ENERGY INEQUALITIES FOR ISOLATED SYSTEMS AND HYPERSURFACES MOVING BY THEIR CURVATURE

GERHARD HUISKEN
Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Golm-Potsdam, Germany
E-mail: huisken@aei.mpg.de

TOM ILMANEN*
Departement Mathematik, ETH Zentrum, Rämistrasse 101, CH-8092 Zürich, Switzerland
Home-Page: http://www.math.ethz.ch/~ilmanen

The total energy of an isolated gravitating system in General Relativity is described by a geometric invariant of asymptotically flat Riemannian 3–manifolds. One–parameter families of two-dimensional hypersurfaces foliating such a manifold and obeying natural curvature conditions can be used to encode and study geometrical and physical properties of the 3–manifold such as mass, quasi-local mass, the center of mass and energy inequalities. The article describes recent results on Penrose inequalities, inverse mean curvature flow, constant mean curvature surfaces and their interconnections.

1 The mass of asymptotically flat 3–manifolds

The classical description of an isolated gravitating system such as a star, black hole or galaxy is given by a Lorentzian 4–manifold $(L^4, \bar{g})$ of signature $(-+++)$ satisfying Einsteins equations

$$R^4_{\alpha\beta} - \frac{1}{2} R^4 \bar{g}_{\alpha\beta} = 8\pi T_{\alpha\beta} \tag{1}$$

and approaching flat Minkowski space near infinity in an appropriate sense. We denote by $\{R^4_{\alpha\beta}\}$, $0 \leq \alpha, \beta \leq 3$, and R^4 the Ricci tensor and the scalar curvature of the Lorentzian metric $\bar{g}$ respectively. The energy momentum tensor $\{T_{\alpha\beta}\}$ is often assumed to satisfy a physically motivated positivity condition such as the weak energy condition

$$T_{\alpha\beta} X^\alpha X^\beta \geq 0 \qquad \text{for timelike } X. \tag{2}$$

The standard approach to the study of $(L^4, \bar{g})$ is a decomposition of L into a family of spacelike 3–dimensional hypersurfaces, see York[31] and

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Christodoulou-Klainerman[9]. The 3-dimensional spacelike slices have the structure of asymptotically flat complete Riemannian 3–manifolds (M^3, g, K) carrying an induced metric $g = \{g_{AB}\}$, $1 \leq A, B \leq 3$, and a second fundamental form $K = \{K_{AB}\}$. Rewriting the Einstein equations as a Hamiltonian system for g and K it is well known that the energy condition (2) implies a lower bound for the scalar curvature of the induced metric g of the hypersurface in view of the constraint equations.

It was shown by Bartnik[2] that under very general assumptions on L the hypersurfaces (M^3, g, K) can be chosen to be maximal, ie with vanishing trace of the second fundamental form K. In this gauge the energy condition (2) implies that the scalar curvature R^3 of the induced metric g is nonnegative, $R^3 \geq 0$. If the second fundamental K vanishes completely, the hypersurface (M^3, g) is called time-symmetric and all geometric information is encoded in the metric g. In this special case the horizon of a black hole is modelled by a two dimensional minimal surface in (M^3, g).

Having this motivation in mind we will now concentrate on one spacelike slice and try to understand how much geometric information is already encoded in complete Riemannian 3–manifolds (M^3, g) of nonnegative scalar curvature approaching the geometry of Euclidean $\mathbb{R}^3$ near infinity. We use the following notions of asymptotically flat manifolds and exterior region.

