Elliptic Regularization and Partial Regularity for Motion by Mean Curvature

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February, 1993

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Abstract

I. We study Brakke's motion of varifolds by mean curvature in the special case that the initial surface is an integral cycle, giving a new existence proof by mean of *elliptic regularization*. Under a uniqueness hypothesis, we obtain a weakly continuous family of currents solving Brakke's motion.

II. These currents remain within the corresponding level-set motion by mean curvature, as defined by Evans-Spruck and Chen-Giga-Goto. Now let T_0 be the reduced boundary of a bounded set of finite perimeter in R^n. If the level-set motion of the support of T_0 does not develop positive Lebesgue measure, then there corresponds a unique integral n-current T, $\partial T = T_0$, whose time-slices form a unit density Brakke motion. Using Brakke's Regularity Theorem, $\operatorname{spt} T$ is smooth $\mathcal{H}^n$-almost everywhere. In consequence, almost every level-set of the level-set flow is smooth $\mathcal{H}^n$-almost everywhere in space-time.

Subject Classifications: 35, 53, 82

Keywords: Mean Curvature, Phase Transitions, Level-Set Flow, Weak Convergence

Introduction

A surface moving so as to decrease its area most efficiently is said to be *moving by mean curvature*. In certain physical phenomena, for example, the energy of an interface between two phases of matter is proportional to the surface area. If momentum, pressure, and other forces can be ignored, then the interface is flowing by mean curvature.

From another point of view, the mean curvature flow is the only possible isotropic, homogeneous, quasilinear parabolic equation for the motion of a surface in space. As such it has great geometric significance and beauty and is closely related to such other geometric flows as Hamilton's Ricci flow [Ha1, Ha2, Ha3], the harmonic map heat flow Chen [Ch], Chen-Struwe [CS], Struwe [St], and the Yang-Mills flow [Do].

Brakke initiated the study of the generalized mean curvature flow in his 1978 book [B], using methods of geometric measure theory. In the early eighties, Gage [Ga] and Gage-Hamilton [GH] showed that convex curves evolve smoothly, becoming asymptotically round as they disappear. Grayson extended the smoothness to general embedded curves in [Gr1, Gr2], solving a problem posed by Gauss. Angenent has shown how to evolve immersed curves in [An1, An2, An3]. Huisken established the asymptotic behavior of convex bodies in all dimensions in [H1], and studied general singularity formation in [H2]. Comprehensive estimates for graphs evolving by mean curvature appear in Ecker-Huisken [EH1, EH2]. These papers employ classical differential geometry and the maximum principle.

Allen and Cahn [AC] conjectured mean curvature motion as the singular limit of a reaction-diffusion (phase-field) equation. This idea has been developed by de Mottoni and Schatzman [MS1, MS2, MS3], Bronsard-Kohn [BK], and many others. Following Osher-Sethian's numerical work [OS], Chen-Giga-Goto [CGG] and Evans-Spruck [ES1, ES2, ES3, ES4] introduced the level-set-flow, in which the moving surface is the zero-set of a function, all of whose level-sets move by mean curvature. The phase-field and level-set approaches are reconciled in the paper of Evans-Soner-Souganidis [ESS], and unified with Brakke's work in Ilmanen [I3]. Other interesting papers along these lines include Altschuler-Angenent-Giga [AAG] and Barles-Soner-Souganidis [BSS].

In the work of Luckhaus [Lu], Almgren-Wang [AW], and Almgren-Taylor-Wang [ATW], the flow is approximated by a time-step minimization method that is broadly

Received by the editor January 3, 1992.

related to the method of this paper. There are also stochastic approximations, discrete approximations, etc.

In this work we introduce a new approximation scheme for Brakke's mean curvature flow of varifolds. Using geometric measure theory, we pass to limits and in the process sharpen and clarify his definition. Next, we synthesize Brakke's flow with the level-set flow. Turning to maximum principle (viscosity) techniques, we succeed in partly closing an annoying gap in Brakke's regularity theory, yielding (generic) almost-everywhere regularity for motion by mean curvature.

The reader should start with the introductions to Chapters One and Two, then proceed directly to §2, referring to §1 as needed.

I would like to thank Jacki Jae-Kyung K. Noh, Lisa Anne Rhode, and Edith Hooker Ilmanen for their faith. I am grateful for mathematical conversations over the years with Steven Altschuler, Sigurd Angenent, Andrew Casson, Yoshikazu Giga, Richard Hamilton, Michael D. Hirsch, Bruce Kleiner, Robert Kusner, Helena Lopes, Yong-Bin Ruan, H. Mete Soner, Takis Souganidis, Lang-Fang Wu, and Yu-Xi Zheng. Also I would like to thank A. Chorin for support from the National Science Foundation, and the Institute for Advanced Study for hospitality during the final preparation.

This work relies heavily on the ideas of Brakke in his 1978 book [B], which are only now beginning to be understood and applied.

My special thanks go to my adviser, L. C. Evans, whose arrival at Berkeley came at a critical moment for me. Our conversations and his stellar notes and classes have contributed greatly to my mathematical development. Without him this work would not have been possible.

Typeset by $\mathcal{AMS}$-TEX.

Notation

$\mathbf{M}[T]$, $\mathbf{M}[\mu]$	mass of current, Radon measure
$\Lambda_k M^n$	bundle of alternating k-vectors on M
$\Lambda^k M^n$	bundle of alternating k-forms on M
P_A	$P\lfloor M \times A$, $A \subseteq \mathbf{R}$
P_z	$\partial(P\lfloor M \times [z, \infty))$
$S(x)$	approximate tangent k-plane of the Radon measure μ at the point x
$S \cdot v$	projection of the vector v onto the k-plane S
$S : T$	inner product of linear transformations S and T
$\mathbf{R}^+$	$[0, \infty)$
$\bar{D}f(t)$	upper derivate, i.e. $\limsup_{s\to t}\ (f(s) - f(t))/(s-t)$
$D^+f(t)$	right upper derivate, i.e. $\limsup_{s\downarrow t}(f(s) - f(t))/(s-t)$
$D^-f(t)$	left upper derivate, i.e. $\liminf_{s\uparrow t}(f(s) - f(t))/(s-t)$
$\nabla, \bar{\nabla}, D, \bar{D}$	covariant derivatives
ω	$\partial/\partial z$
$\lvert A\rvert$	$\mathcal{L}^1(A)$

Conventional Symbols

Γ_t	level set
T, P	current
μ, ν	Radon measure
V	varifold
X	rectifiable set
ϕ, ψ	test function
Φ	diffeomorphism
Y	vector field
U	open set
u	function
M^n	Riemannian manifold of dimension n
g	metric

Chapter One. Elliptic Regularization

In order to model the flow of surfaces through singularities, Brakke [B] developed a theory of varifolds moving by mean curvature. It consists of three parts: definition, existence, and regularity.

Brakke's theory suggests two general questions about motion by mean curvature. First, there is the task (see §6) of sharpening Brakke's definition to resolve difficulties concerning the continuity of the motion and the attainment of initial conditions. In fact, Brakke's moving varifolds seem to be (measure-theoretic) *subsolutions*, just as the level-sets of Evans-Spruck and Chen-Giga-Goto are (set-theoretic) *supersolutions.*

Second, there is the question of the lacuna between Brakke's existence theory and his regularity theory. If the moving varifolds are of *unit density,* then Brakke proves regularity almost everywhere. However, his existence construction may not satisfy this hypothesis, even if the initial surface has unit density.

Brakke's theory is valid for surfaces with junctions. If we restrict ourselves to the smaller class of orientable surfaces without boundary (but still possibly singular), then it is possible to provide an alternate existence proof for Brakke's motion, by means of *elliptic regularization.* The basic idea is to add a small second derivative in the time direction.

This construction has several advantages. Speaking generally, it is elegant and suggestive, and might be useful in other parabolic problems of low regularity. In the problem at hand, it yields not only a family of integer rectifiable varifolds moving by mean curvature in the sense of Brakke, but also an integral current, T, in space-time, whose boundary is the initial surface.

If it happens that T is *unique,* then, as we prove in §9, the time-slices of T themselves form a family moving by mean curvature in the sense of Brakke.

Let us describe the construction in more detail. Suppose T_0 is a locally integral current in $M \times \{t = 0\}$, where M is a manifold. In §2, we introduce a functional I^ε defined by

$$I^\varepsilon[P] = \frac{1}{\varepsilon} \int_P e^{-z/\varepsilon} d\mu_P(x, z)$$

where (x, z) are coordinates for $M \times \mathbf{R}$ and P is a locally integral $(k+1)$-current whose boundary is the initial surface T_0. A stationary point P^ε for I^ε is a *translating soliton* for motion by mean curvature. To be precise, if P^ε begins to evolve by mean curvature in $M \times \mathbf{R}$, P^ε will translate with velocity vector $-(1/\varepsilon)(\partial/\partial z)$.

Let us write T^ε for P^ε rescaled by the transformation $t = \varepsilon z$. The fundamental observation in §2 is that T^ε closely approximates the track in space-time of a motion by mean curvature. In §3 we prove the existence of these approximators T^ε. In the two following sections, we turn the fundamental estimate for the mean curvature flow, namely

$$\text{Area}(T_0) = \text{Area}(T_t) + \int_0^t \int_{T_s} H^2 \, d\mu_s \, ds,$$

into the estimates needed to pass to limits as $\varepsilon \downarrow 0$.

Next, in §6, we define Brakke's varifolds moving by mean curvature, and prove a compactness theorem for them in §7. In §8 we apply this theorem to T^ε to obtain, as $\varepsilon \downarrow 0$, a pair

$$(T, \{\mu_t\}_{t \geq 0}).$$

where T is a locally integral current in $\mathbf{M} \times [0, \infty)$ with boundary T_0, and $\{\mu_t\}_{t\geq 0}$ is a family of mass measures of varifolds moving by mean curvature in the sense of Brakke. Let us write T_t for the time slice $\partial(T \lfloor (M \times [t, \infty))$ and μ_{T_t} for the mass measure of T_t. Then for all $t \geq 0$, μ_{T_t} "sits atop" T_t in the sense that

$$\mu_t \geq \mu_{T_t}$$

as Radon measures. Furthermore, $\{T_t\}_{t\geq 0}$ is continuous in the weak topology on currents, and T_t is locally integral for each t.

The possible discrepancy between the two is an inherent flaw in the procedure, which arises due to the lower semicontinuity of mass with respect to weak convergence. In §9 we make the special assumption that T is unique with respect to the properties above. Employing a "restarting procedure", we prove that $\{\mu_{T_t}\}_{t\geq 0}$ is itself a Brakke motion.

Such a current T is called a *matching motion.* Taking this as the definition of motion by mean curvature, we have gone some distance toward clarifying and sharpening Brakke's definition.

In Chapter Two we prove that a codimension one surface (moving in $\mathbf{R}^n$) can be perturbed slightly to one satisfying the uniqueness assumption. Using the results above, we find that the surface has unit density almost everywhere. This in turn is the crucial hypothesis for the application of Brakke's theory of almost everywhere regularity. As a result we prove a generic almost-everywhere regularity theorem for the level-set flow.

§1 Varifolds and Currents

In order to follow the motion of a surface beyond the onset of singularities, we will replace the surface by its surface measure. Thus a motion by mean curvature will be a family $\{\mu_t\}_{t\geq 0}$ of Radon measures, such that for almost every t, μ_t possesses k-dimensional tangent planes of positive integer multiplicity μ_t-almost everywhere. We take the liberty of calling these latter objects *integer k-rectifiable Radon measures*, and denote

$$\mathcal{IM}_k = \text{space of integer } k\text{-rectifiable Radon measures.}$$

These are exactly the mass measures of integer rectifiable k-varifolds.

We will be needing the following spaces from geometric measure theory:

$$\begin{aligned}
\mathcal{M}(M) &= \text{space of Radon measures on } M, \\
\mathbf{IV}_k(M) &= \text{space of integer rectifiable } k\text{-varifolds on } M, \\
\mathbf{V}_k(M) &= \text{space of general } k\text{-varifolds on } M, \\
\mathbf{I}_k^{loc}(M) &= \text{space of locally integral } k\text{-currents on } M, \\
\mathcal{D}_k(M) &= \text{space of } k\text{-currents on } M.
\end{aligned}$$

The basic references for these ideas are Federer [F], Almgren [A], and Allard [Al]; see also Morgan [M]. Our own source is mainly Simon [S]. We will devote this section to defining the above spaces and some other terms of geometric measure theory.

1.1 Radon Measures, Rectifiable Sets. Let (M, g) be a complete Riemannian manifold, assumed to be isometrically embedded in a high-dimensional Euclidean space. A *Borel regular measure* on M is an (outer) measure on M such that all Borel sets are measurable and every set $A \subseteq M$ is contained in a Borel set of equal measure. A *Radon measure* on M is a locally finite Borel regular measure. The set of all Radon measure is denoted $\mathcal{M}(M)$.

For any Radon measure μ, define a linear functional μ on $C_c^0(M, \mathbf{R})$ by

$$\mu(\phi) = \int_M \phi(x)\, d\mu(x), \qquad \phi \in C_c^0(M, \mathbf{R}).$$

This formula sets up a one-to-one correspondence between the Radon measures on M and the set of continuous positive linear functionals on $C_c^0(M, \mathbf{R})$. We endow $\mathcal{M}(M)$ with the topology such that $\mu_i \to \mu$ if and only if

$$\mu_i(\phi) \to \mu(\phi) \qquad \text{for every } \phi \in C_c^0(M, \mathbf{R}).$$

For any set $A \subseteq M$, define the *restriction* of μ to A by

$$(\mu\lfloor A)(B) = \mu(A \cap B) \qquad \text{for } B \subseteq M.$$

If ϕ is a locally μ-integrable function, define

$$(\mu\lfloor\phi)(B) = \int_B \phi \, d\mu \qquad \text{for } B \subseteq M.$$

The *k-dimensional density* of μ at x is

$$\Theta^k(\mu, x) = \lim_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_k r^k},$$

if the limit exists. Recalling that M is embedded in a Euclidean space, for any $\lambda > 0$ define the scaled Radon measure $\mu_{x,\lambda}$ by

$$\mu_{x,\lambda}(A) = \lambda^{-k}\mu(\lambda A + x), \qquad A \subseteq M.$$

Let P be a k-plane in T_xM and $a > 0$. We say that P is the *k-dimensional (approximate) tangent plane* of μ with multiplicity a, written $T_x\mu = P$, provided that

$$\lim_{\lambda \downarrow 0} \mu_{x,\lambda} = a\,\mathcal{H}^k\lfloor P$$

where $\mathcal{H}^k$ is k-dimensional Hausdorff measure. The k-dimensional approximate tangent space of a set $X \subseteq M$ at x is defined by

$$T_xX = T_x(\mathcal{H}^k\lfloor X)$$

if this exists. We call X *countably k-rectifiable* (see Simon [S, 11.1]) provided

(i) $X \subseteq C_0 \cup (\bigcup_{i \geq 1} C_i)$, where $\mathcal{H}^k(C_0) = 0$ and each C_i is an embedded C^1 k-submanifold.

We say X is *locally k-rectifiable* if in addition,

(ii) X has locally finite $\mathcal{H}^k$-measure.

If X is locally k-rectifiable and $\mathcal{H}^k$-measurable, then T_xX exists $\mathcal{H}^k\lfloor X$-a.e. (see [S, 11.6]).

1.2 k-Rectifiable Radon Measures. Let X be an $\mathcal{H}^k$-measurable subset of M, and let $\theta : M \to \mathbf{Z}^+$ be locally $\mathcal{H}^k$-integrable, with $X = \{\theta > 0\}$ $\mathcal{H}^k$-a.e. Define $\mu(X, \theta)$ to be the Radon measure

$$\mu(X, \theta) = \mathcal{H}^k \lfloor \theta.$$

Define μ to be a *k-rectifiable Radon measure* if either of the following equivalent conditions is satisfied:

(a) μ has k-dimensional tangent planes (of positive multiplicity) μ-a.e.

(b) $\mu = \mu(X, \theta)$ for X $\mathcal{H}^k$-measurable and countably k-rectifiable, and θ locally $\mathcal{H}^k$-integrable.

We write $\mathcal{M}_k$ for the set of k-rectifiable Radon measures. To see that (a) implies (b), apply [S, 11.8]. For the converse, see 1.3 below.

We call μ an *integer k-rectifiable Radon measure* if either of the following equivalent conditions is satisfied:

(c) μ has k-dimensional tangent planes of positive integer multiplicity μ-a.e.

(d) $\mu = \mu(X, \theta)$ for X $\mathcal{H}^k$-measurable and locally $\mathcal{H}^k$-rectifiable, and θ locally $\mathcal{H}^k$-integrable with values in $\mathbf{Z}^+ \cup \{0\}$.

We write $\mathcal{IM}_k$ for the set of all such μ. To see that (c) implies (d), observe that $\Theta^k(\mu, x) \geq 1$ for μ-a.e. $x \in M$ and apply [S, 11.8]. For the converse, see the following proposition.

1.3 Structure of $\mu(X, \theta)$. Let μ be a Radon measure of the form $\mu(X, \theta)$, where θ is locally $\mathcal{H}^k$-integrable and $X = \{\theta > 0\}$ is countably k-rectifiable and $\mathcal{H}^k$-measurable. Define

$$\mathcal{X}(\mu) = \{x \in M : \Theta^k(\mu, x) \text{ exists and is positive}\}.$$

Then

(i) $\Theta^k(\mu, x)$ exists and equals $\theta(x)$ for $\mathcal{H}^k$-a.e. $x \in M$.

(ii) $X = \mathcal{X}(\mu) \qquad \mathcal{H}^k$-a.e.

(iii) (a) For $\mathcal{H}^k$-a.e. $x \in M$, either

$$T_x X \text{ exists, } \Theta^k(\mu, x) > 0, \text{ and } x \in X$$

or

$$T_x X \text{ is empty, } \Theta^k(\mu, x) = 0, \text{ and } x \notin X.$$

(b) Suppose $X \subseteq C_0 \cup (\bigcup_{i \geq 1} C_i)$ where $\mathcal{H}^k(C_0) = 0$ and each C_i is an embedded C^1 k-submanifold. Then for $\mathcal{H}^k$-a.e. $x \in X$, $T_x\mu = T_x C_i$ for every $C_i \ni x$.

(c) If $\mathcal{H}^k \lfloor X$ is locally finite, then for $\mathcal{H}^k \lfloor X$-a.e. $x \in M$, $T_x\mu = T_x X$.

(iv) (a) $\mathcal{H}^k \lfloor X$ has the same null-sets as μ.

(b) $\Theta^k(\mu, x) = 0$ for $\mathcal{H}^k$-a.e. $x \in \operatorname{spt}\mu \setminus X$.

(c) $\mu(\operatorname{spt}\mu \setminus X) = 0$. □

1.4 Remarks.

(a) The point of (iv) is to emphasize that $\mathcal{H}^k(\operatorname{spt}\mu \setminus X)$ need not equal zero. In fact, $\operatorname{spt}\mu$ can be all of M, even though X is locally k-rectifiable.

(b) By (i) and (ii), θ and X are intrinsic to μ, up to sets of $\mathcal{H}^k$-measure zero. We have for $\mu \in \mathcal{M}_k(M)$,

$$\mu = \mu(\mathcal{X}(\mu), \Theta^k(\mu, \cdot)).$$

(c) Up to sets of $\mathcal{H}^k$-measure zero, $\mathcal{X}(\mu)$ is the smallest set supporting μ.

1.5 Proof of 1.3. We will use [S, 3.5, 4.7, 11.6].

1. We can assume $X \subseteq C_0 \cup (\cup_{i \geq 1} C_i)$, where $\mathcal{H}^k(C_0) = 0$ and the C_i are disjoint embedded C^1 submanifolds of M. Define the

$$X_i = X \cap C_i, \qquad \mu_i = \mu \lfloor X_i.$$

According to Simon [S, 3.5], if μ is a Borel regular measure and $A \subseteq \mathbf{R}^n$ is μ-measurable with $\mu(A) < \infty$, then $\Theta^k(\mu \lfloor A, x) = 0$ for $\mathcal{H}^k$-a.e. $x \in \mathbf{R}^n \setminus A$. Thus

$$\begin{aligned} \Theta^k(\mu, x) &= \Theta^k(\mu_i, x) && \text{for } \mathcal{H}^k\text{-a.e. } x \in C_i, \\ \Theta^k(\mu, x) &= 0 && \text{for } \mathcal{H}^k\text{-a.e. } x \in \mathbf{R}^n \setminus X. \end{aligned}$$

2. Thus for $\mathcal{H}^k$-a.e. $x \in C_i$,

$$\Theta^k(\mu, x) = \Theta^k(\mu_i, x) = \frac{d\mu_i}{d(\mathcal{H}^k \lfloor C_i)}\, \Theta^k(\mathcal{H}^k \lfloor C_i, x) = \theta(x)$$

by the Lebesgue differentiation theorem and the fact that $\mu_i = \mu(C_i, \theta\lfloor C_i)$. Together with step 1, this yields (i). Since $X = \{\theta > 0\}$ $\mathcal{H}^k$-a.e., (i) implies (ii).

3. By steps 1 and 2, for $\mathcal{H}^k$-a.e. $x \in C_i$,

$$T_x\mu = T_x\mu_i = T_xC_i$$

with multiplicity $\theta(x)$, and $T_x\mu$ is empty for $\mathcal{H}^k$-a.e. $x \in M \setminus X$. This yields (iii)(a), (b). If $\mathcal{H}^k\lfloor X$ is locally finite, then we get (iii)(c) by considering $\tilde{\mu} = \mathcal{H}^k\lfloor X$.

4. Evidently $\mu \ll \mathcal{H}^k\lfloor X$ and conversely since $\theta > 0$ on X, giving (iv)(a). Finally, (iv)(b) and (c) follow from (iii)(a) and the construction of μ, respectively. Q.E.D.

1.6 General Varifolds. A *general k-varifold* V is a Radon measure on G^kM, the Grassmann bundle of all k-planes tangent to M. We write

$$V(\psi) = \int \psi(x, S)\, dV(x, S)$$

where $\psi \in C^0_c(G^kM, \mathbf{R})$ is a function on the total space of G^kM, and for each $x \in M$, S ranges over G^k_xM. We give the set $\mathbf{V}_k(M)$ of all general k-varifolds the topology of convergence of Radon measures.

Associated to V is a Radon measure μ_V on M, defined by $\mu_V = \pi_\#(V)$, where $\pi : G^kM \to M$ is the projection map. Inversely, if μ is a k-rectifiable Radon measure, then by 1.3(iii)(c) $T_x\mu$ is a μ-measurable section of G^kM defined μ-a.e. Therefore it makes sense to define a varifold $V_\mu = V(\mu)$ by the formula

$$V_\mu(\psi) = \int \psi(x, S)\, dV_\mu(x, S) = \int \psi(x, T_x\mu)\, d\mu(x) \qquad \text{for } \psi \in C^0_c(G^kM, \mathbf{R}).$$

Varifolds of this form, for μ an integer k-rectifiable Radon measure, are called *integer rectifiable k-varifolds.* We write $\mathbf{IV}_k(M)$ for the set of integer rectifiable k-varifolds. The functions

$$V : \mathcal{IM}_k \to \mathbf{IV}_k, \qquad \mu : V_k \to \mathcal{M}$$

set up a one-to-one correspondence between $\mathcal{IM}_k(M)$ and $\mathbf{IV}_k(M)$. Note that μ is a continuous map but V is not.

