/2} \mathrm{vol}(S^n)$.[2]

(5) $\Theta(\mathbb{CP}^N) = \left(\frac{N+1}{\pi e}\right)^N \frac{\mathrm{vol}(S^{2N+1})}{2\pi}$.

(6) The positive Kähler-Einstein manifold $M = \mathbb{CP}^2 \# k(-\mathbb{CP}^2)$, $k = 0, 3, \ldots, 8$, has $\Theta(M) = (9-k)/2e^2$.

[2]Following an observation of White, we note (tantalizingly) that the respective limits as $n \to \infty$ are $\sqrt{2/e}$ and $\sqrt{2}$.

(7) $\Theta(M \times N) = \Theta(M)\Theta(N)$.

We say that one shrinker *decays* to another if there is a small perturbation of the first whose Ricci flow develops a singularity modelled on the second. Because the ν-invariant is monotone during the flow, decay can only occur from a shrinker of lower density to one of higher density. This creates a "decay lowerarchy". (It should be a partial order.)

We have computed the following density values in dimension 4. Note that the conclusion of Theorem 3.4 holds for all our examples, though not all are smooth enough to satisfy the hypotheses.

Shrinker	**Type**	Θ	Θ
$\mathbb{R}^4$	flat	1	1.000
S^4	positive Einstein	$6/e^2$	.812
$S^3 \times \mathbb{R}$	product	$2\left(\pi/e^3\right)^{1/2}$	.791
$S^2 \times \mathbb{R}^2$	product	$2/e$	.736
$L(2,-1)$	blowdown shrinker [FIK04]	$e^{\sqrt{2}-2}(1+\sqrt{2})/2$	.672
$\mathbb{CP}^2$	positive Einstein	$9/2e^2$	.609
$S^2 \times S^2$	product	$4/e^2$	.541
$\mathbb{CP}^2\#(-\mathbb{CP}^2)$	Koiso metric [K90, C94]	$3.826/e^2$	.518
$\mathbb{CP}^2\#(-\mathbb{CP}^2)$	Page metric [Pa78]	$3.821/e^2$	.517
$C(\mathbb{RP}^3)$	Ricci flat cone	$1/2$	.500
$C(\mathbb{RP}^2) \times \mathbb{R}$	product	$1/2$	.500
$\mathbb{RP}^4$	positive Einstein	$3/e^2$	.406
$\mathbb{CP}^2\#3(-\mathbb{CP}^2)$	positive Einstein	$3/e^2$	.406
$\mathbb{RP}^3 \times \mathbb{R}$	product	$\left(\pi/e^3\right)^{1/2}$	.396
$\mathbb{RP}^2 \times \mathbb{R}^2$	product	$1/e$	.368
$\mathbb{CP}^2\#4(-\mathbb{CP}^2)$	positive Einstein	$5/2e^2$	.338
$C(S^3/\mathbb{Z}_3)$	Ricci flat cone	$1/3$	.333
$\mathbb{CP}^2\#5(-\mathbb{CP}^2)$	positive Einstein	$2/e^2$	.271
$\mathbb{CP}^2\#6(-\mathbb{CP}^2)$	positive Einstein	$3/2e^2$	.203
$\mathbb{CP}^2\#7(-\mathbb{CP}^2)$	positive Einstein	$1/e^2$	.135
$\mathbb{CP}^2\#8(-\mathbb{CP}^2)$	positive Einstein	$1/2e^2$	.068

All manifolds in the table are created from Einstein manifolds except for $L(2,-1)$ and the Koiso metric. The computations for these metrics will be detailed in [CHI]. The volume of the Page metric is computed in [Pa78].

The *blowdown shrinker* $L(n,-1)$ is a Kähler shrinker defined on the total space of the tautological holomorphic line bundle $o_{N-1}(-1)$ over $\mathbb{CP}^{N-1}$, that is, on $\mathbb{C}^N$ blown up at $z=0$. The metric of $L(N,-1)$ is $U(N)$ invariant, complete, and conelike at infinity, satisfying quadratic decay for the curvature. As $t \nearrow 0$, the exceptional divisor $\mathbb{CP}^{N-1}$ shrinks to a point and elsewhere the metric converges smoothly to a cone metric on $\mathbb{C}^N \setminus \{0\}$ whose metric completion has a vertex at 0. For positive time, the flow can continue by a smooth, $U(N)$-invariant Kähler expander on $\mathbb{C}^N$. See [FIK].

The Koiso metric [K90, C94] and the Page metric [Pa78] are both $U(2)$-invariant metrics on $\mathbb{CP}^2 \#(-\mathbb{CP}^2)$. The former, but not the latter, is Kähler. The remarks following Example 2.3 show that the Koiso metric has one direction of instability (in a Kähler direction). On the other hand, the Page metric may well decay to the Koiso metric. By the discussion in [FIK04], this leads us to conjecture that either metric decays to $\mathbb{CP}^2$ via a $\mathbb{CP}^1$ pinches off.

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