Definition 1.1 *i) A 3–dimensional asymptotically flat end is a 3–manifold (M^3, g) with Riemannian metric g, which is diffeomorphic to $\mathbb{R}^3 \setminus K$ for some compact set K, such that in Euclidean coordinates $\{x^A\}$ the metric approaches the flat metric $\delta = \{\delta_{AB}\}$ for $r = |x| \to \infty$:*

$$|g_{AB} - \delta_{AB}| \leq \frac{C_0}{r}, \quad |\partial_A g_{BC}| \leq \frac{C_1}{r^2}, \quad Ric \geq -\frac{C_2}{r^2}. \tag{3}$$

ii) A complete 3–manifold (M^3, g) is called asymptotically flat if it can be decomposed as a disjoint union of a compact set K and finitely many asymptotically flat ends (M_i^3, g), $1 \leq i \leq q$:

$$M^3 = K \cup M_1^3 \cup \cdots \cup M_q^3. \tag{4}$$

iii) An exterior region is a complete, connected asymptotically flat 3-manifold (M^3, g) with one end having non-negative scalar curvature $R(g) \geq 0$, such that the boundary of (M^3, g) is compact and smooth, consisting of finitely many minimal surfaces, with no other minimal surfaces contained in (M^3, g).

Apart from empty space $\mathbb{R}^3$ the most important example of an asymptotically flat 3–manifold is the spatial Schwarzschild manifold given by $({}^S M^3, g^S) = (\mathbb{R}^3 - \{0\}, g^S)$ with

$$g^S_{AB} = \delta_{AB}(1 + \frac{m}{2r})^4, \quad 0 < m = \text{const.} \tag{5}$$

It arises as the $\{t = 0\}$–slice of the Schwarzschild space–time modelling a spherically symmetric, static vacuum black hole of mass m. It has two asymptotically flat ends for $r \to \infty$ and $r \to 0$, which are isometric under reflection in the totally geodesic 2–sphere $\{r = m/2\}$. The two regions $\{r \geq m/2\}$ and $\{r \leq m/2\}$ are both exterior regions in the sense of the definition above.

For a general asymptotically flat end (M^3, g) as in definition 1.1 the total energy or ADM–mass in each end is defined according to Arnowitt, Deser and Misner[1] by a flux integral at infinity,

$$m_{ADM}(M^3, g) := \lim_{R \to \infty} \frac{1}{16\pi} \int_{\partial B_R(0)} (g_{ii,j} - g_{ij,i}) n^j \, d\mu, \tag{6}$$

in agreement with the spatial Schwarzschild manifold above: $m_{ADM}({}^S M^3, g^S) = m$. Due to work of Bartnik[3] and Chrusciel[10] it is known that m_{ADM} is a geometric invariant of the manifold (M^3, g). It is finite precisely when the scalar curvature R^3 is bounded in L^1. Physically the mass defined in this way is a total energy and measures both the matter content of the system and the gravitational energy.

The "Positive Mass Theorem" by Schoen and Yau[29] is a crucial result showing that the local energy condition (2) is consistent with this concept of global energy, a basic version states:

Theorem 1.2 (Schoen-Yau 1979) *If (M^3, g) is an asymptotically flat 3–manifold with non-negative scalar curvature such that the mean curvature of its boundary is nonnegative with respect to the outward normal, then the mass of each end is nonnegative. If the mass is zero in one end, then (M^3, g) is isometric to flat space $(\mathbb{R}^3, \delta)$.*

The proof of Schoen and Yau is based on the fundamental insight that the stability of two-dimesional minimal surfaces in a 3-manifold can be controlled by the scalar curvature of the ambient space. The results discussed in the following sections also exploit this deep relation between the second variation of area for a two dimensional hypersurface and the scalar curvature of the surrounding 3-manifold. Asymptotically flat 3–manifolds and the question of positivity for the mass were also studied by many other authors including Geroch, Jang, Kijowski, Jezierski, Chrusciel, Witten, Choquet–Bruhat, Parker–Taubes, Reula, Tod, Bray, Herzlich and Lohkamp. See[18] and the survey article of Lee–Parker[26] for additional references to asymptotically flat 3–manifolds and the concept of total energy.

2 Hawking mass and hypersurfaces of constant mean curvature

Consider a smooth embedding of a sphere $F : S^2 \mapsto (M^3, g)$ into an asymptotically flat 3-manifold (M^3, g). From the physical information present in (M^3, g) we will try to capture as much as possible in terms of the geometry of suitable hypersurface embeddings F.