1.7 First Variation. (Simon [S, §16, §39]) If S is a k-dimensional subspace of a Euclidean space E, we will also use S to denote projection onto S, and

$$\begin{aligned} S \cdot v &= \text{projection of the vector } v \text{ onto } S, \\ w \cdot v &= \text{inner product of vectors } w, v, \\ A : B &= \text{inner product of bilinear forms } A \text{ and } B. \end{aligned}$$

Let $V \in \mathbf{V}_k(M)$ and $\mu = \mu_V$. Let $\Phi : M \to M$ be C^1. The *push-forward* $\Phi_\#(V)$ is defined by

$$\Phi_\#(V)(\psi) = \int \psi(x,S)\, d(\Phi_\#(V))(x,S) = \int \psi(x,S)\, |J_S\Phi(x)|\, dV(x,S)$$

for $\psi \in C_c^0(G^k M, \mathbf{R})$, where the Jacobian is given by

$$|J_S\Phi(x)| = \sqrt{\det[(d\Phi(x)|S)^T \circ d\Phi(x)|S]}, \qquad x \in M, \quad S \in G_x^k M.$$

The *first variation* δV of V is the linear functional on sections $Y \in C_c^1(TM)$ given by

$$\delta V(Y) = \frac{d}{dt}\bigg|_0 \Phi^t_\#(V)\ (U),$$

where $\operatorname{spt} Y \subset\subset U \subset\subset M$ and $\{\Phi^t\}_{t\in(-\varepsilon,\varepsilon)}$ is a family of diffeomorphisms supported in U with $(\partial/\partial t)|_0 \Phi^t = Y$, $\Phi^0 = \mathrm{id}$. Letting ∇ denote covariant differentiation on M, we have the formula (see e.g. Simon [S, §9])

$$\frac{\partial}{\partial t}\bigg|_0 |J_S\Phi^t(x)| = S : \nabla Y(x) = \sum_{i=1}^k \langle \nabla_{e_i} Y(x), e_i\rangle \tag{1}$$

for any orthonormal basis $\{e_1, \dots, e_k\}$ of S. (This is usually denoted in the literature by $\operatorname{div}_V Y$.) Thus

$$\delta V(Y) = \int_V S : \nabla Y(x)\ dV(x,S).$$

For $U \subseteq M$ open, define the *total first variation* by

$$|\delta V|(U) = \sup\{\delta V(Y) : Y \in C_c^1(TM|U),\ |Y| \le 1\}.$$

If $|\delta V|$ is a Radon measure, then δV is a TM-valued Radon measure with $|\delta V|$ as total variation measure. In this case we can decompose δV as

$$\delta V = -\mu \lfloor \vec{H} + \delta V_{sing}$$

where $\vec{H}$ is a locally μ-integrable section of TM and δV_{sing} is supported on a set of μ-measure zero.

1.8 Abbreviations. When integrating over a k-rectifiable Radon measure μ, we will adopt the special notation

$$S = S(x) = T_x\mu.$$

As observed above, $S(x)$ is a μ-measurable section of G^kM defined μ-a.e. We will also write $\vec{H}$ for $\vec{H}_{V_\mu}$ in this context, and suppress the variable of integration whenever possible. Thus the first variation formula becomes

$$\delta V_\mu(Y) = \frac{d}{dt}\bigg|_0 \Phi^t_\#(V)\ (U) = \int S : \nabla Y\, d\mu = \int -\vec{H}\cdot Y\, d\mu + \delta V_{sing}(Y).$$

1.9 Compactness for $\mathcal{IM}_k(\mathbf{M})$. (Allard) Let $\{\mu_i\}_{i\geq 1} \subseteq \mathcal{IM}_k(M)$ with

$$\sup_{i\geq 1}\,(\mu_i(U) + |\delta V_{\mu_i}|(U)) < \infty \qquad \text{for each } U \subset\subset M.$$

Then there exists $\mu \in \mathcal{IM}_k(M)$ and a subsequence $\{\mu_{i_j}\}_{j\geq 1}$ such that

(i) *Convergence of measure.*

$$\mu_{i_j} \to \mu \qquad \text{as Radon measures on } M.$$

(ii) *Convergence of tangent planes.*

$$V_{\mu_{i_j}} \to V_\mu \qquad \text{as Radon measures on } G^kM.$$

(iii) *Convergence of first variation.*

$$\delta V_{\mu_{i_j}} \to \delta V_\mu \qquad \text{as } TM\text{-valued Radon measures.}$$

(iv) *Lower semicontinuity of total first variation.*

$$|\delta V_\mu| \leq \liminf_{i_j} |\delta V_{\mu_{i_j}}| \qquad \text{as Radon measures.} \quad \square$$

The theorem is a restatement of Allard's compactness theorem [Al, 6.4] in our notation. Note that (ii) implies (iii) by the first variation formula and (iii) implies (iv) because $|\cdot|$ is convex.

1.10 Currents. Let $\Lambda_k M$ denote the bundle of alternating k-vectors on M, and $\Lambda^k M$ the corresponding bundle of k-forms. A *k-current* $T \in \mathcal{D}_k(M)$ is a continuous linear functional on the space $C_c^\infty(\Lambda^k M)$ of smooth k-forms of compact support. To any current T we associate the *mass measure*

$$\mu_T(U) = \sup\left\{T(\vartheta) : \vartheta \in C_c^\infty(\Lambda^k M|U),\ |\vartheta| \leq 1\right\}, \qquad U \subseteq M \text{ open},$$

where $|\vartheta|$ is the norm of ϑ with respect to the metric on $\Lambda_x^k M$, normalized so that $|\eta^1 \wedge \cdots \wedge \eta^k| = 1$ for an orthonormal set $\{\eta^1, \ldots, \eta^k\}$ in $T_x^* M$. The *mass* of T is

$$\mathbf{M}[T] \equiv \mu_T(M).$$

If μ_T is locally finite, then T is of *locally finite mass.* Such currents are also said to be *representable by integration.* In fact, by the Riesz representation theorem, μ_T is a Radon measure and

$$T(\vartheta) = \int \langle \vartheta(x), \xi(x) \rangle \, d\mu_T(x)$$

for some locally μ_T-integrable section ξ of $\Lambda_k M$. Thus in this case T is nothing more than a $\Lambda_k M$-valued Radon measure. (Here $\langle \cdot, \cdot \rangle$ is the dual pairing.)

1.11 Locally Integer Rectifiable Currents. A locally integer rectifiable current is an integer k-rectifiable Radon measure endowed with a measurable choice of orientations for the tangent planes. Suppose we are given an integer k-rectifiable Radon measure $\mu = \mu(X, \theta)$. Let ξ be a μ-measurable section of $\Lambda_k M|X$ such that

$$\xi(x) = e_1 \wedge \ldots \wedge e_k, \qquad \text{for some orthonormal basis } \{e_1, \ldots, e_k\} \text{ of } T_x\mu$$

μ-almost everywhere. Then $\xi(x)$ is called an *orientation* for $T_x\mu$. Define the current $T = \tau(X, \theta, \xi)$ by

$$T(\vartheta) = \int_X \langle \vartheta(x), \xi(x) \rangle \, d\mu(x) \qquad \text{for } \vartheta \in C_c^\infty(\Lambda^k M).$$

Such a current is called a *locally integer rectifiable current*, and we denote by $\mathcal{R}_k^{loc}(M)$ the space of all such currents.

Note that $\mu_T = \mu$, and the map

$$\mu : \mathcal{R}_k^{loc}(M) \to \mathcal{IM}_k(M)$$

is many-to-one and onto.

1.12 Boundaries. The boundary $\partial T \in \mathcal{D}_{k-1}(M)$ of a current $T \in \mathcal{D}_k(M)$ is defined by

$$\partial T(\vartheta) = T(d\vartheta), \qquad \vartheta \in C_c^\infty(\Lambda^k M).$$

A *boundary* is a current that can be expressed in this form, and a *cycle* is a current without boundary.

Note that if $T \in \mathcal{R}_k^{loc}(M)$, ∂T need not be locally rectifiable or even of locally finite mass. However, locally finite mass of the boundary implies local integer rectifiability, as follows.

1.13 Boundary Rectifiability Theorem. (Simon [S, 30.3]) Suppose $T \in \mathcal{R}_k^{loc}(M)$ and ∂T is of locally finite mass. Then $\partial T \in \mathcal{R}_{k-1}^{loc}(M)$. $\square$

A current $T \in \mathcal{R}_k^{loc}(M)$ with $\partial T \in \mathcal{R}_{k-1}^{loc}(M)$ is called a *locally integral current.* We write $\mathbf{I}_k^{loc}(M)$ for the set of locally integral k-currents.

If N^k is an oriented smooth submanifold of M, we can associate to N a locally integral current, denoted $[N]$, by integrating a given k-form ϑ with respect to the oriented volume form of N.

1.14 Weak Topology for Currents. If $\{T_i\}_{i\geq 1},\ T \subseteq \mathcal{D}_k(M)$, we say $\{T_i\}_{i\geq 1}$ converges to T in the weak topology on $\mathcal{D}_k(M)$, written

$$T_i \rightharpoonup T$$

provided that $T_i(\vartheta) \to T(\vartheta)$ for each $\vartheta \in C_c^\infty(\Lambda^k M)$. Evidently, $\partial : \mathcal{D}_k(M) \to \mathcal{D}_{k-1}(M)$ is continuous in the weak topology.

Suppose S, $T \in \mathbf{I}_k^{loc}(M)$. Fix $U \subseteq M$ open. We define a pseudometric d_U by

$$d_U(S,T) = \inf \{\mathbf{M}_U[P] + \mathbf{M}_U[Q] : S - T = P + \partial Q,\ \ P \in \mathcal{R}_k^{loc}(M),\ \ Q \in \mathcal{R}_{k+1}^{loc}(M)\}.$$

The topology generated by these pseudometrics as U ranges over all the open sets of M is called the *flat metric topology.* The two topologies are equivalent under the following circumstance.

1.15 Equivalent Topologies. (Simon [S, 31.2]) Let $\{T_i\}_{i\geq 1}$, $T \subseteq \mathbf{I}_k^{loc}(M)$ with

$$\sup_{i\geq 1} (\mathbf{M}_U[T_i] + \mathbf{M}_U[\partial T_i]) < \infty \quad \text{for all } U \subset\subset M.$$

Then $T_i \rightharpoonup T$ if and only if $d_U(T_i, T) \to 0$ for each $U \subset\subset M$. $\square$

Note: The "if" part is trivial and holds for all currents.

1.16 Weak Lower Semicontinuity of Mass. If $T_i \rightharpoonup T$ in $\mathcal{D}_k(M)$, then

$$d\mu_T \leq \liminf_{i\to\infty} \; d\mu_{T_i}$$

as Radon measures. $\square$

In fact, the function $\mu : \mathcal{R}_k^{loc}(M) \to \mathcal{IM}_k(M)$ (or μ : {currents of locally finite mass} $\to \mathcal{M}(M)$) is not weakly continuous. The inequality comes partly from the possibility that oppositely oriented portions may cancel in the limit. In contrast to the case of functions, the cancellation can occur even without large derivatives.

1.17 Compactness Theorem for $\mathbf{I}_k^{loc}(M)$. (Simon [S, 27.3]) Let $\{T_i\}_{i\geq 1} \subseteq \mathbf{I}_k^{loc}(M)$ satisfy

$$\sup_{i\geq 1} (\mathbf{M}_U[T_i] + \mathbf{M}_U[\partial T_i]) < \infty \qquad \text{for all } U \subset\subset M.$$

Then there is $T \in \mathbf{I}_k^{loc}(M)$ and a subsequence $\{T_{i_j}\}_{j\geq 1}$ such that $T_{i_j} \rightharpoonup T$. $\square$

1.18 Slices and Boundaries. In this paper, we will often be dealing with a current $P \in \mathbf{I}_{k+1}^{loc}(M^n \times \mathbf{R})$ with $\partial P \in \mathbf{I}_k^{loc}(M \times \{0\})$. Define

$$P_z = \partial(P \lfloor (M \times [z, \infty))).$$

Let us write $(x, z) \in M \times \mathbf{R}$, $\bar{\nabla}$ for the gradient on $M \times \mathbf{R}$, and D for the gradient on P, that is, the projection μ_P-a.e. of $\bar{\nabla}$ onto $T_{(x,z)}P$. Let

$$\zeta : M \times \mathbf{R} \to \mathbf{R}$$

be the projection map. For each $z \in \mathbf{R}$, define

$$X_z = X \cap (M \times \{z\}), \qquad \theta_z = \theta | X_z \qquad \xi_z = \begin{cases} \xi \lfloor \dfrac{D\zeta}{|D\zeta|} & \text{if defined} \\ \text{anything} & \text{otherwise,} \end{cases}$$

where $\xi \lfloor v$ signifies the contraction of the alternating $(k+1)$-form ξ with the covector $D\zeta/|D\zeta|$ to produce an alternating k-form. Then define

$$\langle P, \zeta, z\rangle = \tau(X_z, \theta_z, \xi_z) \in \mathbf{I}_k^{loc}(M),$$

whenever X_x, θ_z, ξ_z are of the appropriate kind for such a definition. The following lemma says that this happens for a.e. z.

1.19 Slice Lemma. (Simon [S, 28.1, 28.5]) For $\mathcal{L}^1$-a.e. $z \in \mathbf{R}$, we have

(i) X_z is a locally k-rectifiable subset of $M \times \{z\}$.

(ii) $\mathcal{H}^k \lfloor \theta_z$ is a Radon measure.

(iii) For $\mathcal{H}^k$-a.e. $(x, z) \in X_z$, both $T_{(x,z)} X_z$ and $T_{(x,z)} X$ exist and

$$T_{(x,z)} X = T_{(x,z)} X_z \oplus^{\perp} \operatorname{Span}(D\zeta)$$

(which implies $D\zeta \neq 0$) and ξ_z is a simple k-vector of unit length orienting $T_{(x,z)} X_z$.

(iv) $\langle P, \zeta, z\rangle$ is properly defined, and $P_z = \langle P, \zeta, z\rangle \in \mathbf{I}_k^{loc}(M \times \{z\})$. □

1.20 Co-Area Formula. (Simon [S, 28.1, 28.5]) For a μ_P-integrable function $h = h(x, z)$ on X of compact support we have

$$\int_P h\,|D\zeta|\,d\mu_P = \int_{-\infty}^{\infty} \int_{P_z} h\ d\mu_{P_z} dz$$

where we ignore the integrand $\int_{P_z} h\ d\mu_{P_z}$ on the measure zero set of z for which $P_z \notin \mathbf{I}_k^{loc}(M \times \{z\})$. In particular,

$$\int_{P\lfloor(M\times[z,\infty))} h\ |D\zeta|\,d\mu_P$$

is an absolutely continuous function of z. □

1.21 Slicing Level-Sets. As a special case of 1.19, let $u : U \to \mathbf{R}$ be Lipschitz, where $U \subseteq M$ is an oriented open set. Then graph(u) is locally n-rectifiable (see Simon [S, §11]). We form the n-current $[u] \in \mathbf{I}_n^{loc}(U \times \mathbf{R})$ by

$$[u] = \tau(\operatorname{graph}(u), 1, \xi)$$

where $\xi(x, z) \in (\Lambda_k)_x(M \times \mathbf{R})$ orients $T_{(x,z)}\operatorname{graph}(u)$ in agreement with the orientation induced from M, for $\mathcal{H}^n$-a.e. $(x, z) \in \operatorname{graph}(u)$.

Then for a.e. $z \in \mathbf{R}$, $u^{-1}(z)$ is locally n-rectifiable and

$$\langle [u], \zeta, z\rangle = \tau(u^{-1}(z), 1, \xi_z) = \partial([u]\lfloor U \times [z, \infty))\lfloor(U \times \mathbf{R}) = \partial[\{u > z\}]\lfloor U$$

up to rigid motion, where ξ_z is as above, and $[\{u > z\}] \in \mathbf{I}_n^{loc}(U)$ is the current $\tau(\{u > z\}, 1, e_1 \wedge \cdots \wedge e_n)$ for an oriented orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_x M$.

1.22 Chart.

$$\begin{array}{ccccccc}
\mathcal{D}_k & \supseteq & \{\Lambda_k M\text{-valued Radon measures}\} & \supseteq & \mathcal{R}_k^{loc} & \supseteq & \mathbf{I}_k^{loc} \\
 & & \mu \Big\downarrow \text{onto} & & \mu \Big\downarrow \text{onto} & & \\
 & & \mathcal{M} & \supseteq & \mathcal{IM}_k & & \\
 & & \mu \Big\uparrow \text{onto} & & \Big\| & & \\
 & \mathbf{V}_k \equiv & \{\text{Radon measures on } G^k M\} & \supseteq & \mathbf{IV}_k & \supseteq & \{V \text{ with } |\delta V| \text{ Radon}\}.
\end{array}$$

§2 Translative Functional

2.1 The Functional. Let (M^n, g) be a complete Riemannian manifold and fix k, $0 \leq k \leq n-1$. Denote points in $M \times \mathbf{R}$ by $w = (x, z)$. Fix $\varepsilon > 0$. We define the following functional for a current $Q \in \mathbf{I}_{k+1}^{loc}(M \times \mathbf{R})$:

$$I^\varepsilon[Q] = \frac{1}{\varepsilon} \int e^{-z/\varepsilon} d\mu_Q(x, z).$$

We shall see that a stationary point P^ε of I^ε is a *translative soliton* of the mean curvature flow with velocity $-(1/\varepsilon)\partial/\partial z$. That is, if permitted to move by mean curvature, P^ε will translate in the direction $-\partial/\partial z$ with speed $1/\varepsilon$.

2.2 Heuristics of Approximation by Elliptic Regularization. Fix a locally integral cycle $T_0 \in \mathbf{I}_k^{loc}(M \times \{0\})$ of finite mass. We wish to approximate the motion of T_0 by mean curvature. To do this, minimize I^ε over all currents $Q \in \mathbf{I}_{k+1}^{loc}(M \times \mathbf{R})$ with $\partial Q = T_0$, to obtain a minimizer P^ε. For small ε, the fact that $e^{-z/\varepsilon}$ dies off so quickly will cause P^ε to have great height, approximately C/ε. In fact, it will turn out that P^ε is nearly a cylinder, in any fixed finite region.

Let P_z^ε denote the slice of P^ε at the level z and denote the respective mean curvature vectors by $\vec{H}_{P_z^\varepsilon}$, $\vec{H}_{P^\varepsilon}$. Then

$$\vec{H}_{P_z^\varepsilon} \approx \vec{H}_{P^\varepsilon}$$

for ε small, as shown in the picture. Since P^ε is a translative soliton,

$$\text{Motion by } \vec{H}_{P^\varepsilon_z} \text{ nearly produces motion (of } P^\varepsilon\text{) by } -(1/\varepsilon)\partial/\partial z.$$

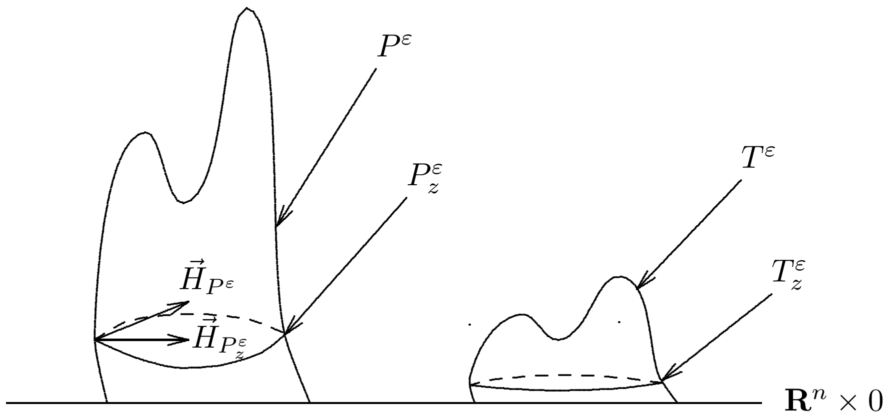

Figure 1. P^ε and T^ε.

Define the *rescaling function* $\kappa_\varepsilon : M \times \mathbf{R} \to M \times \mathbf{R}$ by

$$\kappa_\varepsilon(x, z) = (x, \varepsilon z) \equiv (x, t).$$

(Note that z and ε have units of length.) Define

$$T^\varepsilon = (\kappa_\varepsilon)_\#(P^\varepsilon), \qquad T^\varepsilon_t = \text{slice of } T^\varepsilon \text{ at height } t.$$

Scaling, we find

$$\text{Motion by } \vec{H}_{T^\varepsilon_t} \text{ nearly produces motion (of } T^\varepsilon\text{) by } -\partial/\partial t.$$

That is to say, T^ε is roughly the space-time track of T_0 moving by mean curvature. We will pass to a subsequential limit $T^{\varepsilon_i} \rightharpoonup T$ and attempt to interpret T as the motion of T_0 by mean curvature. Making this precise and exploring its consequences is the subject of this work.

2.3 Provenance. The idea of stretching out in spacetime so that $\vec{H}_{P^\varepsilon}$ approximates $\vec{H}_{P^\varepsilon_z}$ is due to Evans and Spruck. However, all their P's are smooth graphs. In fact, in the case $H > 0$, our T^ε is the graph of the function v^ε solving the equation [ES1, 7.16], namely

$$-(g - \frac{\nabla v^\varepsilon \otimes \nabla v^\varepsilon}{|\nabla v^\varepsilon|^2}) : \nabla^2 v^\varepsilon = 1.$$

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This was the author's starting point for this work.

The variational formulation of the soliton equation, for the case of homothetic solitons, was independently recorded by Angenent in [An4], based on the work of Huisken [H2].

2.4 Weighted First Variation. Let M be a manifold, and assume $\mu \in \mathcal{IM}_k(M)$, $Y \in C^1_c(TM)$, and $\{\Phi^s\}_{s\in(-\delta,\delta)}$ is a smooth family of diffeomorphisms with

$$\frac{\partial}{\partial s}\bigg|_0 \Phi^s = Y, \quad \Phi^0 = id, \quad \Phi^s|(M\setminus U) = id \quad \text{for some } U \subset\subset M.$$

Consider the functional

$$I[V_\mu] = \int_M f(x)\,d\mu(x), \qquad f > 0 \text{ smooth}.$$

The first variation of I is

$$\begin{aligned}
\frac{\partial}{\partial s}\bigg|_0 I[\Phi^s_\#(V_\mu\lfloor U)] &= \frac{\partial}{\partial s}\bigg|_0 \int f(\Phi^s(x))|J_\mu\Phi^s(x)|\,d\mu(x)\\
&= \int \frac{\partial}{\partial s}\bigg|_0 f(\Phi^s(x)) + f(x)\,\frac{\partial}{\partial s}\bigg|_0 |J_\mu\Phi^s(x)|\,d\mu\\
&= \int \nabla f\cdot Y + fS : \nabla Y\,d\mu \qquad \text{by 1.7(1)}\\
&= \int \nabla f\cdot S^\perp\cdot Y + S : \nabla(fY)\,d\mu\\
&= \delta V_\mu(fY) + \int \nabla f\cdot S^\perp\cdot Y\,d\mu.
\end{aligned}$$

We obtain the *weighted first variation formula*

$$\frac{\partial}{\partial s}\bigg|_0 I[\Phi^s_\#(V_\mu\lfloor U)] = \int(-f\vec{H} + \nabla f\cdot S^\perp)\cdot Y\,d\mu \tag{1}$$

if $|\delta V|$ is absolutely continuous with respect to μ.

2.5 Notation. Let us write

$$\nabla = \nabla^M, \quad \bar\nabla = \bar\nabla^{M\times\mathbf{R}}, \quad D = D^P, \quad \bar D = \bar{D^T}$$

for the gradient operators. Define $\omega = \partial/\partial z$.

We now specialize 2.4 to I^ε, replacing x by (x,z), ∇ by $\bar\nabla$ and

$$f(x,z) = \frac{1}{\varepsilon} e^{-z/\varepsilon}, \qquad \bar\nabla f = -\frac{1}{\varepsilon^2} e^{-z/\varepsilon}\omega.$$

2.6 Euler-Lagrange Equation for $\mathbf{I}^\varepsilon$. Assume that $P^\varepsilon \in \mathbf{I}^{loc}_{k+1}(M \times \mathbf{R})$ is stationary with respect to I^ε on $U \subseteq M \times \mathbf{R}$, and $\mu^\varepsilon = \mu_{P^\varepsilon}$. Then for all $Y \in C^1_c(TU)$, we have

$$\begin{aligned} 0 &= \int e^{-z/\varepsilon}(S : \bar\nabla Y - \frac{1}{\varepsilon}\omega \cdot Y)\, d\mu^\varepsilon \\ &= \int S : \bar\nabla(e^{-z/\varepsilon} Y) - \frac{1}{\varepsilon} e^{-z/\varepsilon}\omega \cdot S^\perp \cdot Y\, d\mu^\varepsilon \\ &= \int S : \bar\nabla Y - \frac{1}{\varepsilon}\omega \cdot S^\perp \cdot Y\, d\mu^\varepsilon \qquad\qquad \text{via } Y \leftarrow e^{z/\varepsilon} Y \\ &= \int (-\vec H - \frac{1}{\varepsilon} S^\perp \cdot \omega) \cdot Y\, d\mu^\varepsilon, \end{aligned}$$

including absolute continuity of $\delta V_{\mu^\varepsilon}$ with respect to μ^ε. Therefore

(i) *Equation:* $\quad \vec H + S^\perp \cdot \omega/\varepsilon = 0 \quad \mu^\varepsilon$-a.e.

asserting that P^ε is a translative soliton with velocity vector $-\omega/\varepsilon$.