If ν is a choice of unit normal for the hypersurface $N^2 = F(S^2)$, let $A = \{h_{ij}\}, 1 \leq i,j \leq 2$, be the second fundamental form and $H = tr_g A$ be the mean curvature of N^2 such that $-H\nu$ is the mean curvature vector. In an exterior region we choose ν to be the exterior normal, such that the mean curvature of a sphere in Euclidean space is positive.

The definition of the ADM-mass (6) shows how 2-dimensional hypersurfaces can be used to define the total energy in terms of an asymptotic flux integral. Hawking[14] introduced a geometric quantity for 2-dimensional surfaces N^2 now called the Hawking-mass with the aim of capturing the energy content of the region bounded by N^2:

$$m_H(N^2) := \frac{|N^2|^{1/2}}{(4\pi)^{3/2}} \left(16\pi - \int_{N^2} H^2 \, d\mu \right). \tag{7}$$

Let us collect some of the properties of this geometric quantity: A simple computation shows that

$$m_H(\partial B_r^S(0)) \equiv m, \quad r > 0, \quad B_r(0) \subset ({}^S M^3, g^S), \tag{8}$$

i.e. m_H yields the desired result for the contained energy of all centered coordinate spheres in the spatial Schwarzschild metric. In general it can be shown[18] that asymptotically m_H yields the total mass in general asymptotically flat ends when evaluated on large coordinate spheres:

$$\lim_{r \to \infty} m_H(\partial B_r(0)) = m_{ADM}. \tag{9}$$

On the other hand m_H is rarely positive, for hypersurfaces in Euclidean space we have

$$N^2 \subset \mathbb{R}^3 : \qquad \int_{N^2} H^2 \, d\mu \geq 16\pi, \tag{10}$$

with equality only on round spheres. In fact, by introducing "wrinkles" in a given surface, one can make the Hawking mass as much negative as one likes. The geometric quantity m_H therefore can only be related to "energy" if the surface N^2 is chosen well. As was first observed by Christodoulou and Yau[8], stable surfaces of constant mean curvature are a good class of surfaces: Here a surface of constant mean curvature is called stable, if the second variation of area with respect to all volume preserving variations is non-neagtive, i.e. if it is a stable critical point of the isoperimetric problem in (M^3, g). It can be seen that a constant mean curvature surface is stable for the area functional with respect to volume preserving variations exactly when the inequality

$$\int_{N^2} \big(|A|^2 + Ric(\nu,\nu)\big) f^2 \, d\mu \leq \int_{N^2} |\nabla f|^2 \, d\mu \tag{11}$$

holds for all functions f satisfying $\int f \, d\mu = 0$. Strict stability means that the first eigenvalue of the Jacobi operator

$$Lu = -\Delta u - u\big(|A|^2 + Ric(\nu,\nu)\big) \tag{12}$$

when restricted to functions f with $\int f \, d\mu = 0$ is strictly positive.
Theorem 2.1 (Christodoulou-Yau 1986) *If $F : S^2 \mapsto (M^3, g)$ is a smooth, stable immersion of constant mean curvature in a Riemannian 3-manifold (M^3, g) of non-negative scalar curvature, then $m_H(N^2) \geq 0$.*
This result fits nicely with the reverse inequality in Euclidean space (10), since the only stable constant mean curvature surfaces in Euclidean space are round spheres. Notice that the centered coordinate spheres of the spatial Schwarzschild manifold $({}^S M^3, g^S)$ are strictly stable constant mean curvature surfaces, with smallest eigenvalue of the stability operator (12) on $\partial B_r(0)$ of order $6m/r^3$ as $r \to \infty$[21].