(ii) *Bound on $\vec H$:* $\quad |\vec H| \le 1/\varepsilon \quad \mu^\varepsilon$-a.e.

(iii) *Perpendicularity:* $\quad S^\perp \cdot \vec H = \vec H \quad \mu^\varepsilon$-a.e.

Now calculate, with $\zeta(x,z) \equiv z$,

$$1 = |\bar\nabla\zeta|^2 = |\bar\nabla\zeta \cdot S^\perp|^2 + |\bar\nabla\zeta \cdot S|^2 = |\omega \cdot S^\perp|^2 + |D\zeta|^2,$$

so by (i), we obtain the useful formula

(iv) *Formula:* $\quad \varepsilon^2 H^2 + |D\zeta|^2 = 1 \quad \mu^\varepsilon$-a.e.

§3 Existence of Approximators.

Let M be complete. Consider initial surfaces of the following kind.

3.1 Initial Surface. Assume that $T_0 \in \mathbf{I}^{loc}_k(M \times \{0\})$ is a cycle of finite mass. □

With T_0 as boundary, we will find stationary surfaces for I^ε by minimization. Thereafter, only the stationary property will be used. The role of the minimizing property is under investigation.

3.2 Existence of Stationary Surface with Boundary $\mathbf{T}_0$. Assume M is complete and T_0 satisfies 3.1. Then there exists $P^\varepsilon \in \mathbf{I}_{k+1}^{loc}(M \times \mathbf{R})$ such that

(i) $\operatorname{spt} P^\varepsilon \subseteq M \times [0, \infty)$,

(ii) $I^\varepsilon[P^\varepsilon] \leq \mathbf{M}[T_0]$,

(iii) P^ε is stationary for I^ε in $M \times \mathbf{R} \setminus \operatorname{spt} T_0$. $\square$

We will prove this after 3.4.

3.3 Remarks. (a) As a matter of fact, it is easy to show that statements (ii) and (iii) imply (i). On the other hand, if $I^\varepsilon[P^\varepsilon] = \infty$, then it is possible that $\operatorname{spt} P^\varepsilon \not\subseteq M \times [0, \infty)$. For example, if T_0 is stationary for the ordinary area integrand, take $P = -T_0 \times [(-\infty, 0]]$. Another example is the $z < 0$ portion of any complete translating soliton, such as the grim reaper curve (see Angenent [An4] for an explanation) and the higher dimensional, rotationally symmetric translating solitons observed by Hamilton.

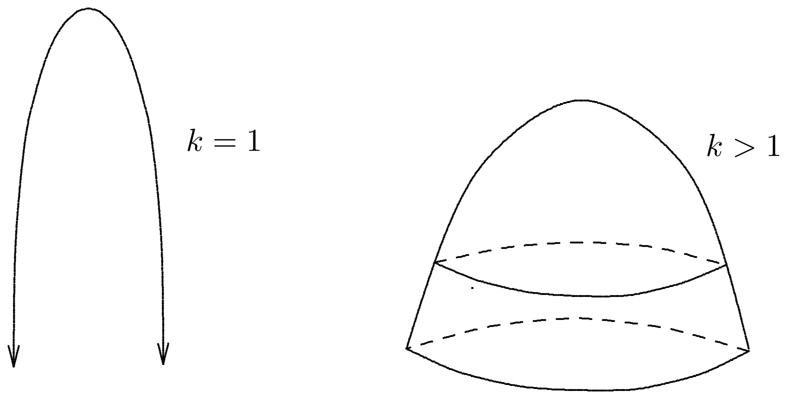

Figure 2. Complete Translating Solitons.

(b) There are examples in which $I^\varepsilon[P^\varepsilon] >> \mathbf{M}[T_0]$. For example, if we puncture a disk by a dense collection of very small disks, then what remains serves as P^ε.

(c) Another question is under what conditions (ii) becomes equality (for minimizers) as $\varepsilon \downarrow 0$. If T_0 is smooth, this holds.

3.4 Geometric Properties of I^ε. Note that I^ε is the area functional for the metric $\bar{g} = e^{-2z/((k+1)\varepsilon)}(g \oplus dz^2)$ where $g \oplus dz^2$ is the product metric on $M \times \mathbf{R}$. The

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curvature of $\bar{g}$ is rather negative due to the convexity of the coefficient. When $M = \mathbf{R}^n$, the sectional curvature of $\bar{g}$ is

$$K(\pi) = \begin{cases} 0 & \text{for } \pi \text{ containing } \partial/\partial z \\ -\dfrac{1}{n^2\varepsilon^2} e^{2z/((k+1)\varepsilon)} & \text{for } \pi \text{ parallel to } \mathbf{R}^n \times \{0\}, \end{cases}$$

where π is a 2-plane in $\mathbf{R}^n \times \mathbf{R}$.

The metric $\bar{g}$ is not complete, since $\int_{z_0}^{\infty} e^{-z/((k+1)\varepsilon)} dz < \infty$. In fact, the minimizers of I^ε can have a noncompact part trailing off the $z \gg 0$ edge of $M \times \mathbf{R}$. Such minimizers approximate solutions of the mean curvature flow that exist for all time.

Proof of 3.2. The proof is standard. Let $\mathcal{A} = \{Q \in \mathbf{I}_{k+1}^{loc}(M \times \mathbf{R}) : \partial Q = T_0\}$. Now $\partial(T_0 \times [[0,\infty)]) = T_0$ and

$$I^\varepsilon[T_0 \times [[0,\infty)]] = \frac{1}{\varepsilon}\int_{T_0}\int_0^\infty e^{-z/\varepsilon}\, dz\, d\mu_{T_0} = \mathbf{M}[T_0] < \infty,$$

and therefore

$$\inf_{Q \in \mathcal{A}} I^\varepsilon[Q] \leq \mathbf{M}[T_0] < \infty.$$

Now let $\{P_i\}_{i \geq 1}$ be a sequence of currents in $\mathcal{A}$ with

$$I^\varepsilon[P_i] \downarrow \inf_{Q \in \mathcal{A}} I^\varepsilon[Q].$$

From this we obtain local mass estimates for $\{P_i\}_{i\geq 1}$. Hence by the compactness theorem 1.17, there exists $P \in \mathbf{I}_{k+1}^{loc}(M \times \mathbf{R})$ with

$$P_i \rightharpoonup P, \qquad \partial P = T_0, \qquad \mu_P \leq \liminf \mu_{P_i} \quad \text{as Radon measures,}$$

and therefore

$$I^\varepsilon[P] \leq \liminf_i I^\varepsilon[P_i] = \inf_{Q \in \mathcal{A}} I^\varepsilon[Q] \leq \mathbf{M}[T_0].$$

Thus P satisfies 3.2(ii) and minimizes I^ε in $\mathcal{A}$. Therefore P is stationary for I^ε in $M \times \mathbf{R} \setminus \operatorname{spt} T_0$. Also note that if $\mu_P(M \times (-\infty, 0)) > 0$, the projection $\pi^+(x,z) \equiv (x, \max(z,0))$ reduces $I^\varepsilon[P]$, so $\operatorname{spt} P^\varepsilon \subseteq M \times [0,\infty)$. Q.E.D.

3.5 Compactness of Support. Assume the Ricci curvature of M is bounded below and that $\operatorname{spt} T_0$ is compact. Let P^ε be stationary with respect to I^ε, $T^\varepsilon = (\kappa_\varepsilon)_\#(P^\varepsilon)$. Then for any $\tau > 0$,

$$\operatorname{spt} T^\varepsilon \cap (M \times [0,\tau]) \subseteq K$$

where $K = K(\operatorname{spt} T_0, \tau) \subseteq M$ is a compact set independent of ε. $\square$

The idea is to prove the maximum principle for stationary varifolds touching smooth barriers (see Solomon-White [SW] or §10]), then to use the curvature hypothesis and the Bishop Comparison Theorem to construct barriers that resemble large, shrinking spheres disjoint from T_0 (see [I1, §6]). Since we do not need this result, we will not prove it here.

3.6 Remarks. (a) Without the curvature hypothesis, a surface can rush off to infinity in finite time.

(b) Without the curvature hypothesis, a surface can blossom at infinity and rush inwards, on an initially empty manifold. While such surfaces are legitimate motions by mean curvature, they have very large mass (see 6.8) and cannot arise from minimizers P^ε.

The following proposition can be proven by similar computations.

3.7 Exterior Sphere Barrier. Let $M = \mathbf{R}^n$. Suppose $M = \mathbf{R}^n$, $1 \le k \le n-1$, and $T_0 \subseteq B_R(0)$. Then $\operatorname{spt} T^\varepsilon \subseteq \{0 \le z \le (R^2 - |x|^2)/(2k)\}$. $\square$

This is consistent with the vanishing of a surface trapped within a shrinking sphere. It helps us to justify the scaling $T^\varepsilon = (\kappa_\varepsilon)_\#(P^\varepsilon)$.

In the case $k = 0$, P^ε can consist of rays $\pm[\{x\} \times [0,\infty)]$, or various grim reaper segments connecting points of T_0. These approximate points moving by mean curvature.

§4 $\mathbf{H}^2$ Estimate

If $\{T_t\}_{t\ge 0}$ is a family of surfaces moving smoothly by mean curvature, then by inserting $Y = \vec{H}$ in the first variation formula 1.8, we obtain

$$\frac{d}{dt}\mathbf{M}[T_t] = -\int H^2 \, d\mu$$

which yields the fundamental estimate for motion by mean curvature:

$$\mathbf{M}[T_t] + \int_0^t \int_{T_s} H^2 \, d\mu_{T_s} ds = \mathbf{M}[T_0].$$

We wish to derive an ε-analogue of this formula.

Let M be complete, let T_0 satisfy 3.1, let $P^\varepsilon \in \mathbf{I}^{loc}_{k+1}(M \times \mathbf{R})$ satisfy 3.2, and let $\mu^\varepsilon = \mu_{P^\varepsilon}$. Recall that $\zeta(x,z) = z$, $D\zeta = S \cdot \omega$. We will suppress the dependence on ε except in stating theorems.

4.1 Lemma. It is possible to insert $Y = \xi(z)\omega$ in 2.6 (first line), where ξ is Lipschitz. That is,

$$0 = \int e^{-z/\varepsilon}(\xi_z \omega \cdot S \cdot \omega - \frac{1}{\varepsilon}\xi)\, d\mu^\varepsilon$$

for $\xi : \mathbf{R} \to \mathbf{R}$ Lipschitz with $\operatorname{spt} \xi \subset\subset [0,\infty)$. $\square$

Proof. First assume that $\xi \in C^1_c((0,\infty),\mathbf{R})$. Fix a point $x_0 \in M$. Let $\sigma_R \in C^1_c(M,\mathbf{R}^+)$ satisfy $\sigma_R = 1$ on $B_R(x_0)$, $\sigma_R = 0$ off $B_{2R}(x_0)$, $0 \le \sigma \le 1$, and $|\nabla \sigma_R| \le 2/R$. Inserting $Y = \sigma_R(x)\xi(z)\omega$ in 2.6, we obtain

$$\begin{aligned} 0 &= \int e^{-z/\varepsilon}(S : \bar{\nabla} Y - \frac{1}{\varepsilon}\omega \cdot Y)\, d\mu \\ &= \int e^{-z/\varepsilon}(\xi\omega \cdot S \cdot \bar{\nabla}\sigma_R + \sigma_R \xi_z \omega \cdot S \cdot \omega - \frac{1}{\varepsilon}\xi\sigma_R)\, d\mu. \end{aligned}$$

Now

$$\left| \int e^{-z/\varepsilon}\xi\omega \cdot S \cdot \bar{\nabla}\sigma_R \, d\mu \right| \le \frac{2}{R} \max|\xi| \varepsilon I^\varepsilon[P]$$

so the first term disappears as we let $R \to \infty$. For the rest of the integrand,

$$\left| e^{-z/\varepsilon}(\sigma_R \xi_z \omega \cdot S \cdot \omega - \frac{1}{\varepsilon}\xi\sigma_R) \right| \le (\varepsilon \max|\xi_z| + \max|\xi|)\frac{1}{\varepsilon}e^{-z/\varepsilon},$$

a fixed function in $L^1(\mu)$. Letting $R \to \infty$, we obtain by the dominated convergence theorem,

$$0 = \int e^{-z/\varepsilon}(\xi_z \omega \cdot S \cdot \omega - \frac{1}{\varepsilon}\xi)\, d\mu, \tag{1}$$

for smooth ξ.

2. Now let $\xi : \mathbf{R} \to \mathbf{R}$ be Lipschitz with $\operatorname{spt}\xi \subset\subset [0,\infty)$. Approximate ξ by $\xi^i \in C^1_c((0,\infty))$ such that

$$\xi^i \le \xi, \quad \xi^i \to \xi \text{ uniformly}, \quad \xi^i_z \rightharpoonup \xi_z \text{ weakly-}* \text{ in } L^\infty(\mathbf{R}).$$

We can immediately pass to limits in the second term of (1). We use the co-area formula 1.20 to rewrite the first term:

$$\int e^{-z/\varepsilon}\xi_z \omega \cdot S \cdot \omega \, d\mu = \int e^{-z/\varepsilon}\xi_z |D\zeta|^2 \, d\mu = \int_{-\infty}^{\infty} \xi_z \int_{P_z} e^{-z/\varepsilon}|D\zeta| d\mu_{P_z} dz.$$

where $P_z = \partial(P\lfloor M \times [z,\infty))$. Now the function $z \mapsto \int_{P_z} e^{-z/\varepsilon}|D\zeta|\, d\mu_{P_z}$ is in $L^1(\mathbf{R})$, so we can pass to limits in the right-hand side as well. Q.E.D.

Replace ξ by $\xi e^{z/\varepsilon}$ in 4.1 to obtain

$$\begin{aligned} 0 &= \int (\frac{1}{\varepsilon}\xi + \xi_z)\, \omega \cdot S \cdot \omega - \frac{1}{\varepsilon}\xi\, d\mu \\ &= \int -\frac{1}{\varepsilon}\xi\omega \cdot S^\perp \cdot \omega + \xi_z \omega \cdot S \cdot \omega\, d\mu \end{aligned}$$

By 2.6(i) we have $S^\perp \cdot \omega = -\varepsilon\vec{H}$, and $\omega \cdot S = D\zeta$, yielding

4.2 Lemma. Let $\xi : \mathbf{R} \to \mathbf{R}$ be Lipschitz with $\operatorname{spt}\xi \subset\subset [0,\infty)$. Then

$$\int \varepsilon\xi H^2 d\mu^\varepsilon = \int \xi_z |D\zeta|^2 d\mu^\varepsilon. \quad \square$$

We apply lemma 4.2 with ξ defined by

$$\xi(z) = \begin{cases} 0 & \text{on } (-\infty, z_1] \\ 1 & \text{on } [z_1+\delta, z_2] \\ 0 & \text{on } [z_2+\delta, \infty), \end{cases}$$

and linearly interpolated between. We obtain (note that by the co-area formula, it makes no difference whether we use open or closed endpoints):

4.3 $\mathbf{H}^2$ Estimate, First Version. For $0 \le z_1 \le z_2 - \delta$,

$$\frac{1}{\delta}\int_{P^\varepsilon(z_1,z_1+\delta)} |D\zeta|^2\, d\mu^\varepsilon - \frac{1}{\delta}\int_{P^\varepsilon(z_2,z_2+\delta)} |D\zeta|^2\, d\mu^\varepsilon = \int \varepsilon\xi H^2\, d\mu^\varepsilon.$$

where $P(a,b)$ denotes $P\lfloor(M \times (a,b))$. $\square$

The lemma implies that for any fixed δ, the function

$$f_\delta(z) \equiv \frac{1}{\delta}\int_{P(z,z+\delta)} |D\zeta|^2\, d\mu$$

is a decreasing function of z. Next we would like to estimate the quantity $f_\delta(0)$. Apply Lemma 4.1 with

$$\xi = \xi_{L,\delta}(z) = \begin{cases} 0 & \text{on } (-\infty, 0] \\ 1 & \text{at } z = \delta \\ 0 & \text{on } [\delta+L, \infty), \end{cases}$$

and linearly interpolated between. We have

$$\xi_z \omega \cdot S \cdot \omega = \xi_z |D\zeta|^2 = \begin{cases} \dfrac{1}{\delta}|D\zeta|^2 & 0 \le z \le \delta \\ -\dfrac{1}{L}|D\zeta|^2 & \delta \le z \le \delta + L. \end{cases}$$

and 4.1 becomes

$$0 = \frac{1}{\delta}\int_{P(0,\delta)} e^{-z/\varepsilon}|D\zeta|^2\, d\mu - \frac{1}{L}\int_{P(\delta,\delta+L)} e^{-z/\varepsilon}|D\zeta|^2\, d\mu - \int_P \frac{1}{\varepsilon} e^{-z/\varepsilon}\xi\, d\mu.$$

The second term is bounded by $(\varepsilon/L)I^\varepsilon[P]$, and therefore vanishes when $L \to \infty$. Sending $L \to \infty$, then $\delta \to 0$, we obtain

$$\text{(1)} \qquad \lim_{\delta \downarrow 0} \frac{1}{\delta} \int_{P^\varepsilon(0,\delta)} e^{-z/\varepsilon}|D\zeta|^2\, d\mu^\varepsilon \le I^\varepsilon[P^\varepsilon].$$

This leads to

4.4 Lemma. For any measurable set $A \subseteq \mathbf{R}$,

$$\frac{1}{|A|}\int_{P^\varepsilon_A} |D\zeta|^2\, d\mu^\varepsilon \le I^\varepsilon[P^\varepsilon],$$

where $P_A = P\lfloor M \times A$ and $|A|$ is the measure of A. In particular $f_\delta(0) \le I^\varepsilon[P^\varepsilon]$. $\square$

Proof of 4.4. The inequality holds for any open interval A by decomposing into many small intervals and using (1) together with the fact that f_δ is nonincreasing. Therefore it holds for all A. Q.E.D.

Define

$$f(z) = \int_{P_z} |D\zeta|\, d\mu_{P_z}, \qquad z \in Z_1,$$

where Z_1 is the full measure subset of $[0,\infty)$ on which $P_z = \langle P, \zeta, z\rangle$ and the other good properties in the slice lemma 1.19 hold. By the co-area formula,

$$f_\delta(z) = \frac{1}{\delta}\int_z^{z+\delta} f(z')\, dz'.$$

By the Lebesgue differentiation theorem and the decreasing property of f_δ, there is a full measure set $Z \subseteq Z_1$ such that $\lim_{\delta\downarrow 0} f_\delta(z) = f(z)$ for $z \in Z$. Therefore we can pass to limits in 4.3, apply 4.4 and recall that $I^\varepsilon[P] \le \mathbf{M}[T_0]$ to obtain

4.5 $\mathbf{H}^2$ Estimate, Second Version. For $z_1, z_2 \in Z$ with $0 \leq z_1 \leq z_2$, we have

$$\int_{P^\varepsilon_{z_2}} |D\zeta| \, d\mu_{P^\varepsilon_{z_2}} + \int_{P^\varepsilon(z_1,z_2)} \varepsilon H^2 \, d\mu^\varepsilon = \int_{P^\varepsilon_{z_1}} |D\zeta| \, d\mu_{P^\varepsilon_{z_1}} \leq I^\varepsilon[P^\varepsilon] \leq \mathbf{M}[T_0]. \quad \square$$

§5 Space-Time Mass Bound

Let M be complete, and let T_0, P^ε, μ^ε, κ_ε, T^ε be as in 3.1, 3.2, 2.2. For $A \subseteq \mathbf{R}$, define

$$P_A = P \lfloor M \times A, \qquad T_B = T \lfloor M \times B, \qquad B = \varepsilon A.$$

We will obtain local mass estimates on $\mathbf{M}[T^\varepsilon_B]$ in order to pass to limits as $\varepsilon_i \downarrow 0$. Let us suppress the dependence on ε except in stating theorems.

To motivate the estimate, suppose for a moment that T were the track in space-time of a smooth surface $\{T_t\}_{t \geq 0}$ evolving by mean curvature. Then by the co-area formula,

$$\mathbf{M}[T(t, t+\delta)] = \int_t^{t+\delta} \int_{T_s} \frac{1}{|\bar{D}\tau|} d\mu_{T_y} \, ds$$

where $\tau(x,t) = t$, $\bar{D} = \bar{D}^T$. If ds is a line element on T pointing in the direction of fastest time increase, we have

$$1/|\bar{D}\tau| = ds/dt = \sqrt{(dx/dt)^2 + 1} = \sqrt{H^2 + 1},$$

since a point $x \in T_t$ is moving by $\partial x / \partial t = \vec{H}$.

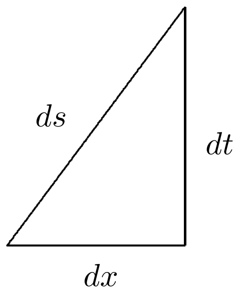

Figure 3. Line Elements

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Recalling from §4 the estimate

$$\mathbf{M}[T_t] + \int_0^t \int_{T_s} H^2 \, d\mu_{T_s} ds = \mathbf{M}[T_0],$$

calculate

$$\begin{aligned}
\mathbf{M}[T(t, t+\delta)] &= \int_t^{t+\delta} \int_{T_s} \sqrt{H^2 + 1} \, d\mu_{T_s} ds \\
&\leq \left(\int_t^{t+\delta} \int_{T_s} H^2 + 1 \; d\mu_{T_s} ds \right)^{1/2} \left(\int_t^{t+\delta} \mathbf{M}[T_s] \, ds \right)^{1/2} \\
&\leq \left(\int_0^{\infty} \int_{T_s} H^2 \, d\mu_{T_s} ds + \delta \mathbf{M}[T_0] \right)^{1/2} (\delta \mathbf{M}[T_0])^{1/2} \\
&\leq (1+\delta)^{1/2} \delta^{1/2} \mathbf{M}[T_0].
\end{aligned}$$

The above is the basic $C^{1/2}$ estimate on the $(k+1)$-volume crossed by T_t, combined with unit speed in the time direction.

Duly motivated, let us first estimate $\mathbf{M}[P_A]$. Applying 2.6(iv), 4.4 and 4.5, we have with $D = D^P$, $|A| = \mathcal{L}^1(A)$,

5.1 Mass Bound for $\mathbf{P}^\varepsilon$. Let M be complete, let T_0, P^ε satisfy 3.1, 3.2, and let $A \subseteq \mathbf{R}$ be measurable. Then

$$\mathbf{M}[P_A^\varepsilon] = \int_{P_A^\varepsilon} |D\zeta|^2 + \varepsilon^2 H^2 \, d\mu \leq (|A| + \varepsilon) \, I^\varepsilon[P^\varepsilon]. \quad \square$$

Now let $\pi : M \times \mathbf{R} \to M$ be the projection. Let us calculate $\mathbf{M}[\pi_\#(T_B)]$, where $B = \varepsilon A$. We have

$$\mathbf{M}[\pi_\#(T_B)] = \mathbf{M}[\pi_\#(P_A)] \leq \int_{P_A} |J_P \pi| \, d\mu$$

where $|J_P\pi|$ is the determinant of the map $d\pi|T_{(x,t)}P : T_{(x,t)}P \to T_x M$. Now, $d\pi|T_{(x,t)}P$ has eigenvalues $\lambda_1 = \lambda_2 = \cdots \lambda_{k-1} = 1$, $\lambda_k = |J_P\pi|$. From the diagram, evidently $|J_P\pi|^2 + |D\zeta|^2 = 1$, but by 2.6(iv) we have $\varepsilon^2 H^2 + |D\zeta|^2 = 1$, so

$$|J_P \pi| = \varepsilon |\vec{H}|.$$

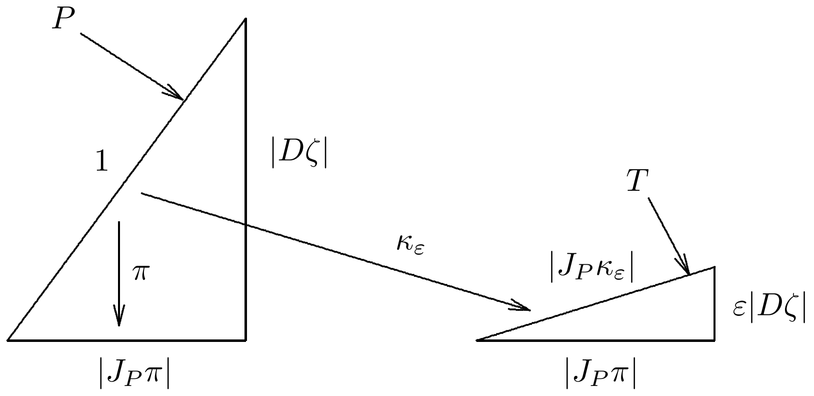

Figure 4. Finding $|J_P\pi|$, $|J_P\kappa_\varepsilon|$.

Assuming that B is a finite union of open intervals with endpoints in the full measure good set Z of 4.5, calculate

$$\begin{aligned}
\mathbf{M}[\pi_\#(T_B)] &\le \int_{P_A} \varepsilon|\vec{H}|\, d\mu^\varepsilon \\
&\le (\varepsilon\mathbf{M}[P_A])^{1/2}\left(\int_{P_A} \varepsilon H^2\, d\mu\right)^{1/2} \\
&\le (\varepsilon(\varepsilon+|A|)I^\varepsilon[P])^{1/2} I^\varepsilon[P]^{1/2} \qquad \text{by 5.1, 4.5} \\
&= (\varepsilon^2+|B|)^{1/2} I^\varepsilon[P].
\end{aligned}$$

The result holds for any measurable $B\subseteq \mathbf{R}$ by approximation. Thus

5.2 C$^{1/2}$ Estimate on (k+1)-Volume Crossed. For $B\subseteq\mathbf{R}$ measurable,

$$\mathbf{M}[\pi_\#(T_B^\varepsilon)] \le \int_{P_A^\varepsilon} |J_{P^\varepsilon}\pi|\, d\mu^\varepsilon \le (\varepsilon^2+|B|)^{1/2} I^\varepsilon[P^\varepsilon].$$

In particular, for the flat metric distance we have the $C^{1/2}+\varepsilon^2$ estimate

$$\operatorname{dist}(\pi_\#(T_t^\varepsilon), \pi_\#(T_{t+\delta}^\varepsilon)) \le (\varepsilon^2+\delta)^{1/2} I^\varepsilon[P^\varepsilon]$$

where $T_t^\varepsilon = \partial(T[t,\infty))$. $\square$

We now estimate $\mathbf{M}[T_B]$ by taking into account the time direction. By the Area Formula [S, 12.4],

$$\mathbf{M}[T_B] = \int_{P_A} |J_P\kappa_\varepsilon|\, d\mu$$

where $|J_P\kappa_\varepsilon|$ is the determinant of

$$d\kappa_\varepsilon|T_{(x,z)}P : T_{(x,z)}P \to T_{(x,t)}T.$$

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We calculate $|J_P \kappa_\varepsilon|$ according to Figure 4,

$$|J_P\kappa_\varepsilon| = \sqrt{|J_P\pi|^2 + \varepsilon^2|D\zeta|^2} \leq |J_P\pi| + \varepsilon|D\zeta| \leq |J_P\pi| + \varepsilon.$$

Thus

$$\begin{aligned}\mathbf{M}[T_B] &\leq \int_{P_A} \varepsilon + |J_P\pi|\, d\mu \\ &\leq \varepsilon(\frac{|B|}{\varepsilon} + \varepsilon)I^\varepsilon[P] + (|B| + \varepsilon^2)^{1/2} I^\varepsilon[P] \qquad \text{by 5.1 and 5.2,}\end{aligned}$$

and so we have proven,

5.3 Space-Time Mass Bound. Let M be complete. For T_0, P^ε, T^ε satisfying 3.1, 3.2, 2.2we have

$$\mathbf{M}[T_B^\varepsilon] \leq (|B| + \varepsilon^2 + (|B| + \varepsilon^2)^{1/2})I^\varepsilon[P^\varepsilon]$$

for $B \subseteq \mathbf{R}$ measurable. $\square$

§6 The Moving Varifolds of Brakke

K. Brakke introduced varifolds moving by mean curvature in his 1978 book [B]. While his method has some drawbacks, it is very general and has an impressive regularity theory.

Our definition involves Radon measures on M in place of the varifolds of [B]. For our purposes, nothing is lost by doing this, for V can be reconstructed from μ_V wherever it is k-rectifiable, and elsewhere V quickly loses mass independent of the choice of "abstract" tangent planes. We will check that our definition implies Brakke's.

6.1 Motivation. Our staring point is the formula

$$\frac{d}{dt}\int_{\Gamma_t} \phi\, d\mathcal{H}^k = \int_{\Gamma_t} -\phi H^2 + \nabla\phi \cdot S^\perp \cdot \vec{H}\, d\mathcal{H}^k, \tag{1}$$

valid for a smooth family $\{\Gamma_t\}_{t\geq 0}$ of submanifolds evolving by mean curvature, where $\phi : M \to \mathbf{R}$ is a smooth ambient test function. (Of course when M is smooth, we have $S^\perp \cdot \vec{H} = \vec{H}$). To derive this formula, apply the weighted first variation formula 2.4 with $f = \phi$ and $\partial x/\partial t = Y = \vec{H}$ (smoothly extended to M). We now define a quantity $\mathcal{B}(\mu, \phi)$ to stand in for the right hand side of (1).

6.2 Definition of $\mathcal{B}(\mu,\phi)$. Let M be complete, μ a Radon measure on M, and $\phi \in C_c^2(M,\mathbf{R}^+)$.

Singular cases. In the following cases

(i) $\mu\lfloor\{\phi>0\}$ is not a k-rectifiable Radon measure,

(ii) $|\delta V|\lfloor\{\phi>0\}$ is not a Radon measure on $\{\phi>0\}$, where $V = V_\mu\lfloor\{\phi>0\}$),

(iii) $|\delta V|\lfloor\{\phi>0\}$ is singular with respect to $\mu\lfloor\{\phi>0\}$,

(iv) $\int \phi H^2\,d\mu = \infty$, where $\vec{H} = d(\delta V)/d\mu\lfloor\{\phi>0\}$,

define $\mathcal{B}(\mu,\phi) = -\infty$.

Nonsingular case.

(v) Otherwise, define

$$\mathcal{B}(\mu,\phi) = \int -\phi H^2 + \nabla\phi\cdot S^\perp\cdot\vec{H}\,d\mu,$$

where $\nabla = \nabla^M$ and $S = S(x) \equiv T_x\mu$ for $\mathcal{H}^k$-a.e. $x \in \{\phi>0\}$. □

Note that the failure of each case renders the succeeding case meaningful.

The *upper derivate* of $f:\mathbf{R}\to\mathbf{R}$ is defined by

$$\bar{D}_{t_0} f(t) = \limsup_{t\to t_0} \frac{f(t)-f(t_0)}{t-t_0}.$$

6.3 Definition of Brakke Motion. Let M be complete and $\{\mu_t\}_{t\geq 0}$ a family of Radon measures on M. We call $\{\mu_t\}_{t\geq 0}$ a *Brakke motion* provided that for all $t\geq 0$ and all $\phi\in C_c^2(M,\mathbf{R}^+)$,

$$\bar{D}_t\mu_t(\phi) \leq \mathcal{B}(\mu_t,\phi). \quad \square$$

6.4 Discussion of Technical Points.

(a) It follows trivially from 6.2(i) and 7.2 that a Brakke motion satisfies $\mu_t \in \mathcal{M}_k(M)$ for a.e. $t\geq 0$, provided the mass stays locally bounded.

(b) Note that $\mathcal{B}(\mu,\phi)$ depends only on $\mu\lfloor\{\phi>0\}$.

(c) Note that $|\delta V|\lfloor\{\phi>0\} = |\delta V\lfloor\{\phi>0\}|$.

(d) It follows from lemma 6.6 below and Holder's inequality that $\mathcal{B}(\mu,\phi)$ is finite in the nonsingular case 6.2(v).

(e) Brakke included the $S^\perp$ before he had proven that for μ of locally finite total variation, $S^\perp \cdot \vec{H} = 0$ μ-a.e., see [B, §5.8]. We choose to retain it.

6.5 Advantages and Disadvantages of the Moving Varifolds of Brakke.

1. *Inequality.* It certainly seems preferable to have equality in 6.3. The inequality arises from the lower semicontinuity of $\int H^2 d\mu$ in passing to a limit. In fact, there are moving varifolds where inequality is necessary. For example, if μ_0 is stationary (i.e. $\delta V_{\mu_0} = 0$), then equality in 6.3 would imply that $\mu_t = \mu_0$ for all $t \geq 0$. But a stationary, nonminimizing varifold really can begin moving, as shown in the figure.

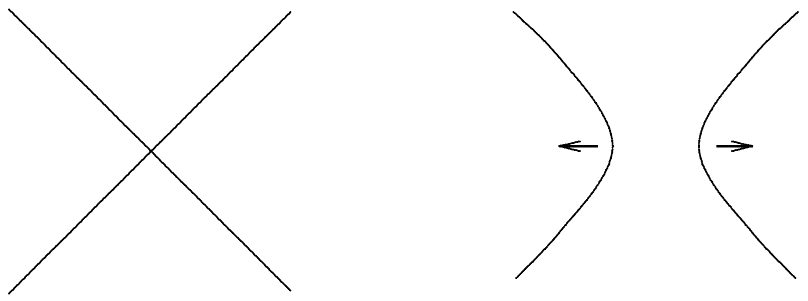

Figure 5. Stationary Cross Begins to Move.

2. *Vanishing.* The inequality permits μ_t to vanish abruptly, even gratuitously, or otherwise dwindle faster than expected. Part and parcel of the vanishing phenomenon is the question: in what sense does $\{\mu_t\}_{t\geq 0}$ assume the initial conditions?

In fact, certain varifolds must vanish abruptly. For example, Brakke [B, Fig. 9] gives the example of a spoon-shaped homothetic soliton, as in the figure. The varifold shrinks self-similarly toward the indicated point, then vanishes.

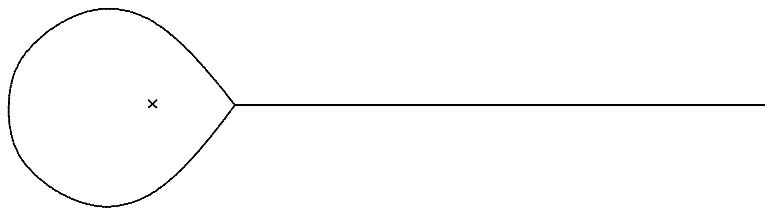

Figure 6. Homothetic Spoon.

Conjecture. Sudden loss of mass never occurs for the motion of cycles (as distinct from surfaces with flanges, etc.). □

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3. *Nonuniqueness.* Another characteristic of Brakke's approach is nonuniqueness of the motion. The problem of gratuitous vanishing leads to much inessential nonuniqueness. However, nonuniqueness is a true property of motion by mean curvature, at least for nonsmooth initial data. The example in the figure above can be rotated to give nonuniqueness; Brakke also gives examples. Nonuniqueness occurs exactly when the corresponding level-set flow fattens, to be made precise in Chapter Two.

Conjecture. (Evans-Spruck; DeGiorgi) If Γ_0 fattens under the level-set flow, then it fattens instantly. In particular, a smooth flow never fattens. □

4. *Multiple density.* Brakke's regularity theory requires the hypothesis:

For each $t \geq 0$, μ_t has unit density μ_t-a.e.

However, Brakke's existence theory does not provide this hypothesis.

We turn next to properties we will need.

6.6 Lemma. If $\phi \in C_c^2(M, \mathbf{R}^+)$, then

$$\frac{|\nabla\phi|^2}{\phi} \leq 2 \max |\nabla^2\phi| \qquad \text{on } \{\phi > 0\}.$$

In particular, $\max |\nabla\phi^{1/2}| < \infty$. □

Proof. (Y.-X. Zheng) Define $f^\varepsilon = |\nabla\phi|^2/(\phi+\varepsilon)$. If $\phi \not\equiv 0$, then f^ε will attain a maximum on $\{\phi > 0\}$. At the maximum we have

$$0 = \nabla f^\varepsilon = \frac{2\nabla^2\phi \cdot \nabla\phi}{\phi+\varepsilon} - \frac{|\nabla\phi|^2\nabla\phi}{(\phi+\varepsilon)^2}.$$

Since $\nabla\phi \neq 0$ at the maximum of f^ε, we get

$$\frac{|\nabla\phi|^2}{\phi+\varepsilon} = 2\left|\nabla^2\phi \cdot \frac{\nabla\phi}{|\nabla\phi|}\right| \leq 2\max|\nabla^2\phi|.$$

Letting $\varepsilon \downarrow 0$, we obtain the desired inequality on $\{\phi > 0\}$. Q.E.D.

6.7 Growth Bound. ([B, 3.4]) Let μ be a Radon measure and suppose $\phi \in C_c^2(M, \mathbf{R}^+)$. Then

$$\mathcal{B}(\mu, \phi) \leq C_2(\phi)\, \mu(\{\phi > 0\}) < \infty$$

where $C_2(\phi) \equiv \max|\nabla^2\phi|/2$. □

Proof. If $\mathcal{B}(\mu,\phi) > -\infty$, then

$$\begin{aligned}\mathcal{B}(\mu,\phi) &= \int -\phi H^2 + \nabla\phi \cdot S^\perp \cdot \vec{H}\, d\mu \\ &= \int -\phi \left| \vec{H} - \frac{1}{2}\frac{\nabla\phi : S^\perp}{\phi} \right|^2 + \frac{1}{4}\frac{|\nabla\phi : S^\perp|^2}{\phi} d\mu \\ &\le \frac{1}{4}\int \frac{|\nabla\phi|^2}{\phi} d\mu \\ &\le (\max|\nabla^2\phi|/2)\, \mu(\{\phi > 0\}) \qquad \text{by 6.6.}\end{aligned}$$

Q.E.D.

6.8 Decreasing Property of Mass. If $\int_0^t \mathbf{M}[\mu_s]\, ds < \infty$ then

$$\mathbf{M}[\mu_t] \le \mathbf{M}[\mu_0]. \quad \square$$

Proof. Fix $x_0 \in M$. Let $\phi = \phi_R \in C_c^2(M, \mathbf{R}^+)$ satisfy $\phi = 1$ on $B_R(x_0)$, $\phi = 0$ off $B_{2R}(x_0)$, $0 \le \phi \le 1$, $|\nabla\phi| \le 2/R$. Then by 6.3 and Cauchy's inequality,

$$\int \phi^2\, d\mu_t - \int \phi^2 d\mu_0 \le \int_0^t \mathcal{B}(\mu_s, \phi^2)\, ds \le \int_0^t \int |\nabla\phi|^2\, d\mu_s\, ds \le \frac{4}{R^2}\int_0^t \mathbf{M}[\mu_s]\, ds.$$

Letting $R \to \infty$, we obtain the desired result. Q.E.D.

This shows, for example, that on those highly curved manifolds on which a surface can blossom at infinity, the surface must be pretty big at first.

6.9 Time-Varying Test Functions. Let $\{\mu_t\}_{t\ge 0}$ be a Brakke motion. Suppose $\phi \in C_c^1(M \times [0,T], \mathbf{R}^+)$. Then

$$D_t^+ \mu_t(\phi(\cdot,t)) \le \mathcal{B}(\mu_t, \phi(\cdot,t)) + \int \frac{\partial}{\partial t}\phi(x,t)\, d\mu_t(x). \quad \square$$

Here $D_{t_0}^+ f(t) = \limsup_{t \downarrow t_0}(f(t) - f(t_0))/(t - t_0)$. The proof is the same as [B, 3.5] except for the change of notation.

6.10 Brakke's Original Definition. (adapted to M) Let $V \in \mathbf{V}_k(M)$ and $\phi \in C_c^2(M, \mathbf{R}^+)$. Brakke [B, 3.2] defines $\delta(V,\phi)(\vec{H}_V)$ as follows. If

(i) $|\delta V| \lfloor \phi$ is not a Radon measure,

(ii) $|\delta V|_{sing} \lfloor \phi \ne 0$, where $|\delta V|_{sing}$ is calculated with respect to μ_V, or (iii) $\int \phi H_V^2 \, d\mu_V = \infty$, where $\vec{H}_V = d(\delta V \lfloor \{\phi > 0\})/d\mu_V$, then

$$\delta(V, \phi)(\vec{H}_V) = -\infty.$$

(iv) Otherwise

$$\delta(V, \phi)(\vec{H}_V) = -\int \phi(x) H^2(x) \, d\mu_V(x) + \int \nabla\phi(x) \cdot S^\perp \cdot \vec{H}(x) \, dV(x, S).$$

A family $\{V_t\}_{t \geq 0} \subseteq \mathbf{V}_k(M)$ is said to be moving by mean curvature provided

$$\bar{D}_t \mu_{V_t}(\phi) \leq \delta(V_t, \phi)(\vec{H}_{V_t})$$

for all $t \geq 0$ and $\phi \in C_c^2(M, \mathbf{R}^+)$. $\square$

Let us see that our definition implies Brakke's original definition. Given a Radon measure μ_t on M, define a varifold $V_t \in \mathbf{V}_k(M)$ as follows.

(i) On the largest open set $U \subseteq \mathbf{R}^n$ such that $\mu_t \lfloor U \in \mathcal{M}_k(U)$, define $V_t \lfloor (G^k M | U) = V(\mu_t \lfloor U)$.

(ii) Elsewhere we extend V_t to be any varifold such that $\mu_{V_t} = \mu_t$.

If $\mu \lfloor \{\phi > 0\} \in \mathcal{M}_k(\{\phi > 0\})$, then, using Holder's inequality to handle 6.10(i), $\mathcal{B}(\mu, \phi) = \delta(V, \phi)(\vec{H}_V)$. Otherwise, $\mathcal{B}(\mu, \phi) = -\infty$. Therefore,

6.11 Implication Between the Definitions. If $\{\mu_t\}_{t \geq 0}$ satisfies 6.3, then any associated family $\{V_t\}_{t \geq 0}$, defined as above, satisfies 6.10. $\square$

The converse is not quite true, but this will not bother us.

§7 Convergence Lemma for Integral Brakke Motions

Like any other self-respecting elliptic or parabolic equation, Brakke's motion has a compactness theorem. The proof is an adaptation of ideas in [B, Chap. 4]. We call a Brakke motion *integral* if $\mu_t \in \mathcal{IM}_k(M)$ for a.e. $t \geq 0$.

7.1 Compactness for Integral Brakke Motions. Let M be complete. Let $\{\mu_t^i\}_{t \geq 0}$, $i = 1, 2, \ldots$ be a sequence of integral Brakke motions in M. Suppose

(i) $\sup_{i,t} \mu_t^i(U) \leq C_1(U) < \infty$

for each $U \subset\subset M$. Then

(ii) There is a subsequence $\{\mu_t^{i_j}\}_{t\geq 0}$, $j = 1, 2, \ldots$ and an integral Brakke motion $\{\mu_t\}_{t\geq 0}$ such that

$$\mu_t^{i_j} \to \mu_t \quad \text{as Radon measures for each } t \geq 0.$$

(iii) For a.e. $t \geq 0$, there is a subsequence $\{i'_j\}_{j\geq 1}$ of $\{i_j\}_{j\geq 1}$ (depending on t) such that

$$\lim_{j\to\infty} V(\mu_t^{i'_j}) = V(\mu_t) \quad \text{as varifolds.} \quad \square$$

The proof appears in 7.6 after a few lemmas.

Remarks. 1. Hypothesis (i) nearly always follows from mass bounds at the initial time $t = 0$. See 6.8.

2. The theorem is equally valid for (real) k-rectifiable Brakke motions with a lower density bound

$$\Theta^k(\mu_t, x) \geq \theta_0 > 0 \qquad \mu_t\text{-a.e. (for a.e. } t).$$

7.2 Semidecreasing and Continuity Properties. Let $\{\mu_t\}_{t\geq 0}$ be a Brakke motion satisfying $\mu_t(U) \leq C_1(U)$ for $t \geq 0$. Then

(i) $\mu_t - C_3(\phi)t$ is nonincreasing for $\phi \in C_c^2(M, \mathbf{R}^+)$, where $C_3(\phi) = C_2(\phi)C_1(\{\phi > 0\})$.

(ii) The right and left limits always exist and satisfy

$$\lim_{s\uparrow t} \mu_s(\phi) \geq \mu_t(\phi) \geq \lim_{s\downarrow t} \mu_s(\phi).$$

(iii) There is a co-countable set $B_2 \subseteq [0, \infty)$ with μ_t continuous at all $t \in B_2$.

(iv) There is a full measure set $B_3 \subseteq [0, \infty)$ with $\bar{D}_t\mu_t(\phi) > -\infty$ for all $\phi \in C_c^2(M, \mathbf{R}^+)$ and all $t \in B_3$.

(v) $\mu_t \in \mathcal{M}_k(M)$ for a.e. $t \geq 0$. $\square$

Proof. (i) By 6.3 and 6.7, we have

$$\bar{D}_t\mu_t(\phi) \leq \mathcal{B}(\mu_t, \phi) \leq C_2(\phi)C_1(\{\phi > 0\}).$$

(ii) is clear from (i).

(iii) Let Ψ be a countable dense subset of $C_c^2(M,\mathbf{R}^+)$. By (i), for each $\psi \in \Psi$ there is a co-countable set $B_\psi \subseteq [0,\infty)$ such that $\mu_t(\psi)$ is continuous on B_ψ. Define $B_2 = \bigcap_{\psi\in\Psi} B_\psi$, so that $\mu_t(\psi)$ is continuous at each $t \in B_2$ for each $\psi \in \Psi$. The result follows by uniform approximation.

(iv) For each $\psi \in \Psi$, there is a full measure set $\tilde{B}_\psi$ such that $\bar{D}_t\mu_t(\phi) > -\infty$ on $\tilde{B}_\psi$. Define $B_3 = \bigcap\limits_{\psi\in\Psi} \tilde{B}_\psi$. Let $\phi \in C_c^2(M,\mathbf{R}^+)$ and choose $\psi \in \Psi$, $\psi \geq \phi$. For $t \in B_3$,

$$-\infty < \bar{D}_t\mu_t(\psi) \leq \bar{D}_t\mu_t(\phi) + \bar{D}_t\mu_t(\psi-\phi) \leq \bar{D}_t\mu_t(\phi) + C_3(\psi-\phi)$$

by (1), so $\bar{D}_t\mu_t(\phi) > -\infty$ for $t \in B_3$.

(v) Therefore by definition of $\mathcal{B}(\mu_t,\phi)$, we have $\mu_t \lfloor \{\phi > 0\} \in \mathcal{M}_k(\{\phi > 0\})$ for all $t \in B_3$. Q.E.D.

7.3 Upper Semicontinuity of $\mathcal{B}(\mu,\phi)$. ([B], 4.28]) Let $\phi \in C_c^2(M,\mathbf{R}^+)$. Let $\{\mu_i\}_{i\geq 1} \subseteq \mathcal{IM}_k(M)$ be a sequence satisfying

$$\text{(i)} \qquad \mathcal{B}(\mu_i,\phi) \geq -C_4, \quad \mu_i(\{\phi > 0\}) \leq C_4 \qquad \text{for all } i \geq 1.$$

Then there is a subsequence $\{\mu_{i_j}\}_{j\geq 1}$ and a Radon measure $\mu \in \mathcal{IM}_k(\{\phi > 0\})$ such that

$$\text{(ii)} \qquad \begin{aligned} \mu_{i_j}\lfloor\{\phi > 0\} &\to \mu && \text{as Radon measures on } \{\phi > 0\} \\ V(\mu_{i_j}\lfloor\{\phi > 0\}) &\to V(\mu) && \text{as varifolds over } \{\phi > 0\}. \end{aligned}$$

Furthermore, for any convergent sequence $\mu_i \to \mu$ of Radon measures on $\{\phi > 0\}$ with $\sup\limits_{i\geq 1} \mu_i(\{\phi > 0\}) < \infty$, we have

$$\text{(iii)} \qquad \limsup_i \mathcal{B}(\mu_i,\phi) \leq \mathcal{B}(\mu,\phi). \quad \square$$

To prove this, we need the following elementary lemma.