The existence of constant mean curvature surfaces in an asymptotically flat end of a Riemannian 3-manifold has first been adressed by Huisken and Yau [21] assuming that the end is *strongly asymptotically flat*, i.e. assuming that for large r the metric g has the form

$$g_{AB} = \delta_{AB}\big(1 + \frac{m}{2r}\big)^4 + P_{AB}, \tag{13}$$

$$|P_{AB}| \leq \frac{C_0}{r^2}, \; |\partial^j P_{AB}| \leq \frac{C_j}{r^{2+j}}, \; j = 1, \cdots, 4. \tag{14}$$

It was shown that there is a smooth foliation by unique stable constant mean curvature spheres outside some compact set, provided the mass m is strictly positive:
Theorem 2.2 (Huisken-Yau 1996) *Let (M^3, g) be a strongly asymptotically flat end of a Riemannian 3–manifold with strictly positive mass $m > 0$. Then there are constants $R_0 > 0$, $\tau_0 > 0$ depending only on the mass m and the constants in (14) together with a family of 2–spheres $\Sigma^2_\tau, 0 < \tau < \tau_0$, of constant mean curvature τ providing a regular foliation of the exterior region $M^3 \setminus B_{R_0}(0)$. Furthermore, for each $1/2 < q \leq 1$ there is a constant $\tau_1 > 0$ depending only on q, m and the constants in (14) such that for each $0 < \tau < \tau_1$*

the surface Σ^2_τ is the only stable constant mean curvature surface contained in $M^3 \setminus B_{\tau^{-q}}(0)$.

The proof of the existence result uses the L^2-gradient flow of the area functional subject to a volume constraint to deform an initial coordinate sphere to the desired constant mean curvature surface. The uniqueness result exploits that the stability inequality (11) yields a priori estimates for the difference of the two principal curvatures on a constant mean curvature surface. Notice that Ye[30] has an alternative approach to the existence part of the theorem.

The existence and uniqueness result above extends the notion of "center of mass" from Newtonian mechanics to the relativistic setting: The family of 2-spheres Σ_τ provides an asymptotic center of mass for the infinitely far observer. Such a family of unique 2-spheres seems to be the most natural concept of a "center", since no distinguished Euclidean structure can be refered to on (M^3, g) in a natural way and since already the spatial Schwarzschild manifold illustrates that a "center" cannot be provided by a point of the manifold. The family Σ_τ thus breaks the translation invariance in the asymptotically flat region and provides a natural geometric radial coordinate there. It can easily be employed to define a canonical coordinate system near spatial infinity using a harmonic gauge in angular direction.

The relation of constant mean curvature surfaces to the Hawking-mass has been further clarified by Bray [6], who studies the isoperimetric problem in (M^3, g) from the variational point of view and observes in particular that the Hawking-mass is monotone along a foliation of stable constant mean curvature surfaces.

Theorem 2.3 (Bray 1997) *Let (M^3, g) be a Riemannian 3-manifold of nonnegative scalar curvature. Then the Hawking mass is monotonically increasing along a foliation of expanding stable constant mean curvature spheres in (M^3, g).*

In the special case where (M^3, g) is an exterior region admitting a foliation by stable constant mean curvature spheres between a single boundary component and spatial infinity, the monotonicity above and (9) yield an inequality between the ADM-mass and the area of the boundary known as the Penrose inequality. In general such a foliation of constant mean curvature spheres will not exist in (M^3, g). The next section shows how a different family of 2-spheres with monotone Hawking-mass can be constructed for general exterior regions.

3 Inverse mean curvature flow and the Penrose inequality

Let (M^3, g) be an exterior region as in definition 1.1. In theorem 1.2 Schoen and Yau show that the total energy of an isolated system given by the ADM-mass is non-negative and that in some sense Minkowski space is the absolute ground state of minimum energy. It was already conjectured earlier by Penrose [28] in 1973 that a stronger result holds, namely there should be a positive lower bound for the total energy in terms of the size of the outermost black hole present in the system, with the Schwarzschild metric as the sharp limiting ground state for such an inequality.

In recent work by the authors[17,18] such a lower bound for the ADM-mass is proved in general exterior regions:

Theorem 3.1 (Huisken-Ilmanen 1997) *Let (M^3, g) be an asymptotically flat exterior region. Then the total mass m of (M^3, g) is non-negative and*

$$16\pi m^2 \geq |\Sigma^2|,$$

where $|\Sigma^2|$ is the area of any connected component of ∂M^3. Equality holds if and only if M^3 is one-half of the spatial Schwarzschild manifold $({}^S M^3, g^S)$.