7.4 $\mathbf{L}^2$ Approximation Lemma. Let μ be k-rectifiable. Then for any vector field $Z \in L^2(\mu,TM)$ and any $\varepsilon > 0$, there is a vector field $Y \in C_c^1(TM)$ such that

$$\|Z - Y\|_{L^2(\mu,TM)} \leq \varepsilon. \quad \square$$

Proof. We can assume $\mu = \mu(X,\theta)$ where $X \subseteq \bigcup_{i=1}^\infty C_i$ and the C_i are disjoint, precompact C^1 submanifolds of M. In $L^2(\mu,TM)$ we have $Z = \sum_i Z|C_i$. Each function

$Z|C_i$ can be well approximated in $L^2(\mathcal{H}^k \lfloor C_i \lfloor \theta, TM)$ by a function Z_i with $\operatorname{spt} Z_i \subset\subset C_i$. Then any finite assemblage $Z_1, \dots, Z_p$ can be approximated in $L^2(\mathcal{H}^k \lfloor C_i \lfloor \theta, TM)$ by functions $Y_1, \dots, Y_p \in C^1_c(TM)$ with disjoint supports. Then $Y_1 + \cdots + Y_p$ approximates Z in $L^2(\mu, TM)$. Q.E.D.

Proof of 7.3. 1. Let μ_i, $i = 1, 2, \dots$ satisfy inequalities (i). Then V_{μ_i}, $\vec{H}_{\mu_i}$ make sense and using Cauchy's inequality,

$$-C_4 \le \int -\phi H^2 + \nabla\phi \cdot S^\perp \cdot \vec{H}\, d\mu_i \le \int -\frac{1}{2}\phi H^2 + \frac{1}{2}\frac{|\nabla\phi|^2}{\phi}\, d\mu_i,$$

from which we obtain

$$\int \phi H^2\, d\mu_i \le 2C_4 + 4C_4 C_2(\phi) = C(C_4, \phi). \tag{1}$$

and by Cauchy-Schwarz, for each $U \subset\subset \{\phi > 0\}$,

$$|\delta V_{\mu_i}|(U) \le C(C_4, \phi, U). \tag{2}$$

Since also $\mu_i(\{\phi > 0\}) \le C_4$, Allard's compactness theorem 1.9 implies that there is a subsequence $\{\mu_{i_j}\}_{j\ge1}$ and a Radon measure $\mu \in \mathcal{IM}_k(\{\phi > 0\})$ such that

$$\begin{aligned} \mu_{i_j} \lfloor \{\phi > 0\} &\to \mu && \text{as Radon measures on } \{\phi > 0\}, \\ V(\mu_{i_j} \lfloor \{\phi > 0\}) &\to V(\mu) && \text{as varifolds over } \{\phi > 0\}, \end{aligned}$$

proving (ii).

2. For (iii), let $\mu_i \to \mu$ as Radon measures on $\{\phi > 0\}$ with $\sup_{i\ge1} \mu_i(\{\phi > 0\}) < \infty$. If $\limsup_i \mathcal{B}(\mu_i, \phi) = -\infty$, we are done. Otherwise assume, by taking a subsequence, that $\lim_{i\to\infty} \mathcal{B}(\mu_i, \phi)$ exists and is finite. Therefore hypothesis (i) holds for $\{\mu_i\}_{i\ge1}$. By (ii), we can replace $\{\mu_i\}_{i\ge1}$ by a subsequence and assume (after relabelling) that

$$V(\mu_i \lfloor \{\phi > 0\}) \to V(\mu \lfloor \{\phi > 0\}) \tag{3}$$

as varifolds over $\{\phi > 0\}$.

3. We cannot prove (iii) directly for ϕ. We will prove it first for $\psi \in C^2_c(\{\phi > 0\}, \mathbf{R}^+)$. Note by (1) and (2) that both $\int -\psi H^2\, d\mu_i$ and $\int \nabla\psi \cdot S^\perp \cdot \vec{H}\, d\mu_i$ are bounded independent of i.

4. For the first of these terms, by the rectifiability of μ and the L^2 Approximation Lemma we have

$$\left(\int \psi H^2\, d\mu\right)^{1/2} = \sup\left\{\int \psi^{1/2}\vec{H}\cdot Y\, d\mu : \|Y\|_{L^2(\mu,TM)} = 1,\ \ Y \in C^1_c(\{\phi>0\}, TM)\right\}.$$

Recalling that $\psi^{1/2} \in C^1$ by 6.6, calculate

$$\begin{aligned}\int \psi^{1/2}\vec{H}\cdot Y\, d\mu &= -\delta V_\mu(\psi^{1/2}Y)\\ &= \lim_{i\to\infty} -\delta V_{\mu_i}(\psi^{1/2}Y) \qquad\qquad \text{by (3)}\\ &= \lim_{i\to\infty}\int \psi^{1/2}\vec{H}_i\cdot Y d\mu_i\\ &\le \liminf_{i\to\infty}\left(\int \psi H_i^2\, d\mu_i\right)^{1/2}\left(\int |Y|^2 d\mu_i\right)^{1/2}\end{aligned}$$

from which it follows that

$$\int \psi H^2\, d\mu \le \liminf_{i\to\infty}\int \psi H_i^2\, d\mu_i. \tag{4}$$

5. For the second term, note that $\nabla\psi(x)\cdot S^\perp(x)$ is a μ-integrable vector field in $L^\infty(\mu, TM)$. Since μ is k-rectifiable, by the L^2 Approximation Lemma we can choose $Y \in C^1_c(\{\phi>0\}, \mathbf{R}^n)$ with

$$\|Y - \nabla\psi\cdot S^\perp\|_{L^2(\mu,TM)} \le \delta.$$

Then

$$\begin{aligned}\int \nabla\psi\cdot S^\perp\cdot\vec{H}\, d\mu - \int \nabla\psi\cdot S_i^\perp\cdot\vec{H}_i\, d\mu_i = &\int (\nabla\psi\cdot S^\perp - Y)\cdot\vec{H}\, d\mu && \text{(A)}\\ &-\delta V_\mu(Y) + \delta V_{\mu_i}(Y) && \text{(B)}\\ &+\int (Y - \nabla\psi\cdot S_i^\perp)\cdot\vec{H}_i\, d\mu_i. && \text{(C)}\end{aligned}$$

Now $\lim_{i\to\infty} B = 0$ by (3) and

$$|A|^2 \le \delta^2\int_{\{\psi>0\}} H^2 d\mu \le \delta^2 \sup_{\{\psi>0\}}\left(\frac{1}{\phi}\right)\int \phi H^2 d\mu \le C(\phi,\psi)\delta^2,$$

and again by (3),

$$\begin{aligned}\lim_{i\to\infty}|C|^2 &\le \sup_{\{\psi>0\}}\left(\frac{1}{\phi}\right)\sup_{i>0}\int \phi H_i^2\,d\mu_i\cdot\lim_{i\to\infty}\int_{\{\psi>0\}}|Y(x)-\nabla\psi(x)S^\perp|^2 dV_{\mu_i}(x,S)\\ &\le C(\phi,\psi)\int\limits_{\{\psi>0\}}|Y(x)-S^\perp\cdot\nabla\psi(x)|^2 dV_\mu(x,S)\\ &\le C(\phi,\psi)\delta^2.\end{aligned}$$

We pass $\delta\to 0$ in (A), (B), (C) to obtain

$$\lim_{i\to\infty}\int \nabla\psi\cdot S_i^\perp\cdot\vec{H}_i\,d\mu_i=\int\nabla\psi\cdot S^\perp\cdot\vec{H}\,d\mu.$$

Combining with (4), we get

$$\limsup_{i\to\infty}\mathcal{B}(\mu_i,\psi)\le\mathcal{B}(\mu,\psi)\qquad\text{for }\psi\in C_c^2(\{\phi>0\},TM). \tag{5}$$

6. Now choose a sequence $\{\psi_j\}_{j\ge 1}$ in $C_c^2(\{\phi>0\},\mathbf{R}^+)$ such that $\psi_j\le\phi$ and $\psi_j\to\phi$ in C^2. Then

$$\lim_{j\to\infty}\int\psi_j H^2\,d\mu=\int\phi H^2\,d\mu$$

by the dominated convergence theorem. Also

$$\left|\int\nabla\psi_j\cdot S^\perp\cdot\vec{H}\,d\mu-\int\nabla\phi\cdot S^\perp\cdot\vec{H}\,d\mu\right|^2\le\int\frac{|\nabla\psi_j-\nabla\phi|^2}{|\psi_j-\phi|}d\mu\ \int|\psi_j-\phi|H^2 d\mu.$$

Each of the terms on the right goes to zero as $j\to\infty$ since $\int\phi H^2 d\mu<\infty$ and $\psi_j\to\phi$ in C^2 (see 6.6). Combining these,

$$\lim_{j\to\infty}\mathcal{B}(\mu,\psi_j)=\mathcal{B}(\mu,\phi). \tag{6}$$

Then by (5) and 6.7,

$$\begin{aligned}\limsup_{i\to\infty}\mathcal{B}(\mu_i,\phi)&\le\limsup_{i\to\infty}\mathcal{B}(\mu_i,\psi_j)+\sup_{i>0}\mathcal{B}(\mu_i,\phi-\psi_j)\\ &\le\mathcal{B}(\mu,\psi_j)+2\|\phi-\psi_j\|_{C^2}\sup_{i>0}\mu_i(\{\phi>0\}).\end{aligned}$$

Then by (6), sending $j\to\infty$,

$$\limsup_{i\to\infty}\mathcal{B}(\mu_i,\phi)\le\mathcal{B}(\mu,\phi)$$

as desired. Q.E.D.

7.5 Measurability. Let $\{\mu_t\}_{t\geq 0}$ be a Brakke motion satisfying $\mu_t(U) \leq C_1(U)$ for $U \subset\subset M$. Then

(i) $t \mapsto \bar{D}_t\mu_t(\phi)$ is measurable.

(ii) $t \mapsto \mathcal{B}(\mu_t, \phi)$ is upper semicontinuous on the set B_2 of points at which $\{\mu_t\}_{t\geq 0}$ is continuous. $\square$

Proof. (i) follows from 7.2(i). For (ii), if $t_i \to t$ and $\mathcal{B}(\mu_{t_i}, \phi) \geq \alpha$, then by the upper semicontinuity result 7.3, there is a Radon measure μ on M and a subsequence with

$$\mu_{t_{i_j}} \to \mu, \qquad \mathcal{B}(\mu, \phi) \geq \alpha.$$

If $t \in B_2$, then $\mu = \mu_t$. Q.E.D.

7.6 Proof of 7.1. Idea of proof: We will first establish subsequential convergence of the measure μ_t^i on a countable dense set of times. Convergence for all times follows by the semidecreasing property in 7.2. The upper semicontinuity 7.3 allows Brakke's inequality 6.3 to pass to limits. The estimate $\iint H^2\, d\mu_t\, dt < \infty$ and Allard's compactness theorem 1.9 proves integer rectifiability for a.e. t.

1. Let B_1 be a countable dense set of times. For each $t \in B_1$, the mass bound 7.1(i) and the compactness of Radon measures allows us to choose a subsequence $\{\mu_t^{i_j}\}_{j\geq 1}$ and a Radon measure μ_t on M such that $\mu_t^{i_j} \to \mu_t$ as Radon measures. By choosing successive subsequences and diagonalizing, we can find a single subsequence, also to be called $\{\mu_t^{i_j}\}_{j\geq 1}$, such that

$$\mu_t^{i_j} \to \mu_t \qquad \text{for all } t \in B_1.$$

For any $\phi \in C_c^2(M, \mathbf{R}^+)$, by 7.2(i) we know that $\mu_t^{i_j}(\phi) - C_3(\phi)t$ is nonincreasing in t. Passing to limits as $i_j \to \infty$ we find

$$\mu_t(\phi) - C_3(\phi)t \quad \text{is nonincreasing for } t \in B_1.$$

2. For $t \in [0, \infty) \setminus B_1$, let $\{\mu_t^{i'_k}\}_{k\geq 1}$ be any convergent subsequence of $\{\mu_t^{i_j}\}_{j\geq 1}$, depending on t. Define the Radon measure μ_t by

$$\mu_t = \lim_{k\to\infty} \mu_t^{i'_k}.$$

So μ_t is defined for all $t \in [0, \infty)$. Applying step 1 to $B_1 \cup \{t\}$ separately for each $t \in [0, \infty) \setminus B_1$, we find

$$\mu_t(\phi) - C_3(\phi)t \quad \text{is nonincreasing for each } \phi \in C^2_c(M, \mathbf{R}^+).$$

By 7.2(iii), there is a co-countable set B_2 such that μ_t is continuous at all $t \in B_2$. For such t, μ_t is uniquely determined, independent of the subsequence $\{i'_k\}_{k \geq 1}$ chosen for that t, and hence the full sequence converges:

$$\mu_t^{i_j} \to \mu_t \quad \text{for } t \in B_2.$$

Because $[0, \infty) \setminus B_2$ is countable, we can choose further subsequences and diagonalize again to obtain a single subsequence, also to be called $\{i_j\}_{j \geq 1}$, such that $\lim_{i_j \to \infty} \mu_t^{i_j}$ exists for all $t \geq 0$. For $t \in [0, \infty) \setminus B_2$, replace the old values of μ_t by these new ones. Thus we have constructed a family $\{\mu_t\}_{t \geq 0}$ of Radon measures such that

$$\begin{aligned} &\mu_t^{i_j} \to \mu_t \quad \text{for all } t \geq 0, \\ &\mu_t(\phi) - C_3(\phi)t \quad \text{is nonincreasing for all } \phi \in C^2_c(M, \mathbf{R}^+). \end{aligned} \tag{1}$$

3. Let us prove that $\{\mu_t\}_{t \geq 0}$ is an integral Brakke motion. Fix $\phi \in C^2_c(M, \mathbf{R}^+)$ and $t_0 \geq 0$. We aim to show that $\bar{D}_{t_0}\mu_t(\phi) \leq \mathcal{B}(\mu_{t_0}, \phi)$. Now $\bar{D}_{t_0}\mu_t(\phi) = \max(D^-_{t_0}\mu_t(\phi), D^+_{t_0}\mu_t(\phi))$, where D^- and D^+ are the left and right upper derivates, respectively. We will prove that $D^+_{t_0}\mu_t(\phi)) \leq \mathcal{B}(\mu_{t_0}, \phi)$; the other case is similar.

Without loss of generality, assume $D^+_{t_0}\mu_t(\phi) > -\infty$. Then there exist $t_q \downarrow t_0$, $\varepsilon_q \downarrow 0$ with

$$D^+_{t_0}\mu_t(\phi) - \varepsilon_q \leq \frac{\mu_{t_q}(\phi) - \mu_{t_0}(\phi)}{t_q - t_0}. \tag{2}$$

By the convergence $\mu_t^{i_j} \to \mu_t$, there is for each q a number r_q, selected from the sequence $\{i_j\}_{j \geq 1}$, such that

$$D^+_{t_0}\mu_t(\phi) - 2\varepsilon_q \leq \frac{\mu^{r_q}_{t_q}(\phi) - \mu^{r_q}_{t_0}(\phi)}{t_q - t_0} \leq \frac{1}{t_q - t_0} \int_{t_0}^{t_q} \bar{D}_t \mu_t^{r_q}(\phi)\, dt.$$

Note that $\bar{D}_t\mu_t^{r_q}(\phi)$ is measurable and bounded above by $C_3(\phi)$. We can assume that $r_q \to \infty$ as $q \to \infty$. It follows that there is at least one point $s_q \in [t_0, t_q]$ such that $\mu^{r_q}_{s_q} \in \mathcal{IM}_k(\{\phi > 0\})$ and

$$D^+_{t_0}\mu_t(\phi) - 2\varepsilon_q \leq \bar{D}_t\mu^{r_q}_{s_q}(\phi) \leq \mathcal{B}(\mu^{r_q}_{s_q}, \phi).$$

By the mass bound 7.1(i) and the upper semicontinuity lemma 7.3, there is a subsequence $\{\mu^{r'_q}_{s'_q}\}_{q\geq 1}$ and a Radon measure $\mu \in \mathcal{IM}_k(\{\phi > 0\})$ such that

$$r'_q \to \infty, \qquad s'_q \downarrow t_0, \qquad \mu^{r'_q}_{s'_q} \to \mu \quad \text{as Radon measures} \tag{3}$$

and

$$D^+_{t_0}\mu_t(\phi) \leq \limsup_{q\to\infty} \mathcal{B}(\mu^{r'_q}_{s'_q}, \phi) \leq \mathcal{B}(\mu, \phi). \tag{4}$$

4. We must check that

$$\mu = \mu_{t_0} \lfloor \{\phi > 0\}.$$

Let $\psi \in C^2_c(\{\phi > 0\}, \mathbf{R}^+)$. Assume by scaling that $\psi \leq \phi$. Then

$$-\infty < D^+_{t_0}\mu_t(\phi) \leq D^+_{t_0}\mu_t(\psi) + D^+_{t_0}\mu_t(\phi - \psi) \leq D^+_{t_0}\mu_t(\psi) + C_3(\phi - \psi)$$

by 7.2(i). Thus $D^+_{t_0}\mu_t(\psi) > -\infty$, and using 7.2(i) to control the times not covered by the lim sup,

$$\mu_{t_0}(\psi) \leq \lim_{t\downarrow t_0} \mu_t(\psi). \tag{5}$$

Next we will show that $\mu_{t_0}(\psi) = \mu(\psi)$. Fix $t > t_0$, then for q large enough we have $t_0 < s'_q < t$. Since $\mu^i_t(\psi) - C_3(\psi)t$ is nonincreasing in t,

$$\mu^{r'_q}_{t_0}(\psi) + C_3(\psi)(s'_q - t_0) \geq \mu^{r'_q}_{s'_q}(\psi) \geq \mu^{r'_q}_t(\psi) + C_3(\psi)(s'_q - t).$$

Sending $r'_q \to \infty$ and $s'_q \downarrow t_0$, we obtain by (3) and (1),

$$\mu_{t_0}(\psi) \geq \mu(\psi) \geq \mu_t(\psi) + C_3(\psi)(t_0 - t)$$

for any $t > t_0$. Sending $t \downarrow t_0$, we obtain by (5)

$$\mu_{t_0}(\psi) \geq \mu(\psi) \geq \lim_{t\downarrow t_0} \mu_t(\psi) \geq \mu_{t_0}(\psi).$$

Thus $\mu_{t_0}(\psi) = \mu(\psi)$. This was true for arbitrary $\psi \in C^2_c(\{\phi > 0\}, \mathbf{R}^+)$, so by (4) and 6.4(b),

$$D^+_{t_0}\mu_t(\phi) \leq \mathcal{B}(\mu, \phi) = \mathcal{B}(\mu_t, \phi).$$

The case of $D^-_{t_0}\mu_t(\phi)$ is similar, and so $\bar{D}_{t_0}\mu_t(\phi) \le \mathcal{B}(\mu_t, \phi)$. Together with (1), we have proven 7.1(ii).

5. Let us prove (iii). Let Ψ be a countable subset of $C^2_c(M, \mathbf{R}^+)$ such that the open sets $\{\phi > 0\}$ for $\phi \in \Psi$ together cover M. Fix $T < \infty$. For each $\phi \in \Psi$, by the mass bound 7.1(i)

$$-C \le \mu^{i_j}_T(\phi) - \mu^{i_j}_0(\phi) \le \int_0^T \bar{D}_t \mu^{i_j}_t(\phi)\, dt \le \int_0^T \mathcal{B}(\mu^{i_j}_t, \phi)\, dt$$

where $C = C(\phi, T)$. Then

$$-C \le \limsup_{j\to\infty} \int_0^T \mathcal{B}(\mu^{i_j}_t, \phi)\, dt \le \int_0^T \limsup_{j\to\infty} \mathcal{B}(\mu^{i_j}_t, \phi)\, dt.$$

Thus for a.e. $t \ge 0$,

$$\limsup_{j\to\infty} \mathcal{B}(\mu^{i_j}_t, \phi) > -\infty.$$

In fact, for a.e. $t \ge 0$, the above statement holds for every $\phi \in \Psi$. Then by 7.1(i), (ii) and the $\mathcal{B}$-compactness lemma 7.3, there is a subsequence $\{i'_j\}_{j\ge1}$ of $(i_j\}_{j\ge1}$, depending on t, such that

$$\mu^{i'_j}_t \to \mu_t \in \mathcal{IM}_k(M), \qquad V(\mu^{i'_j}_t) \to V(\mu_t).$$

We have proven 7.1(iii). Q.E.D.

§8 Enhanced Motion

Given an initial surface T_0, we will use the estimates in §4-§5 and the Brakke convergence theorem of §7 to produce a pair

$$(T, \{\mu_t\}_{t\ge0})$$

that we call "enhanced motion." Here T is a locally integral current with boundary T_0 and $\{\mu_t\}_{t\ge0}$ is a Brakke motion "sitting atop" the slices T_t of T. There can be a discrepancy between μ_t and μ_{T_t} due to weak lower semicontinuity of mass. The analogy in "straight" analysis is $|\lim Du| \le \lim |Du|$ (as Radon measures).

8.1 Existence Theorem for Enhanced Motion. Let M^n be complete. Let $T_0 \in \mathbf{I}^{loc}_k(M)$ satisfy 3.1, namely

(i) $\partial T_0 = 0, \quad \mathbf{M}[T_0] < \infty$.

Then there exists $T \in \mathbf{I}^{loc}_{k+1}(M \times [0,\infty))$ and a family $\{\mu_t\}_{t\geq 0}$ of Radon measures, such that

(ii) (a) $\partial T = T_0$.

(b) $\mathbf{M}[T_B]$ is absolutely continuous with respect to $\mathcal{L}^1(B)$ for $B \subseteq \mathbf{R}$.

(iii) (a) $\mu_0 = \mu_{T_0}$, $\mathbf{M}[\mu_t] \leq \mathbf{M}[\mu_0]$ for $t > 0$.

(b) $\{\mu_t\}_{t\geq 0}$ is a Brakke motion.

(iv) $\mu_t \geq \mu_{T_t}$ for each $t \geq 0$. □

The proof begins with 8.6.

8.2 Definition of Enhanced Motion. A pair $(T, \{\mu_t\}_{t\geq 0})$ satisfying the conclusions of 8.1 is called an *enhanced motion* with initial condition T_0. □

We call T the *undercurrent* and $\{\mu_t\}_{t\geq 0}$ the *overflow.*

One advantage of this definition is that μ_t cannot arbitrarily disappear, as we see from 8.1(ii)(a), (b) and (iv). There still might be sudden mass loss.

The only reason for explicitly requiring $\mathbf{M}[\mu_t] \leq \mathbf{M}[\mu_0]$ in the definition is to rule out the creation of mass at infinity as in 3.6(b).

8.3 Additional Properties. The particular solution constructed in the proof is integral and satisfies, for $B \subseteq \mathbf{R}$ measurable,

$$\mathbf{M}[T_B] \leq (|B| + |B|^{1/2})\mathbf{M}[T_0], \qquad \mathbf{M}[\pi_\#(T_B)] \leq |B|^{1/2}\mathbf{M}[T_0].$$

It should be possible to prove this directly from definition 8.2, without recourse to the construction.

8.4 Restartability.

(i) By 8.1(iii)(a), (iv), we have

$$\mathbf{M}[T_t] \leq \mathbf{M}[\mu_t] \leq \mathbf{M}[\mu_0] = \mathbf{M}[T_0].$$

(ii) By the absolute continuity requirement 8.1(ii)(b), $\{T_t\}_{t\geq 0}$ is continuous in the weak topology.

(iii) By 1.17, 1.15, (i), and (ii),

$$T_t \in \mathbf{I}_k^{loc}(M \times \{t\}) \qquad \text{for each } t \geq 0.$$

(iv) Thus T_t satisfies the initial data hypothesis 8.1(i) for each $t \geq 0$. This will come in handy in §9. □

8.5 Product Lemma. Let $\bar{\mu}$ be a Radon measure on $U \times (z_0, \infty)$, where $U \subseteq M$ is open. Suppose $\bar{\mu}$ is translation invariant in the sense that for all $\psi \in C_c^0(U \times (z_0, \infty), \mathbf{R}^+)$ and all $\tau \geq 0$,

$$\bar{\mu}(\psi^\tau) = \bar{\mu}(\psi)$$

where

$$\psi^\tau(z) = \begin{cases} \psi(z-\tau) & \text{for } z - \tau > z_0 \\ 0 & \text{otherwise.} \end{cases}$$

Then

(i) If we choose $\theta \in C_c^0((z_0, \infty), \mathbf{R}^+)$ such that $\int \theta(z)\, dz = 1$ and define a Radon measure μ on U by

$$\mu(\phi) = \bar{\mu}(\theta\phi), \qquad \phi \in C_c^0(U, \mathbf{R}^+),$$

then μ is independent of θ, and

$$\bar{\mu} = \mu \times (\mathcal{L}^1 \lfloor (z_0, \infty)).$$

(ii) If $\bar{\mu} \in \mathcal{M}_{k+1}(U \times (z_0, \infty))$, then $\mu \in \mathcal{M}_k(U)$. If $\bar{\mu}$ is integer rectifiable, so is μ.

(iii) For any $\theta \in C_c^2((z_0, \infty), \mathbf{R}^+)$ and any $\phi \in C_c^2(U, \mathbf{R}^+)$,

$$\mathcal{B}(\bar{\mu}, \theta\phi) = \mathcal{B}(\mu, \phi). \quad \square$$

Proof. We will assume that $z_0 = -\infty$, by extending $\bar{\mu}$ to a translation-invariant measure on all of $U \times \mathbf{R}$.

(i) Let $K = \{\theta \in C_c^0(\mathbf{R}, \infty) : \int \theta\, dz = 1\}$. For each $\theta \in K$, let $K(\theta)$ be the set of all finite convex combinations

$$\theta' = \sum a_\tau \theta^\tau, \quad \sum a_\tau = 1, \quad a_\tau \geq 0.$$

By translation invariance, $\bar{\mu}(\theta'\phi)$ is constant on $K(\theta)$. Two fixed functions θ_1, $\theta_2 \in K$ can be approximated uniformly closely by elements of $K(\theta_\varepsilon)$ for a sufficiently tall, narrow θ_ε, proving $\bar{\mu}(\theta_1\phi) = \bar{\mu}(\theta_2\phi)$ for all $\phi \in C_c^0(U, \mathbf{R}^+)$, as required.

Then for $\rho \in C_c^0(\mathbf{R}, \mathbf{R}^+)$, $\phi \in C_c^0(U, \mathbf{R}^+)$,

$$\bar{\mu}(\rho\phi) = \mu(\phi)\,\mathcal{L}^1(\rho),$$

showing $\bar{\mu} = \mu \times \mathcal{L}^1$.

(ii) Assume $\bar{\mu} \in \mathcal{M}_{k+1}(U \times \mathbf{R})$. We will show $\mu \in \mathcal{M}_k(U)$. First, note that $T_{(x,z)}\bar{\mu} \ni \partial/\partial z$ for $\bar{\mu}$-a.e. (x, z). If not, then by the rectifiability of $\bar{\mu}$, there exists (x, z) and a small C^1 submanifold $N^{k+1} \ni (x, z)$ transverse to $\partial/\partial z$ such that $\bar{\mu}(N) > 0$. Translating N up and down the z-axis contradicts the local finiteness of $\bar{\mu}$.

By the rectifiability of $\bar{\mu}$, for $\bar{\mu}$-a.e. $(x, z) \in U \times \mathbf{R} \times [0, \infty)$ we have for all $\phi \in C_c^0(M, \mathbf{R}^+)$,

$$\begin{aligned}
\lim_{\lambda\downarrow 0} \mu_{x,\lambda}(\phi) &= \lim_{\lambda\downarrow 0} \bar{\mu}_{(x,z),\lambda}(\theta\phi) \\
&= \Theta^{k+1}(\bar{\mu}, (x, z)) \int_{T_{(x,z)}\bar{\mu}} \theta(z)\phi(x)\, d\mathcal{H}^{k+1}(x, z) \\
&= \Theta^{k+1}(\bar{\mu}, (x, z)) \int_{T_{(x,z)}\bar{\mu}\cap\mathbf{R}^n} \phi(x)\, d\mathcal{H}^k(x) \qquad \text{since } T_{(x,z)}\bar{\mu} \ni \partial/\partial z,
\end{aligned}$$

so $\mu \in \mathcal{M}_k(U)$ and $T_x\mu \oplus \mathrm{Span}\,(\partial/\partial z) = T_{(x,z)}\bar{\mu}$ for μ-a.e. x and all z.

(iii) By the above, we see that $\mu\lfloor\{\phi > 0\}$ is k-rectifiable if and only if $\bar{\mu}\lfloor\{\phi > 0\}\times\mathbf{R}$ is $(k + 1)$-rectifiable. In this case, let $Y \in C_c^1(T(\{\phi > 0\} \times \mathbf{R}))$, then since $\bar{\mu} = \mu \times \mathcal{L}^1$ and $T_{(x,z)}\bar{\mu} = T_x\mu \oplus \mathrm{Span}\,(\partial/\partial z)$ μ-a.e., we obtain by the first variation formula,

$$\delta V_{\bar{\mu}}(Y) = \delta V_\mu(\tilde{Y}),$$

where $\tilde{Y} = \int T_{(x,z)}\mu \cdot Y(x, z)\, dz$. It follows that

$$\delta V_{\bar{\mu}}\lfloor(\{\phi > 0\} \times \mathbf{R}) = \delta V_\mu\lfloor\{\phi > 0\} \times \mathcal{L}^1.$$

Then 6.2(ii)-(iv) hold for $\bar{\mu}\lfloor(\{\phi > 0\} \times \mathbf{R})$ if and only if they hold for $\mu\lfloor\{\phi > 0\}$. Also, for μ-a.e. $x \in \{\phi > 0\}$

$$\vec{H}_\mu(x) = \vec{H}_{\bar{\mu}}(x, z) \perp \partial/\partial z \qquad \text{for all } z \in \mathbf{R},$$

So in case 6.2(v) holds,

$$\begin{aligned}\mathcal{B}(\bar\mu,\theta\phi) &= \int -\theta\phi H^2 + \vec H\cdot S^\perp_{\bar\mu}\cdot\bar\nabla(\theta\phi)\,d\bar\mu\\ &= \int -\phi H^2 + \vec H\cdot S^\perp_{\mu}\cdot\nabla\phi\,d\mu + 0\\ &= \mathcal{B}(\mu,\phi).\end{aligned}$$

This finishes Lemma 8.5. Q.E.D.

Proof of Theorem 8.1, consisting of 8.6-8.11.

8.6 Construction of Approximating Motion. For each $\varepsilon > 0$, let P^ε be the locally integral current of 3.2 satisfying

$$\partial P^\varepsilon = T_0, \qquad I^\varepsilon[P^\varepsilon] \le \mathbf{M}[T_0],$$
$$P^\varepsilon \text{ stationary for } I^\varepsilon \text{ on } (M\times\mathbf{R})\setminus \operatorname{spt} T_0.$$

Define

$$\sigma_{-t/\varepsilon}(x,z) = (x, z-\frac{t}{\varepsilon}), \qquad P^\varepsilon(t) = \left(\sigma_{-t/\varepsilon}\right)_\#(P^\varepsilon), \qquad \mu^\varepsilon_t = \mu_{P^\varepsilon(t)}.$$

8.7 Soliton Lemma. $\{\mu^\varepsilon_t\}_{t\ge0}$ is an integral Brakke motion on the relatively open subset

$$W^\varepsilon \equiv \{(x,z,t) : z > -\frac{t}{\varepsilon},\ t\ge 0\}$$

of space-time $(M\times\mathbf{R})\times[0,\infty)$. That is to say, μ^ε is a translative soliton, as asserted in §2. □

Proof. Integrality is clear. Note that $\operatorname{spt}(\partial P^\varepsilon(t))\times\{t\}$ lies outside of W^ε, so the following derivation is valid. Fix $t\ge 0$ and $\psi\in C^2_c(M,\mathbf{R}^+)$ with $\operatorname{spt}\psi\times\{t\}\subset W^\varepsilon$. Recall from 2.6(i),(iii) that

$$S^\perp(x,z)\cdot\vec H(x,z) = \vec H(x,z) = \frac{1}{\varepsilon}S^\perp(x,z)\cdot\omega$$

for μ^ε-a.e. (x,z). Inserting $Y = \partial\sigma_{-t/\varepsilon}/\partial t = -\omega/\varepsilon$ in 2.4(1), we obtain

$$\begin{aligned}\frac{d}{dt}\mu^\varepsilon_t(\psi) &= \frac{d}{dt}\int \psi\, d(\sigma_{-t/\varepsilon})_\#(\mu^\varepsilon)\\ &= \int(-\psi\vec H + \nabla\psi\cdot S^\perp)\cdot(-\frac{1}{\varepsilon}\omega)\,d\mu^\varepsilon_t\\ &= \int -\psi H^2 + \nabla\psi\cdot S^\perp\cdot\vec H\,d\mu^\varepsilon_t\\ &= \mathcal{B}(\mu^\varepsilon_t,\psi).\end{aligned}$$

Q.E.D.

Note that W^ε converges as $\varepsilon \downarrow 0$ to $W \equiv \big(M \times \mathbf{R} \times (0,\infty)\big) \cup \big(M \times (0,\infty) \times \{0\}\big)$. We wish to pass to limits as $\varepsilon \downarrow 0$ to obtain an integral Brakke motion $\{\bar{\mu}_t\}_{t \ge 0}$ on W, where the domain of the Radon measure $\bar{\mu}_t$ is $M \times \mathbf{R}$ for $t > 0$, $M \times (0,\infty)$ for $t = 0$.

Fix $t_1 \ge 0$ and $R_1 < R_2$ such that $R_1 \ge 0$ if $t_1 = 0$. Let

$$W_2 = M \times (R_1, R_2) \times [t_1, \infty).$$

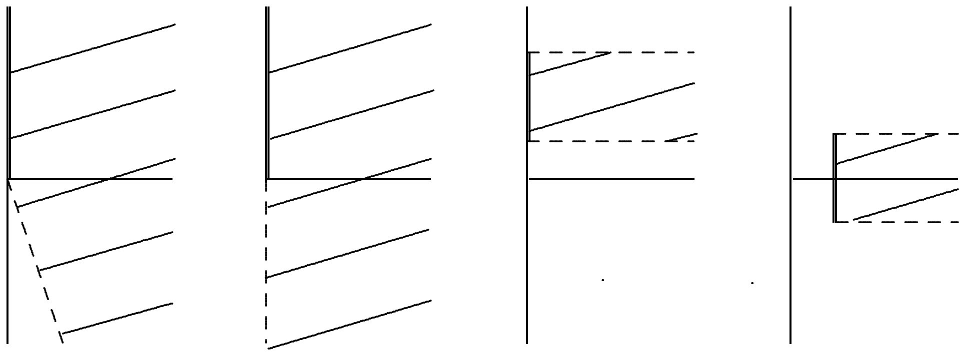

Figure 7. Sets.

For ε small enough, $W_2 \subseteq W^\varepsilon$. For $t \ge t_1$, we have by 5.1 and 3.2(ii)

$$\mu_t^\varepsilon(M \times (R_1, R_2)) \le (R_2 - R_1 + \varepsilon)\mathbf{M}[T_0].$$

Hence the Brakke convergence theorem 7.1 applies to yield a subsequence $\varepsilon_i \downarrow 0$ and an integral Brakke motion $\{\bar{\mu}_t\}_{t \ge 0}$ on W such that, for all $t \ge 0$,

$$(1) \qquad \mu_t^{\varepsilon_i} \to \bar{\mu}_t, \qquad \bar{\mu}_t(M \times (R_1, R_2)) \le (R_2 - R_1)\mathbf{M}[T_0].$$

For $t = 0$, the convergence is on $M \times (0,\infty)$.

8.8 Translation Invariance. Let $t \ge 0$ and $\psi \in C_c^2(M \times \mathbf{R}, \mathbf{R}^+)$. If $t = 0$, assume $\operatorname{spt}\psi \subseteq C_c^2(M \times (0,\infty), \mathbf{R}^+)$. Then

(i) $\bar{\mu}_t(\psi) \ge \bar{\mu}_t(\psi^\tau) \ge \lim_{s \downarrow t} \bar{\mu}_s(\psi) \quad$ for $\tau \ge 0$.

(ii) If $t > 0$, $\lim_{s \uparrow t} \bar{\mu}_s(\psi) \ge \bar{\mu}_t(\psi^\tau) \ge \bar{\mu}_t(\psi) \quad$ for $\tau \le 0$.

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(iii) In particular, by the continuity properties 7.2(i), (ii), we have

$$(\sigma_\tau)_\#(\bar\mu_t) = \bar\mu_t \quad \text{for all } \tau \in \mathbf{R}$$

for all but countably many t. □

Proof. Note that $\bar\mu_t^\varepsilon(\psi^\tau) = \bar\mu_{t+\varepsilon\tau}^\varepsilon(\psi)$. Recall from (1) and 7.2(i) that $\mu_t^\varepsilon(\psi) - C_3(\psi)t$ is nonincreasing. Fix $s > t$ and pass to limits as $\varepsilon_i \downarrow 0$ to obtain

$$\bar\mu_t(\psi) - C_3(\psi)t \geq \bar\mu_t(\psi^\tau) - C_3(\psi)t \geq \bar\mu_s(\psi) - C_3(\psi)s.$$

Therefore

$$\bar\mu_t(\psi) \geq \bar\mu_t(\psi^\tau) \geq \lim_{s\downarrow t} \bar\mu_s(\psi)$$

as desired. (ii) is similar. (iii) is self-explanatory. Q.E.D.

8.9 Construction of $\{\mu_t\}_{t\geq 0}$, proof of 8.1(iii)(b). Fix $\theta \in C_c^2((0,\infty), \mathbf{R}^+)$ such that $\int_{-\infty}^{\infty} \theta \, dz = 1$. Define $\{\mu_t\}_{t\geq 0}$ by

$$\mu_t(\phi) = \bar\mu_t(\theta\phi), \qquad \phi \in C_c^2(M, \mathbf{R}^+).$$

Then

(i) $\bar\mu_t = \mu_t \times \mathcal{L}^1$ for all but countably many $t \geq 0$.

(ii) $\{\mu_t\}_{t\geq 0}$ is an integral Brakke motion on M. □

Proof. (i) follows from lemma 8.8(iii) and the product lemma 8.5.

(ii) Fix $t_0 \geq 0$ and $\phi \in C_c^2(M, \mathbf{R}^+)$. It suffices to prove that both $D_{t_0}^- \mu_t(\theta\phi)$ and $D_{t_0}^+ \mu_t(\theta\phi)$ are less than $\mathcal{B}(\mu_{t_0}, \phi)$, where D^-, D^+ are the left and right upper derivates. Let us treat the latter case, as the two cases are similar.

We can assume

$$D_{t_0}^+ \mu_t(\theta\phi) > -\infty.$$

It follows that for any $\tilde\psi \in C_c^2(\{\theta\phi > 0\}, \mathbf{R}^+)$ with $\tilde\psi \leq \theta\phi$,

$$-\infty < D_{t_0}^+ \bar\mu_t(\theta\phi) \leq D_{t_0}^+ \bar\mu_t(\tilde\psi) + C_3(\theta\phi - \tilde\psi),$$

so $D_{t_0}^+ \bar\mu_t(\tilde\psi) > -\infty$, and then by lemma 8.8(i),

$$\bar\mu_{t_0}(\tilde\psi^\tau) = \bar\mu_{t_0}(\tilde\psi) \qquad \text{for all } \tau \geq 0.$$

Now let $z_0 = \min(\operatorname{spt}\theta)$ and let $\psi \in C^2_c(\{\phi > 0\} \times (z_0,\infty), \mathbf{R}^+)$. Approximating ψ by a sum of translates $\tilde\psi^\tau$, $\tau \ge 0$, of such functions $\tilde\psi$, we can prove

$$\bar\mu_{t_0}(\psi^\tau) = \bar\mu_{t_0}(\psi) \qquad \text{for all } \tau \ge 0.$$

It follows by the product lemma 8.5(i) that

$$\bar\mu_{t_0} \lfloor (\{\phi > 0\} \times (z_0,\infty)) = (\mu_{t_0} \lfloor \{\phi > 0\}) \times (\mathcal{L}^1 \lfloor (z_0,\infty)).$$

Then using 8.5(iii),

$$D^+_{t_0}\mu_t(\phi) = D^+_{t_0}\bar\mu_t(\theta\phi) \le \mathcal{B}(\bar\mu_{t_0},\theta\phi) = \mathcal{B}(\mu_{t_0},\phi).$$

Together with the D^- case, this establishes 8.1(iii)(b). Note that 8.5(iii) also yields the integrality of $\{\mu_t\}_{t\ge0}$. Q.E.D.

8.10 Construction of T, proof of 8.1(ii) and 8.3. Let $T^\varepsilon = (\kappa_\varepsilon)_\#(P^\varepsilon)$ as in 3.2. The mass estimate 5.3 and the I^ε bound 3.2(ii) give the mass bound needed for application of the current compactness theorem 1.17, yielding a subsequence of $\{\varepsilon_i\}_{i\ge1}$, also to be called $\{\varepsilon_i\}_{i\ge1}$, and a current $T \in \mathbf{I}^{loc}_{k+1}(M \times \mathbf{R})$ such that

(i) $T^{\varepsilon_i} \rightharpoonup T$, $\quad \operatorname{spt} T \subseteq M \times [0,\infty)$.

(ii) $\mathbf{M}[T_B] \le (|B| + |B|^{1/2})\mathbf{M}[T_0]$ and $\mathbf{M}[\pi_\#(T_B)] \le |B|\,\mathbf{M}[T_0]$ for all measurable $B \subseteq \mathbf{R}$, by 5.3. $\square$

8.11 Slice Convergence, proof of 8.1(iii)(a) and (iv), completion of 8.1.

(i) As $\varepsilon_i \downarrow 0$, the family $\{T^{\varepsilon_i}_t\}_{t\ge0}$ converges uniformly in the flat metric topology to a $C^{1/2}$ path $\{T_t\}_{t\ge0}$ of currents $T_t = \partial(T \lfloor (M \times [t,\infty)))$.

(ii) $\mu_t \ge \mu_{T_t}$ for all $t \ge 0$.

(iii) $\mu_0 = \mu_{T_0}$, $\mathbf{M}[\mu_t] \le \mathbf{M}[\mu_0]$. $\square$

Proof. (i) Let $\delta > 0$ and choose $\eta \in C^\infty(\mathbf{R})$, $0 \le \eta \le 1$, such that

$$\eta = \begin{cases} 0 & \text{on } (-\infty, t-\delta] \\ 1 & \text{on } [t,\infty). \end{cases}$$

Then for each $t \ge 0$ and $\vartheta \in C^\infty_c(\Lambda^k(M \times \mathbf{R}))$,

$$\begin{aligned} \lim_{i\to\infty} T^{\varepsilon_i}[t,\infty)(\vartheta) &= \lim_{i\to\infty} T^{\varepsilon_i}[t,\infty)(\eta\vartheta) \\ &= \lim_{i\to\infty} T^{\varepsilon_i}(\eta\vartheta) - \lim_{i\to\infty} T^{\varepsilon_i}[0,t)(\eta\vartheta) \\ &= T(\eta\vartheta) \pm (\delta + \delta^{1/2})\mathbf{M}[T_0]\max|\vartheta| \qquad \text{by 5.3.} \end{aligned}$$

Let $\delta \downarrow 0$ and obtain $T^{\varepsilon_i}[t,\infty) \rightharpoonup T[t,\infty)$ and thus, taking boundary,

$$T_t^{\varepsilon_i} \rightharpoonup T_t \qquad \text{for each } t \geq 0.$$

By 1.15, this implies convergence in the flat metric topology. It follows from the mass bound 5.3 and increasingly refined partitions of $[0,\infty)$ that the convergence is uniform. We have (i).

(ii) Fix $t \geq 0$ and let $\phi \in C_c^2(M, \mathbf{R}^+)$. We seek to prove that $\mu_t(\phi) \geq \mu_{T_t}(\phi)$. By essentially the same argument as for (i), we have for each fixed τ,

$$T_{t+\varepsilon_i\tau}^{\varepsilon_i} \rightharpoonup T_t, \qquad \liminf_{\varepsilon_i \downarrow 0} \mu_{T_{t+\varepsilon_i\tau}^{\varepsilon_i}}(\phi) \geq \mu_{T_t}(\phi).$$

Then

$$\begin{aligned}
\mu_t(\phi) &= \bar{\mu}_t(\theta\phi) \\
&= \lim_{\varepsilon_i \downarrow 0} \int \theta(z)\phi(x)\, d\mu_t^{\varepsilon_i}(x,z) \\
&\geq \liminf_{\varepsilon_i \downarrow 0} \int \theta\phi |D\zeta|\, d\mu_t^{\varepsilon_i} \\
&= \liminf_{\varepsilon_i \downarrow 0} \int_0^\infty \theta \int \phi\, d\mu_{\partial(P^{\varepsilon_i}(t) \lfloor M \times [z,\infty))} dz \qquad \text{by co-area} \\
&= \liminf_{\varepsilon_i \downarrow 0} \int_0^\infty \theta \int \phi\, d\mu_{P_{t/\varepsilon_i+z}^{\varepsilon_i}} dz \qquad \text{by 8.6} \\
&= \liminf_{\varepsilon_i \downarrow 0} \int_0^\infty \theta \int \phi\, d\mu_{T_{t+\varepsilon_i z}^{\varepsilon_i}} dz \\
&\geq \int_0^\infty \theta \liminf_{\varepsilon_i \downarrow 0} \mu_{T_{t+\varepsilon_i z}^{\varepsilon_i}}(\phi)\, dz \\
&\geq \mu_{T_t}(\phi).
\end{aligned}$$

(iii) Recall from (1) that

$$\bar{\mu}_t(M \times (R_1, R_2)) \leq (R_2 - R_1)\mathbf{M}[T_0].$$

Approximating θ by step functions, this implies

$$\mathbf{M}[\mu_t] = \int d\mu_t = \int \theta(z)\, d\bar{\mu}_t(x,z) \leq \mathbf{M}[T_0].$$

Since $\mu_0 \geq \mu_{T_0}$ by (ii), we also get $\mu_0 = \mu_{T_0}$, as was desired for (iii). This completes Lemma 8.11 and Theorem 8.1. Q.E.D.

§9 Matching Motion

The problem with the enhanced motion of the previous section is the discrepancy between μ_t and μ_{T_t}. One consequence is that an enhanced motion is not an enhanced motion with respect to later starting times.

It is a remarkable fact that under the hypothesis of *uniqueness,* the discrepancy is removable. On this observation rest the results of Chapter Two, culminating in the partial regularity result.

We define a *matching motion* to be an enhanced motion with zero discrepancy:

9.1 Definition of Matching Motion. Let M be complete and let T_0 satisfy 3.1. We call a current $T \in \mathbf{I}_{k+1}^{loc}(M \times [0,\infty))$ a *matching motion* for T_0 provided

(i) $\partial T = T_0$.

(ii) $\mathbf{M}[T_B] \leq (|B| + |B|^{1/2})\mathbf{M}[T_0]$ for $B \subseteq \mathbf{R}$ measurable.

(iii) $\{\mu_{T_t}\}_{t\geq 0}$ is a Brakke motion. $\square$

If every enhanced motion pair $(T, \{\mu_t\}_{t\geq 0})$ with initial condition T_0 has the same undercurrent T, then we say T is the *unique undercurrent* for T_0.

9.2 Existence of Matching Motion. Let M be complete and let $T_0 \in \mathbf{I}_k^{loc}(M \times \{0\})$ be a cycle of finite mass. If $T \in \mathbf{I}_{k+1}^{loc}(M \times [0,\infty))$ is the unique undercurrent for T_0, then T is a matching motion. $\square$

Such uniqueness is not to be expected in codimension greater than one, as shown in the figure.