Notice that the exclusion of other compact minimal surfaces in the definition of an exterior region is necessary to account for the possibility of large black holes being hidden behind small ones. Also, the theorem provides an alternative proof of the positive mass theorem 1.2 as a special case. An approach to this result based on 2–dimensional surfaces flowing along the inverse of the mean curvature was originally suggested by Geroch[11], see also[22,23]. Other approaches are due to Gibbons[12,13], Herzlich[16], Bartnik[4], Jezierski[24,25] and Bray[6] as mentioned in the previous section. See[17,18] for additional references and a more detailed account. Recently Bray[7] has used a method of conformal deformations and area-minimizing boundaries in conjunction with the positive mass theorem to estimate the ADM-mass from below by the sum of the areas of the boundary components, improving the estimate above.

The approach suggested by Geroch employs smooth solutions $F : S^2 \times [0, T) \mapsto (M^3, g)$ of the inverse mean curvature flow

$$\frac{d}{dt}F(p,t) = \frac{1}{H}\nu(p,t), \quad p \in S^2,\, t \in [0,T), \tag{15}$$

which is a parabolic system for the evolving surfaces $N_t^2 = F(\cdot, t)(S^2)$ if the mean curvature H is strictly positive. Geroch observes that the flow acts monotone on the Hawking-mass (7) if the scalar curvature of the ambient manifold is non-negative:

$$\frac{d}{dt}m_H(N_t^2) \geq 0. \tag{16}$$

Near an initial minimal surface the Hawking-mass yields the appropriate area term and for large coordinate spheres it approaches the ADM-mass as in (9), such that the desired inequality is proven provided the existence of a solution to (15) is established.

Without extra assumptions on the ambient manifold and the initial data it is clear that singularities will occur in general. For example, the solution evolving from a thin symmetric torus cannot exist forever and similar examples can be constructed in the class of 2–spheres, showing that the flow usually has no smooth solutions in general exterior regions.

In[18] the problem of singularities is overcome by using a level–set formulation of the flow where the surfaces N_t^2 solving (15) are replaced by level–sets

$$N_t^2 = \partial\{x \in M^3 | u(x) < t\}$$

of a scalar function $u : M^3 \mapsto \mathbb{R}$. Intuitively $u(x)$ is the time where the expanding surface passes through $x \in M^3$. In the smooth case the function u satisfies a degenerate elliptic boundary value problem

$$\operatorname{div}\Big(\frac{Du}{|Du|}\Big) = |Du|, \quad u\Big|_{N_0^2} = 0 \tag{17}$$

replacing the parabolic equation (15), note that the LHS is the mean curvature of a level-set and the RHS is the inverse speed. To obtain a concept of weak solution suitable for our purposes, we introduce a global variational principle for u requiring that u minimizes an energy depending on u itself,

$$J_u(v) = J_u^K(v) := \int_K |\nabla v| + v|\nabla u|\, dx, \tag{18}$$

amongst all competing functions $v \in C^{0,1}$ with the same initial data N_0^2 and agreeing with u outside some compact set K. This concept of weak solution implies (17) in an integral form as Euler-Lagrange equation, it allows spatial jumps of the surfaces N_t^2 where the mean curvature $H = |Du|$ becomes zero and the level-sets of u "fatten", and it ensures that the surfaces N_t^2 have continuously varying area while allways enclosing the maximal possible volume for the area available at time t. Using elliptic regularisation

$$\operatorname{div}\Big(\frac{Du}{\sqrt{\epsilon^2 + |Du|^2}}\Big) = \sqrt{\epsilon^2 + |Du|^2} \tag{19}$$

it is possible to construct for $\epsilon \to 0$ a unique weak solution of (17) with these properties that still possesses the crucial monotonicity property for the Hawking-mass (16), leading to the proof of theorem 3.1. The proof of the monotonicity of the mass employs geometric measure theory and is one of the main analytic difficulties in this approach, see[18] for the theory of inverse mean curvature flow. The regularised equation (19) has been used by Pasch[27] to compute weak solutions of the flow numerically.