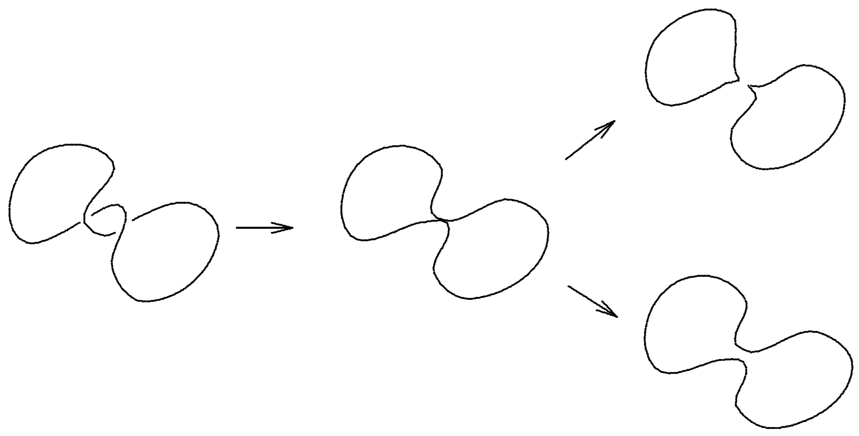

Figure 8. Nonuniqueness.

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Proof of 9.2. 1. Let $(T, \{\mu_t\}_{t\geq 0})$ be an enhanced motion as constructed in §8. By the definition of enhanced motion, 9.1(i) already holds, and 9.1(ii) is true by 8.3.

2. Fix $\delta > 0$. We begin by constructing an enhanced motion $(T, \{\mu_t^\delta\}_{t\geq 0})$ such that

$$\mathbf{M}[\mu_t^\delta] < \mathbf{M}[T_t] + \delta \qquad \text{for } t \geq 0.$$

Let $Z_\delta = \{t : \mathbf{M}[\mu_t] \geq \mathbf{M}[T_t] + \delta\}$. If $Z_\delta = \emptyset$, we can take $\mu_t^\delta = \mu_t$ for all $t \geq 0$. Otherwise, let $t_0 = \inf Z_\delta$. By 8.4(ii), $\{T_t\}_{t\geq 0}$ is weakly continuous, so $\mathbf{M}[T_t]$ is lower semicontinuous as a function of t. Also, $\mathbf{M}[\mu_t]$ is nonincreasing, so we have

$$\mathbf{M}[T_{t_0}] + \delta \leq \liminf_{t\downarrow t_0} \mathbf{M}[T_t] + \delta \leq \lim_{t\downarrow t_0} \mathbf{M}[\mu_t] \leq \mathbf{M}[\mu_{t_0}]$$

so $t_0 \in Z_\delta$.

3. We now *restart* the evolution at time t_0 with initial data T_{t_0}. (Recall from 8.4(iv) that T_{t_0} is legitimate initial data.) By 8.1 there is an enhanced motion $(\tilde{T}, \{\tilde{\mu}_t\}_{t\geq t_0})$ with

$$\tilde{T} \in \mathbf{I}_{k+1}^{loc}(M \times [t_0, \infty)), \qquad \partial\tilde{T} = T_{t_0}, \qquad \tilde{\mu}_{t_0} = \mu_{T_{t_0}} \in \mathcal{M}_k(M).$$

Now define

$$\hat{T} = T - T\lfloor(M \times [t_0, \infty)) + \tilde{T}, \qquad \hat{\mu}_t = \begin{cases} \mu_t & 0 \leq t < t_0 \\ \tilde{\mu}_t & t_0 \leq t. \end{cases}$$

Let us check that $(\hat{T}, \{\hat{\mu}_t\}_{t\geq 0})$ is an enhanced motion for T_0. We have

$$\begin{aligned} \partial\hat{T} &= \partial T - \partial(T\lfloor(M \times [t_0, \infty))) + \partial\tilde{T} \\ &= T_0 - T_{t_0} + T_{t_0} \\ &= T_0. \end{aligned}$$

so $\hat{T}$ satisfies 8.1(ii)(a). In addition, the absolute continuity requirement 8.1(ii)(b) holds for $\hat{T}$ as it did for T, $\tilde{T}$ separately. Evidently we also have

$$\begin{aligned} 8.1(\text{iii})(\text{a}) \ &: \quad \hat{\mu}_0 = \mu_{T_0}, \\ 8.1(\text{iv}) \ &: \quad \hat{\mu}_t \geq \mu_{\hat{T}_t} \qquad \text{for all } t \geq 0. \end{aligned}$$

Let us check 8.1(iii)(b), namely that $\{\hat{\mu}_t\}_{t\geq 0}$ is a Brakke motion. This is clear for $t \neq t_0$ and for $t \downarrow t_0$. Recall

$$D_{t_0}^- f(t) = \limsup_{t\uparrow t_0} \frac{f(t) - f(t_0)}{t - t_0}.$$

We must check that $D^-_{t_0}\hat{\mu}_t(\phi) \le \mathcal{B}(\hat{\mu}_{t_0}, \phi)$ for $\phi \in C^2_c(M, \mathbf{R}^+)$. Recall that

$$\hat{\mu}_{t_0} = \tilde{\mu}_{t_0} = \mu_{T_0} \le \mu_{t_0}.$$

If $\hat{\mu}_{t_0}(\phi) = \mu_{t_0}(\phi)$, then

$$\hat{\mu}_{t_0} \lfloor \{\phi > 0\} = \mu_{t_0} \lfloor \{\phi > 0\}$$

and

$$D^-_{t_0}\hat{\mu}_t(\phi) = D^-_{t_0}\mu_t(\phi) \le \mathcal{B}(\mu_{t_0}, \phi) = \mathcal{B}(\hat{\mu}_{t_0}, \phi)$$

by 6.4(b). On the other hand, if $\hat{\mu}_{t_0}(\phi) < \mu_{t_0}(\phi)$, then by the semidecreasing property 7.2(i), $D^-_{t_0}\hat{\mu}_t(\phi) = -\infty$ and we have $D^-_{t_0}\hat{\mu}_t(\phi) \le \mathcal{B}(\hat{\mu}_{t_0}, \phi)$ in this case as well.

In summary, we have proven that $(\hat{T}, \{\hat{\mu}_t\}_{t\ge 0})$ is an enhanced motion for T_0.

4. Therefore, by the uniqueness hypothesis of the theorem, we have $\hat{T} = T$. As in step 2, we argue

$$\lim_{t\downarrow t_0} \mathbf{M}[\hat{\mu}_t] \le \mathbf{M}[\hat{\mu}_{t_0}] = \mathbf{M}[T_{t_0}] \le \liminf_{t\downarrow t_0} \mathbf{M}[T_t]$$

from which it follows that we have increased the value of $\inf Z_\delta$. Notice also that

$$\mathbf{M}[\hat{\mu}_{t_0}] \le \mathbf{M}[\mu_{t_0}] - \delta \le \mathbf{M}[\hat{\mu}_0] - \delta.$$

Therefore, if we repeat the process, it must terminate after no more than $[\mathbf{M}[T_0]/\delta]$ iterations. We have constructed an enhanced motion $(T, \{\mu^\delta_t\}_{t\ge 0})$ with $\mathbf{M}[\mu^\delta_t] < \mathbf{M}[T_t] + \delta$ for all $t \ge 0$.

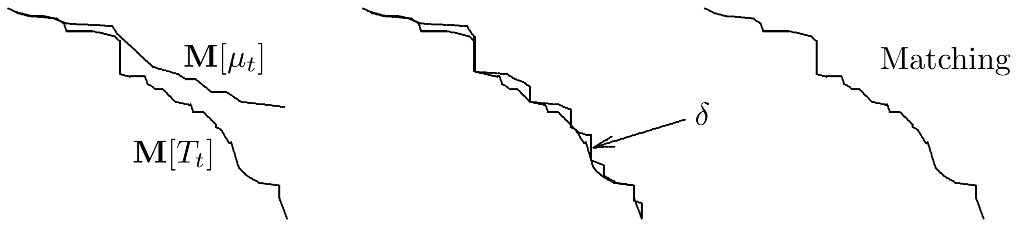

Figure 9. Restarting Procedure.

5. By the Brakke convergence theorem 7.1 there is a Brakke motion $\{\mu'_t\}_{t\ge 0}$ with

$$\mu^{\delta_i}_t \to \mu'_t, \qquad \mu'_t \ge \mu_{T_t} \quad \text{for all } t \ge 0,$$

= 0.880cm

and by step 4, $\mathbf{M}[\mu'_t] = \mathbf{M}[T_t]$ for each $t \geq 0$, so actually,

$$\mu'_t = \mu_{T_t} \quad \text{for all } t \geq 0.$$

Thus we have shown that $\{\mu_{T_t}\}_{t\geq 0}$ is a Brakke motion, establishing 9.2. Q.E.D.

Remark. The construction of μ^δ can be done for each δ without the uniqueness assumption, at the cost of changing T for each δ. But in passing to limits, there could be cancellation of mass.

Chapter Two. Partial Regularity in Codimension One.

Codimension one surfaces have the special property that they avoid each other when moving by mean curvature. That is to say, the distance between disjoint surfaces never decreases. Therefore we can fill space with a family of disjoint surfaces, indexed as the level-sets of a Lipschitz function u, as in the work of Evans-Spruck [ES1] and Chen-Giga-Goto [CGG]. Because the level-sets do not approach one another, $u(\cdot, t)$ remains Lipschitz.

Moreover, we can prove that the support of the enhanced motion $(T, \{\mu_t\}_{t\geq 0})$ defined in §8 remains within the corresponding level set of u. Necessarily, all but countably many level-sets of u have zero $\mathcal{L}^{n+1}$-measure in space-time. For such a "thin" level-set, the current T of 8.2 is uniquely determined as the boundary (in the sense of currents) of the set $\{u > a\}$. It follows by Theorem 9.2 that T is a matching motion, that is to say, $\{\mu_{T_t}\}_{t\geq 0}$ is a Brakke motion.

Furthermore, we will show that $T_t = \partial[\{x : u(x,t) > a\}]$. Then by DeGiorgi's theorem on sets of finite perimeter, μ_{T_t} will have unit density μ_{T_t}-a.e. This, in turn, provides the crucial hypothesis for the application of Brakke's powerful Local Regularity Theorem to the motion. As a result, we will prove the following:

> Let T_0 be the reduced boundary of a open set of finite perimeter in $\mathbf{R}^n$. If the level-set flow of the support of T_0 does not develop positive Lebesgue measure, then T_0 has a unique undercurrent T, whose time-slices form a unit density Brakke motion. Furthermore, $\operatorname{spt} T$ is smooth $\mathcal{H}^n$-almost everywhere.

As a corollary, if u_0 is a Lipschitz function with compact level-sets and u is the corresponding level-set flow, then almost every level-set of u is smooth $\mathcal{H}^n$-almost everywhere.

Recently Evans and Spruck [ES4] have discovered an unsuspected global estimate on the integral of $|H|$, and have used it to improve weak to strong convergence for almost every level of the level-set flow, yielding unit density. This constitutes an alternate pathway to the application of Brakke's regularity theory to the level-set flow.

The organization of the chapter is as follows. In §10 we prove that a Brakke motion never collides with a manifold moving smoothly by mean curvature. It follows by the geometric avoidance principle of [I2] that two Brakke motions, initially disjoint, never collide, and that a Brakke motion remains included within the corresponding level-set motion. In §11, we show that a level-set that doesn't fatten up has a unique undercurrent which is a unit density Brakke motion. In §12, we apply Brakke's Local Regularity Theorem to yield theorems about almost-everywhere regularity for motion by mean curvature.

For simplicity, take $M = \mathbf{R}^n$.

§10 Avoidance and Inclusion for Brakke Motions

Our first task in this section is to sketch a geometric characterization of the level-set flow. Soner did this originally in [So], using the evolving distance function. Our approach, using smooth barrier surfaces, appears in [I2]. It differs slightly from [So] because it applies to (unoriented) closed sets rather than to topological boundaries.

The second step is to show that a Brakke motion avoids these smooth barriers. This is inspired by [B, 3.6]; see also the varifold maximum principle of Solomon-White [SW]. In consequence, a Brakke motion that is initially included in a level-set moving by mean curvature, remains included in it.

10.1 Set-Theoretic Subsolution. ([I2, def. 4A]) A family $\{\Delta_t\}_{t\geq 0}$ of closed sets in $\mathbf{R}^n$ is said to be a *set-theoretic subsolution* of the mean curvature flow provided that for any family $\{Q_t\}_{t\in[t_0,t_1]}$ of smooth, compact hypersurfaces moving by mean curvature,

$$\Delta_{t_0} \cap Q_{t_0} = \emptyset \qquad \text{implies} \qquad \Delta_t \cap Q_t = \emptyset \quad \text{for all } t \in [t_0, t_1]$$

or equivalently,

$$\mathrm{dist}(\Delta_t, Q_t) \geq \mathrm{dist}(\Delta_{t_0}, Q_{t_0}) \qquad \text{for } t \in [t_0, t_1]. \quad \Box$$

Proof of Equivalence. Clearly the second condition implies the first. For the converse, we use the translation invariance of motion by mean curvature. If $\mathrm{dist}(\Delta_t, Q_t) < \mathrm{dist}(\Delta_{t_0}, Q_{t_0})$ ever occurs, then some translate of $\{Q_t\}_{t\in[t_0,t_1]}$ violates 10.1. Q.E.D.

The justifying virtue of set-theoretic subsolutions is that they satisfy the maximum principle with respect to each other.

10.2 Avoidance Lemma for Set-Theoretic Subsolutions. ([I2, Lemma 4E]) Let $\{\Delta_t\}_{t\geq 0}$, $\{\Gamma_t\}_{t\geq 0}$ be set-theoretic subsolutions of the mean curvature flow in $\mathbf{R}^n$. Assume

$$\Delta_0 \cap \Gamma_0 = \emptyset, \quad \Gamma_0 \text{ compact.}$$

Then $\operatorname{dist}(\Delta_t, \Gamma_t)$ is nondecreasing. □

The proof (see [I2]) relies on the translation invariance of the equation, and should apply to "viscosity solutions" in other contexts as well. The basic idea is to interpose a $C^{1,1}$ surface between Γ_t and Δ_t, then use the fact that set-theoretic subsolutions avoid smooth flows.

10.3 Relation to Level-Set Flow. (see [I2]) Let us relate the concept of set-theoretic subsolutions to the level-set flow introduced by Evans-Spruck and Chen-Giga-Goto. In [ES1] and [CGG], the following equation appears together with a (viscosity) definition of its weak solutions.

$$(\star) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = (\delta - \nu \otimes \nu) : \nabla^2 u & \text{on } \mathbf{R}^n \times [0, \infty) \\ u(\cdot, 0) = f(\cdot), & \text{along } \mathbf{R}^n \times \{0\}, \end{cases}$$

where δ is the identity matrix and $\nu = Du/|Du|$. When u is smooth and $Du \neq 0$, equation $(\star)$ says that the level-sets of u are moving by mean curvature. If f is continuous and all but at most one of the level-sets of f are compact, then there exists a unique u weakly solving $(\star)$ (see [I1, §7]). The family of level-sets $\Gamma_t^a \equiv \{x : u(x,t) = a\}$, $t \geq 0$, is unique and is called the *level-set flow* (by mean curvature) of $f^{-1}(a)$. It follows trivially from the definition of weak solution of $(\star)$ (which involves tangency of u with smooth test functions, see [ES1]) that a level-set flow in R^n is a set-theoretic subsolution of mean curvature motion.

In fact, any set-theoretic subsolution that is contained within a level-set flow at $t = 0$ must remain contained within it, because otherwise it would run into some of the other level-sets of u, violating 10.2. (One of the two sets involved in the collision is compact by the hypothesis on f). This proves

10.4 Equivalence of Definitions. A level-set flow in R^n is a *maximal* set-theoretic subsolution of mean curvature motion in the sense that it contains all other set-theoretic subsolutions with the same initial condition. □

Let us now briefly sketch how to derive the existence, uniqueness and properties of u and of the level-set flow directly from 10.2, alternate to the methods of Evans-Spruck, Chen-Giga-Goto, and Soner; most of this was done in [I2]. Let f be a continuous function on $\mathbf{R}^n$ with all level-sets compact (except possibly one). For each $a \in \mathbf{R}$, we can instantly construct the (unique) maximal set-theoretic subsolution $\{\Gamma_t^a\}_{t\geq 0}$ with initial condition $f^{-1}(a)$ by

$$\Gamma_t^a = \overline{\bigcup\{\Delta_t : \{\Delta_s\}_{s\geq 0} \text{ satisfies 10.1, } \Delta_0 = f^{-1}(a)\}}.$$

(Evidently $\{\Gamma_t\}_{t\geq 0}$ satisfies the second equivalent property of 10.1.) By 10.2, the $\{\Gamma_t^a\}_{t\geq 0}$ are all disjoint. In fact, the maximality property implies, with some work, that for each fixed t, the sets Γ_t^a, $a \in \mathbf{R}$, form a partition of space. Using 10.1, it is then easy to show that they constitute the level-sets of a continuous function u weakly solving $(\star)$, establishing the existence and uniqueness of u. Therefore, once again, the maximal set-theoretic subsolution is a level-set flow and vice versa.

Having outlined prior art, let us turn to Brakke motions. The main result of this section is

10.5 Tangency Property of Brakke Motions. The support of any codimension one Brakke motion on $\mathbf{R}^n$ is a set-theoretic subsolution of motion by mean curvature. $\square$

Then by 10.2 we obtain immediately

10.6 Avoidance Lemma for Codimension One Brakke Motions. Let $\{\mu_t\}_{t\geq 0}$, $\{\nu_t\}_{t\geq 0}$ be codimension one Brakke motions on $\mathbf{R}^n$ such that $\operatorname{spt}\nu_0$ is compact and

$$\operatorname{spt}\mu_0 \cap \operatorname{spt}\nu_0 = \emptyset.$$

Then $\operatorname{dist}(\operatorname{spt}\mu_t, \operatorname{spt}\nu_t)$ is nondecreasing. $\square$

By 10.4 we have

10.7 Inclusion Theorem. Let $\{\mu_t\}_{t\geq 0}$ be a codimension one Brakke motion on $\mathbf{R}^n$ and let $\{\Gamma_t\}_{t\geq 0}$ be a level-set flow. Then

$$\operatorname{spt}\mu_0 \subseteq \Gamma_0 \quad \text{implies} \quad \operatorname{spt}\mu_t \subseteq \Gamma_t \quad \text{for all } t \geq 0. \quad \square$$

Proof of 10.5. Let $\{\mu_t\}_{t\geq 0}$ be a Brakke motion and let $\{Q_t\}_{t\in[t_0,t_1]}$ be a family of compact hypersurfaces flowing smoothly by mean curvature, such that $\operatorname{spt}\mu_{t_0} \cap Q_{t_0} = \emptyset$. The idea is to construct a test function for μ_t out of the distance function to Q_t.

1. Let E_t be the compact region bounded by Q_t and define

$$r(x,t) = \begin{cases} -\mathrm{dist}(x, Q_t) & x \in E_t \\ \mathrm{dist}(x, Q_t) & x \notin E_t. \end{cases}$$

Fix $\gamma > 0$ so small that

$$\mathrm{dist}(\mathrm{spt}\,\mu_{t_0}, Q_{t_0}) > \gamma$$

and such that $r(x,t)$ is smooth on the set

$$U \equiv \{(x,t) : -\gamma < r(x,t) < \gamma, \quad t_0 \le t \le t_1\}.$$

Let $\beta > 2$ be a number to be selected later. Define the test function

$$\phi = h(r) \equiv \begin{cases} (\gamma - |r|)^\beta & |r| \le \gamma \\ 0 & |r| \ge \gamma. \end{cases}$$

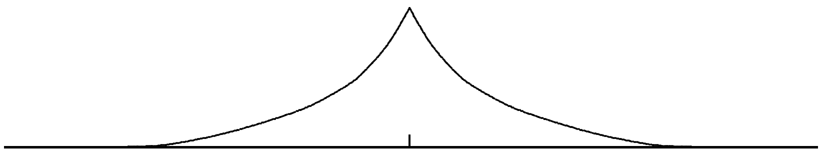

Figure 10. Graph of ϕ.

Then $\phi \in C^0_c(\mathbf{R}^n \times [t_0, t_1])$, ϕ vanishes except on U, and ϕ is C^2 except along Q_t.

Observe that $\mu_{t_0}(\phi) = 0$. We aim to show this remains so for $t \in [t_0, t_1]$. This will complete the proof.

2. Let $t_* = \sup\{t \in [t_0, t_1] : \mu_t(\phi) = 0\}$. Assume, for a contradiction, that $t_* < t_1$. By the growth bound 6.7, we have $\mu_{t_*}(\phi) = 0$. Thus $\mathrm{dist}(\mathrm{spt}\,\mu_{t_*}, Q_{t_*}) \ge \gamma$, so $\mathrm{spt}\,\mu_{t_*}$ is disjoint from any ball $B_{\gamma/2}(x)$ centered at a point $x \in Q_{t_*}$. Now let $B_{\gamma/2}(x)$ begin to flow by mean curvature. By Brakke's Internal Sphere Barrier Lemma [B, 3.6], $\mathrm{spt}\,\mu_t$ remains disjoint from the evolute of $B_{\gamma/2}(x)$ for as long as the latter exists. So there is $\tau > 0$ such that

$$\mathrm{spt}\,\mu_t \cap Q_t = \emptyset \qquad \text{for } t \in [t_*, t_* + \tau].$$

= 0.880cm

Therefore ϕ is C^2 on $\operatorname{spt}\mu_t$.