In [18] we show that the unique weak solution u of inverse mean curvature flow is of class $C^{0,1}$ with level-sets N_t^2 in $C^{1,\alpha}, 0 < \alpha < 1$, of non-negative bounded mean curvature. Heidusch[15] has recently shown that the level sets N_t^2 of the weak solution are allways at least of class $C^{1,1}$, which is the best possible regularity result for the flow. In new work[20] we show that solutions of inverse mean curvature flow in all dimensions are smooth as long as the mean curvature is strictly positive and then go on to show that such a strictly positive lower bound for the mean curvature is allways true on "starshaped" surfaces in Euclidean space. In a forthcoming paper we show that solutions of inverse mean curvature flow in asymptotically flat ends are smooth outside some compact set. If the end is strongly asymptotically flat, the flow approaches the center of mass given by the constant mean curvature foliation in theorem 2.2 at an exponential rate.

4 Quasi-local mass and the Bartnik capacity

Given an open 3-manifold (Ω, g) with compact boundary and non-negative scalar curvature, it is important to have a rigoros mathematical notion of quasi-local mass motivated by the energy content attributable to a subset $(\Omega, g) \subset (M^3, g)$ of an asymptotically flat 3-manifold modelling a spacelike slice of an isolated gravitating system.

One of the first suggestions for a quasi-local mass was the Hawking-mass of the boundary, $m_H(\partial\Omega)$. But as we have seen in the previous sections the Hawking-mass, despite having very desirable properties on certain special two-spheres such as stable constant mean curvature surfaces or solutions of the inverse mean curvature flow, is negative on generic boundaries $\partial\Omega$ and is therefore not suitable as a general measure of energy content. A good concept of quasi-local mass Q should have the following properties[5]:

(i) (Monotonicity) If $\tilde{\Omega} \subset \Omega$, then $Q(\tilde{\Omega}) \leq Q(\Omega)$.
(ii) (Positivity) $Q(\Omega) \geq 0$, and $Q(\Omega) > 0$ unless Ω is locally isometric to $\mathbb{R}^3$.
(iii) (Exhaustion) If Ω_i is a sequence of bounded subsets of an exterior region M^3 such that $\chi_{\Omega_i} \to \chi_M$ locally uniformly, then $\lim Q(\Omega_i) = m_{ADM}(M^3)$.

The following definition is inspired by Bartnik[5] and also used in this form in[18]:

Definition 4.1 *Let (Ω, g) be an open Riemannian 3-manifold with compact boundary and non-negative scalar curvature. We say that an exterior region*

(M^3, g) is an admissible extension of (Ω, g) if there is an isometric embedding $(\Omega, g) \to (M^3, g)$. If Ω has an admissible extension the Bartnik gravitational capacity of Ω is defined as

$$c_B(\Omega) := \inf\{m_{ADM}(M^3, g) \,|\, (M^3, g) \,\text{is an admissible extension of}\ \Omega\}.$$

Notice that with this definition the extension (M^3, g) (and hence (Ω, g)) can have horizons on an inner boundary, which is less restrictive than Bartniks original definition, making our definition less than or equal to Bartniks. The monotonicity property (i) above is satisfied trivially and the weak positivity in (ii) is a consequence of the positive mass theorem 1.2. Using the inverse mean curvature flow and the precise form of the monotonicity for the Hawking-mass (16) we can show[18] that all the properties (i)-(iii) listed above are satisfied for the Bartnik capacity:

Theorem 4.2 (Huisken-Ilmanen 1997) *The Bartnik capacity c_B satisfies the positivity property (ii) and the exhaustion property (iii).*

From the proof it can be seen that in specific models, eg for stars having certain density profiles, the presence of the scalar curvature in the monotonicity formula for the Hawking-mass allows the computation of concrete lower bounds for the Bartnik capacity. It is an open problem whether an admissible domain (Ω, g) with $c_B(\Omega) = 0$ is actually isometric to a subset of $\mathbb{R}^3$. Other interesting questions concern the limiting behaviour of sequences of extensions realizing the infimum in the definition of c_B.

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