3. Thus, for $t \in [t_*, t_*+\tau]$, we can legitimately insert ϕ into the time-varying Brakke inequality 6.9 to obtain, by completing the square and using the first variation formula 1.8,

$$\begin{aligned}
\bar{D}_t\mu_t(\phi) &\leq \int -\phi H^2 + \nabla\phi \cdot S^\perp \cdot \vec{H} + \frac{\partial}{\partial t}\phi \, d\mu_t \\
&= \int -\phi H^2 - \nabla\phi \cdot S \cdot \vec{H} + \nabla\phi \cdot \vec{H} + \frac{\partial}{\partial t}\phi \, d\mu_t \\
&= \int -\left|\phi^{1/2}\vec{H} + \frac{1}{2}\frac{S\cdot\nabla\phi}{\phi^{1/2}}\right|^2 + \frac{1}{4}\frac{|\nabla\phi\cdot S|^2}{\phi} - S : \nabla^2\phi + \frac{\partial}{\partial t}\phi \, d\mu_t \\
&\leq \int \frac{1}{4}\frac{|\nabla\phi\cdot S|^2}{\phi} - S : \nabla^2\phi + \frac{\partial}{\partial t}\phi \, d\mu_t.
\end{aligned}$$

(If $\{\mu_t\}_{t\geq 0}$ is integral, we can use Brakke's result [B, 5.8] that $S \cdot \vec{H} = 0$ μ_t-a.e. to drop a term, but the proof doesn't become any easier.) Plug in $\phi = h(r)$,

$$\nabla\phi = h'(r)\nabla r, \qquad \nabla^2\phi = h''(r)\nabla r \otimes \nabla r + h'(r)\nabla^2 r, \qquad \frac{\partial}{\partial t}\phi = h'(r)\frac{\partial}{\partial t}r$$

to get

$$\bar{D}_t\mu_t(\phi) \leq \int \left(\frac{1}{4}\frac{(h')^2}{h} - h''\right)|\nabla r \cdot S|^2 + h'\left(-S : \nabla^2 r + \frac{\partial}{\partial t}r\right) d\mu_t.$$

Since $\{Q_t\}_{t\in[t_0,t_1]}$ solves the smooth mean curvature flow, a standard calculation (see [ES2]) tells us that

$$r\left(\frac{\partial}{\partial t}r - \Delta r\right) \geq 0,$$

on U, where $\Delta = \Delta^{\mathbf{R}^n}$. Since $h'(r)r \leq 0$, we have

$$h'(r)\left(\frac{\partial}{\partial t}r - \Delta r\right) \leq 0.$$

Now $\Delta r = S : \nabla^2 r + \vec{n}\cdot\nabla^2 r\cdot\vec{n}$, where $\vec{n}$ is the unit normal of S, and therefore

$$\bar{D}_t\mu_t(\phi) \leq \int \left(\frac{1}{4}\frac{(h')^2}{h} - h''\right)(1 - |\nabla r\cdot\vec{n}|^2) + h'\vec{n}\cdot\nabla^2 r\cdot\vec{n}\, d\mu.$$

Define the $(n-1)$-plane field $T(x)$ on U by $T = (\nabla r)^\perp$, note that $\vec{n} = \vec{n}\cdot T + (\vec{n}\cdot\nabla r)\nabla r$, and obtain

$$\begin{aligned}
\bar{D}_t\mu_t(\phi) &\leq \int \left(\frac{1}{4}\frac{(h')^2}{h} - h''\right)|\vec{n}\cdot T|^2 + |h'||\vec{n}\cdot\nabla^2 r\cdot\vec{n}|\, d\mu_t \\
&= \int \left(\frac{1}{4}\frac{(h')^2}{h} - h'' + |h'||T\cdot\nabla^2 r\cdot T|\right)|\vec{n}\cdot T|^2\, d\mu_t
\end{aligned}$$

where we have used the fact that $\nabla r \cdot \nabla^2 r = 0$, as can be seen by differentiating $|\nabla r|^2 = 1$. Now substitute

$$\begin{aligned} h &= (\gamma - |r|)^\beta, \\ h' &= -\mathrm{sgn}\,(r)\beta(\gamma - |r|)^{\beta-1}\mathrm{sgn}\, r, \\ h'' &= \beta(\beta-1)(\gamma - |r|)^{\beta-2}, \\ \frac{(h')^2}{h} &= \beta^2(\gamma - |r|)^{\beta-2}, \end{aligned}$$

valid on spt μ_t, $t \in [t_0, t_1]$, to obtain

$$\bar{D}_t \mu_t(\phi) \le \int_U \beta(\gamma - |r|)^{\beta-2}|\vec{n}\cdot T|^2 \left[1 - \frac{3}{4}\beta + (\gamma - |r|)|T\cdot\nabla^2 r\cdot T|\right] d\mu_t.$$

Now choose $\beta \ge (4/3)(1 + \gamma(n-1)\max_{\overline{U}}|\nabla^2 r|)$. Notice that β is independent of t and t_*. (Alternately fix $\beta = 3$ and choose γ so that $\gamma(n-1)\max_{\overline{U}}|\nabla^2 r| < 5/4$.) Then

$$\bar{D}_t \mu_t(\phi) \le 0 \qquad \text{for all } t \in [t_*, t_* + \tau].$$

and therefore $\mu_t(\phi) = 0$ for $t \in [t_*, t_* + \tau]$, contradicting the definition of t_*. Hence $t_* = t_1$, and $\mu_t(\phi) = 0$ for all $t \in [t_0, t_1]$, proving 10.5. Q.E.D.

§11 Unit Density

We will see that the inclusion theorem of the last chapter, together with the matching theorem 9.2, shows that a solution of the level-set flow that does not fatten up corresponds to a unique Brakke motion of unit density that is the boundary of a set of finite perimeter in space-time.

Define

$$\tau(\theta) = \tau(\{\theta \ne 0\},\ |\theta|,\ \mathrm{sgn}(\theta)\, e_1 \wedge \ldots \wedge e_p)$$

whenever $\theta \in L^1_{loc}(\mathbf{R}^p, \mathbf{Z})$ is a function of locally bounded variation. Then $\tau(\theta) \in \mathbf{I}_p^{loc}(\mathbf{R}^p)$. The *perimeter* of a set $E \subseteq \mathbf{R}^p$ is $\mathbf{M}[\partial[E]]$, where $[E] \equiv \tau(\chi_E)$.

11.1 Relation of Enhanced Motion to the Level-Set Flow.

(i) Let $E_0 \subseteq \mathbf{R}^n$ be a bounded open set of finite perimeter, and let $T_0 = \partial[E_0]$. Then there is a set $E \subseteq \mathbf{R}^n \times [0, \infty)$ of finite perimeter and an enhanced motion $(T, \{\mu_t\}_{t\ge0})$ such that

$$T = \partial[E] + [E_0], \qquad T_t = \partial[E_t],$$

where $E_t = E \cap (M \times \{t\})$, and $\{\chi_{\pi_\#(E_t)}\}_{t\geq 0}$ is absolutely continuous in $L^1(\mathbf{R}^n)$.

(ii) Let E, $(T, \{\mu_t\}_{t\geq 0})$ be objects satisfying (i). Let $f : \mathbf{R}^n \to \mathbf{R}$ be a continuous function with $E_0 = \{f > 0\}$, and let u be the unique continuous function weakly solving the level-set flow equation $(\star)$ of §10. Then E may be modified on a set of $\mathcal{L}^{n+1}$-measure zero, without affecting the properties of (i), so that

$$\{u > 0\} \subseteq E \subseteq \{u \geq 0\}. \quad \square$$

We will prove this after 11.7.

11.2 Definition of Boundary Motion. Let E_0 be a bounded open set of finite perimeter in $\mathbf{R}^n$ with $T_0 = \partial[E_0]$. A current $T \in \mathbf{I}_n(M \times [0,\infty))$ is called a *boundary motion* for T_0 provided there is a relatively open set $E \subseteq \mathbf{R}^n \times [0,\infty)$ of finite perimeter such that

(i) $T = \partial[E] + [E_0]$, $T_t = \partial[E_t]$ where $E_t = E \cap (M \times \{t\})$, and $\{\chi_{\pi_\#(E_t)}\}_{t\geq 0}$ is absolutely continuous in $L^1(\mathbf{R}^n)$.

(ii) $\{\mu_{T_t}\}_{t\geq 0}$ is a Brakke motion. $\square$

In particular, each E_t is a set of finite perimeter. DeGiorgi's theorem (discussed in §12) implies that $\mu_{\partial[E_t]}$ has unit density almost everywhere. Since this is the crucial requirement for Brakke's Regularity Theory, a boundary motion is a very desirable kind of mean curvature motion.

11.3 Definition of Nonfattening. Let $\{\Gamma_t\}_{t\geq 0}$ be a level-set flow. We say that $\{\Gamma_t)_{t\geq 0}$ satisfies the *nonfattening condition* provided that

$$\mathcal{L}^{n+1}\Big(\bigcup_{t\geq 0} \Gamma_t \times \{t\}\Big) = 0$$

Remark. If $\mathcal{H}^{n-1}(\Gamma_0) < \infty$, then the following are equivalent forms of the nonfattening hypothesis:

(i) $\mathcal{L}^{n+1}\big(\bigcup_{t\geq 0} \Gamma_t \times \{t\}\big) = 0$.

(ii) $\bigcup_{t\geq 0} \Gamma_t \times \{t\}$ has no (space-time) interior.

(iii) $\mathcal{L}^n(\Gamma_t) = 0 \quad$ for a.e. $t \geq 0$.

(iv) For a.e. $t \geq 0$, Γ_t has no spatial interior.

(v) $\mathcal{L}^n(\Gamma_t) = 0$ for all $t \geq 0$.

(vi) For all $t \geq 0$, Γ_t has no spatial interior.

Proof of Equivalence. Clearly (i) implies (ii), (iii) implies (iv), and (v) implies (vi). By Evans-Spruck [ES3, 4.2], $\mathcal{H}^{n-1}(\partial\Gamma_t) \leq C\mathcal{H}^{n-1}(\Gamma_0) < \infty$ for all $t \geq 0$. Thus in fact

$$\text{(iii) is equivalent to (iv),} \qquad \text{(v) is equivalent to (vi).}$$

Furthermore, if Γ_t develops an interior, it retains it for a short time (by considering a shrinking ball embedded in Γ_t). Thus

$$\text{(ii), (iv) and (vi) are equivalent.}$$

Finally,

$$\text{(i) is equivalent to (iii).}$$

by Fubini's theorem, since $\bigcup_{t\geq 0} \Gamma_t \times \{t\} = \{u = 0\}$ is closed. Q.E.D.

11.4 Unit Density Theorem. Let $E_0 \subseteq \mathbf{R}^n$ be a bounded open set of finite perimeter, $T_0 = \partial[E_0]$. Let f be any continuous, proper function with $E_0 = \{f > 0\}$, $\partial E_0 = \{f = 0\}$. Let u solve $(\star)$ for f, and let $\{\Gamma_t\}_{t\geq 0}$ be the corresponding level-set flow. Define $E = \{u > 0\}$, $T = \partial[E] + [E_0]$.

If $\{\Gamma_t\}_{t\geq 0}$ satisfies the nonfattening condition, then T is the unique undercurrent for T_0, T is a boundary motion, and $\{\mu_{T_t}\}_{t\geq 0}$ is a unit density Brakke motion. $\square$

Only a countable number of level-sets of a given function can fatten up, so any level-set can be perturbed to one satisfying the above theorem.

Remark. The nonfattening hypothesis is always fulfilled in the following special cases:

(a) T_0 is smooth of positive mean curvature.

(b) T_0 bounds a smooth star-shaped region (see Soner [So]).

See Soner-Souganidis [SS] for other examples. To proceed, we will need two lemmas.

11.6 Support Lemma. Let T be the undercurrent for an enhanced motion of dimension k in $\mathbf{R}^n$. Then, with $D = D^T$,

(i) The set $Z \equiv \{(x,t) : |D\tau(x,t)| \text{ is undefined or zero}\}$ has $\mu_T(Z) = \sigma(Z) = 0$, where σ is given by

$$\sigma(\phi) = \iint \phi \, d\mu_{T_t} dt, \qquad \phi \in C_c^0(M \times [0,\infty), \mathbf{R}^+).$$

(ii) μ_T and σ are mutually absolutely continuous.

(iii) $\operatorname{spt} T = \overline{\bigcup_{t \geq 0} \operatorname{spt} T_t}$. □

Proof. (i) Let $Z_0 = \{(x,t) : |D\tau(x,t)| = 0\}$. We will prove $\mu_T(Z_0) = 0$. (Note that $\sigma(Z_0) = 0$ is trivial by the co-area formula.) Let $T = \tau(X, \theta, \xi)$. By the rectifiability of X, we have

$$Z_0 \subseteq X \subseteq \bigcup_{i \geq 1} C_i \qquad \mathcal{H}^{k+1}\text{-a.e.}$$

where C_i is a C^1 $(k+1)$-submanifold and $|D_{C_i}\tau| = |D\tau| = 0$ for $\mathcal{H}^{k+1}$-a.e. $(x,t) \in C_i \cap Z_0$. Thus by Sard's theorem for C^1 functions [S, 10.4] (an immediate consequence of the co-area formula)

$$Z_0 = Z_1 \cup Z_2$$

where $\mathcal{H}^{k+1}(Z_1) = 0$ and $\mathcal{L}^1(\tau(Z_2)) = 0$. Then $\mu_T(Z_1) = 0$, and applying the absolute continuity 8.1(ii)(b), we have $\mu_T(Z_2) = 0$. Thus $\mu_T(Z_0) = 0$.

If Z_* is the set where $|D\tau|$ is undefined, then $\mu_T(Z_*) = 0$. Therefore $\mu_T(Z) = \mu_T(Z_0 \cup Z_*) = 0$.

(ii) By the co-area formula, for any W

$$\sigma(W) = \int_W |D\tau| \, d\mu_T.$$

Since $|D\tau| \neq 0$ μ_T-a.e., σ and μ_T are mutually absolutely continuous. In particular $\sigma(Z) = 0$.

(iii) Clearly $\operatorname{spt} T_t \subseteq \operatorname{spt} T$ for each t. For the other direction, let $(x,t) \in \operatorname{spt} T$ and let $U \ni (x,t)$ be open. Then $\mu_T(U) > 0$, so by (ii), $\sigma(U) > 0$, i.e.

$$\int_0^\infty \int \chi_U \, d\mu_{T_s} ds > 0.$$

Thus $\mathbf{M}[T_s \lfloor U] > 0$ for some s near t. Since U was arbitrary, $(x,t) \in \overline{\bigcup_{s \geq 0} \operatorname{spt} T_s}$. □

11.7 Impossibility of Spontaneous Generation. Let $u : \mathbf{R}^n \times [0,\infty) \to \mathbf{R}$ be a continuous function solving §10($\star$). Then each component of $\{u \neq 0\}$ meets $\mathbf{R}^n \times \{0\}$. $\square$

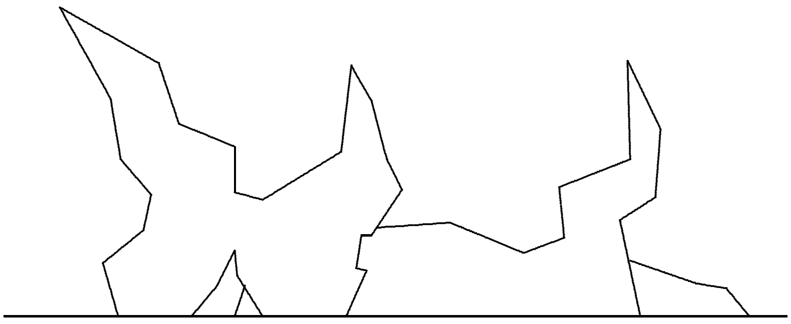

Figure 11. Zero set of u.

Proof. Let U be a component of $\{u \neq 0\}$. By applying the relabelling lemma [I1, 3.2] locally, it is evident that the function u^ε defined by

$$u^\varepsilon = \begin{cases} u & \text{on } \{-\varepsilon \le u \le \varepsilon\} \cup U \\ \varepsilon & \text{on } \{\varepsilon \le u\} \setminus U \\ -\varepsilon & \text{on } \{u \le -\varepsilon\} \setminus U \end{cases}$$

will solve ($\star$). Passing to limits by the uniform convergence lemma [I1, 3.1], we obtain $\tilde{u}$ solving ($\star$) with $U = \{\tilde{u} \neq 0\}$. If $U \cap (\mathbf{R}^n \times \{0\}) = \emptyset$, then we are contradicting uniqueness of the zero solution of ($\star$). Q.E.D.

Proof of 11.1. 1. Let $E_0 \subseteq \mathbf{R}^n$ be a bounded open set of finite perimeter and let $T_0 = \partial[E_0]$. Let $(T, \{\mu_t\}_{t\ge 0})$ be the enhanced motion with initial condition T_0 constructed in the proof of 8.1.

It is not hard to check that the existence procedure for the current T, which takes place in 3.2 and §8, can also be made to yield a set $E \subseteq \mathbf{R}^n \times [0,\infty)$ of finite perimeter such that

$$T = \partial[E] + [E_0].$$

One obtains this result by initially restricting the admissible set $\mathcal{A}$ of 3.2 to consist only of currents of this form, and carrying out convergence of the characteristic function of E in $L^1(\mathbf{R}^n \times [0,\infty))$ by means of the compactness theorem for BV_{loc}.

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2. Let us rename E_0 as E_0^{given}. Define $E_t = E \cap (\mathbf{R}^n \times \{t\})$, $t \geq 0$. (We will worry about consistency of E_0 with E_0^{given} shortly.) By the slice theorem [S, §28] we have for a.e. $0 < t < s$,

$$\begin{aligned} \partial([E]\lfloor(\mathbf{R}^n \times [t,s))) &= \partial[E]\lfloor(\mathbf{R}^n \times [t,s)) + \langle [E], \tau, s\rangle - \langle [E], \tau, t\rangle \\ &= (T - [E_0^{given}])\lfloor(\mathbf{R}^n \times [t,s)) + [E_s] - [E_t] \\ &= T[t,s) + [E_s] - [E_t]. \end{aligned} \tag{1}$$

Project onto $\mathbf{R}^n$ to obtain, since $\pi_\#([E]\lfloor(\mathbf{R}^n \times [t,s))) = 0$,

$$0 = \pi_\# \partial([E]\lfloor(\mathbf{R}^n \times [t,s))) = \pi_\# T[t,s) + \pi_\#[E_s] - \pi_\#[E_t].$$

By the absolute continuity of $\mathbf{M}[T(t,s)]$ with respect to $|t-s|$ (see 8.1(ii)(b)), we find that

$$\mathbf{M}[\pi_\#[E_s] - \pi_\#[E_t]] \downarrow 0 \qquad \text{as } |s-t| \downarrow 0$$

for the full measure set of s, t such that (1) holds. Thus we may change E by a set of $\mathcal{L}^{n+1}$-measure zero so that

$$\{\chi_{\pi(E_t)}\}_{t\geq 0}$$

is an absolutely continuous path in $L^1(\mathbf{R}^n)$.

3. Setting $s = \infty$ and taking the boundary of (1) we obtain

$$T_t = \partial(T[t,\infty)) = \partial[E_t] \qquad \text{a.e. } t > 0.$$

Since both $\{T_t\}_{t\geq 0}$ and $\{[E_t]\}_{t\geq 0}$ are weakly continuous, we obtain

$$T_t = \partial[E_t] \qquad \text{for all } t \geq 0.$$

Now $T_0 = \partial[E_0^{given}]$, so $\partial([E_0] - [E_0^{given}]) = 0$, which shows by the Constancy Theorem [S, 26.27] that the two definitions of E_0 are equal up to a set of $\mathcal{L}^n$-measure zero. This completes the proof of assertion 11.1(i).

4. Let f, u be as in hypothesis 11.1(ii). In particular $\operatorname{spt}\mu_0$ is compact and contained in $\{f = 0\}$. By 11.6(iii), the definition of enhanced motion, and the inclusion lemma 10.7,

$$\operatorname{spt} T = \overline{\bigcup_{t\geq 0} \operatorname{spt} T_t} \subseteq \overline{\bigcup_{t\geq 0} \operatorname{spt}\mu_t \times \{t\}} \subseteq \{u = 0\}.$$

Thus $\operatorname{spt} \partial E = \operatorname{spt}(T - [E_0]) \subseteq \{u = 0\} \cup \{f \geq 0\}$.

Now let U be any component of $\{u > 0\}$ relative to $\mathbf{R}^n \times [0, \infty)$. By the Constancy Theorem and the above, χ_E is constant $\mathcal{L}^{n+1}$-a.e. on U. Since $\{\chi_{E_t}\}_{t \geq 0}$ is continuous in L^1 and U is open relative to $\mathbf{R}^n \times [0, \infty)$, we can redefine E on a set of $\mathcal{L}^{n+1}$-measure zero, and each E_t on a set of $\mathcal{L}^n$-measure zero without affecting the continuity in L^1, so that

$$\chi_E \text{ is constant on } U.$$

By the Spontaneous Generation Lemma 11.7, U meets $\mathbf{R}^n \times \{0\}$ in one or more components of $\{f > 0\} = E_0$, so in fact $U \subseteq E$. Since U was arbitrary, it follows that $\{u > 0\} \subseteq E$.

A similar argument shows that $\{u < 0\} \subseteq \mathbf{R}^n \times [0, \infty) \setminus E$. Thus E is trapped between $\{u > 0\}$ and $\{u \geq 0\}$. This is assertion 11.1(ii). Q.E.D.

Proof of 11.4. 1. Let T be the current constructed in 11.1 and let T^1 be any other undercurrent for T_0, with corresponding Brakke motion $\{\mu_t^1\}_{t \geq 0}$. As in the previous proof, we find that

$$\operatorname{spt} T^1 \subseteq \{u = 0\}.$$

Then $\operatorname{spt}(T^1 - T) \subseteq \{u = 0\}$. Also, $\partial(T^1 - T) = T_0 - T_0 = 0$, so by homology $T^1 - T = \partial F$ for some current $F \in \mathbf{I}_{n+1}^{loc}(\mathbf{R}^n \times \mathbf{R})$ with $\operatorname{spt} \partial F \subseteq \{u = 0\}$. By 11.7, the set $\{u = 0\}$ (which lies in $\mathbf{R}^n \times [0, \infty)$) does not separate $\mathbf{R}^n \times \mathbf{R}$. Hence $F = \tau(\theta)$ for a function θ that is almost everywhere constant, in fact zero, on $\{u \neq 0\}$. By hypothesis $\mathcal{L}^{n+1}(\{u = 0\}) = 0$, so $F = 0$ and $T^1 = T$.

2. This establishes that T_0 has the unique undercurrent T. The set E of 11.1 has $T = \partial[E] + [E_0]$, $T_t = \partial[E_t]$, and $\{\chi_{\pi_\#(E_t)}\}_{t \geq 0}$ weakly continuous in $BV(\mathbf{R}^n)$. Since $\{u > 0\} \subseteq E \subseteq \{u \geq 0\}$ and $\mathcal{L}^{n+1}(\{u = 0\}) = 0$, we can take E to be the open set $\{u > 0\}$.

3. By 9.2 and the uniqueness of T, T is a matching motion, that is, $\{\mu_{T_t}\}_{t \geq 0}$ is a Brakke motion. Q.E.D.

§12 Almost Everywhere Regularity

Let T be a boundary motion and $\mu_t = \mu_{T_t}$. Because $\{\mu_t\}_{t \geq 0}$ has unit density, Brakke's Main Regularity Theorem [B, 6.12] immediately applies to prove a certain kind of partial regularity. Because T is locally a reduced boundary, we will use Brakke's Local Regularity Theorem to prove a slightly different result, namely

$$\mathcal{H}^n(\operatorname{sing}(\operatorname{spt} T)) = 0.$$

This is Theorem 12.9. Corollary 12.11 treats almost-everywhere regularity of almost every level-set of the level-set flow.

Let ω_k be the volume of the unit ball in $\mathbf{R}^k$.

12.1 Brakke's Local Regularity Theorem. ([B, 6.11]) There are constants $0 < c_1 < c_2$ and $\eta_0 > 0$, depending on n and k, such that if $\{\mu_t\}_{t\geq 0}$ is a unit density Brakke motion, that is,

$$\Theta^k(\mu_t, \cdot) = 1 \quad \mu_t\text{-a.e.} \qquad \text{for all } t \geq 0,$$

and if

(i) $R > 0$, $(x_0, t_0) \in \mathbf{R}^n \times [0, \infty)$ and $P \subseteq \mathbf{R}^n$ is a k-plane containing x_0,

(ii) $\displaystyle\int_{B_R(x_0)} |\operatorname{dist}(x, P)|^2 \, d\mu_{t_0} \leq \eta_0 R^{k+2}$,

(iii) $\displaystyle\frac{\omega_k}{2}\left(\frac{R}{2}\right)^k \leq \mu_t(B_{R/2}(x_0)) < \frac{3\omega_k}{2}\left(\frac{R}{2}\right)^k$ for $t \in [t_0, t_0 + c_2 R^2]$,

(iv) $\displaystyle\mu_t(B_R(x_0)) < \frac{3}{2}\omega_k R^k$ for $t \in [t_0, t_0 + c_2 R^2]$,

then the intersection of

$$\bigcup_{t\geq 0} \operatorname{spt} \mu_t \times \{t\}$$

with

$$B_{R/2}(x_0) \times [t_0 + c_1 R^2, t_0 + c_2 R^2]$$

is a C^∞ manifold. $\square$

We will need the following specialization of Brakke's Clearing-Out Lemma.

12.2 Clearing-Out Lemma. ([B, 6.3]) There are constants $\eta_1 > 0$, $0 < c_3 < c_4$ depending on n and k such that for any integral Brakke motion $\{\mu_t\}_{t\geq 0}$, $R > 0$, and $(x_0, t_0) \in \mathbf{R}^n \times [0, \infty)$, if

$$\mu_{t_0}(B_R(x_0)) \leq \eta_1 R^k$$

then

$$\mu_t(B_{R/2}(x_0)) = 0 \qquad \text{for } t \in [t_0 + c_3 R^2, t_0 + c_4 R^2]. \quad \square$$

Recall that the *reduced boundary* $\partial^* E$ of a set E of finite perimeter is defined to be the set of points $x \in \partial E$ such that

(i) $P = T_x(\mathcal{H}^{n-1} \lfloor \partial E)$ exists and is of multiplicity one.

(ii) $\lim_{\lambda \downarrow 0} \chi_E(x + \lambda y) = \chi_{P^+}(y)$ in L^1, where P^+ is a half-space bounded by P.

According to *DeGiorgi's theorem* (see [S, 14.3]), $\partial^* E$ is rectifiable and $\partial[E] = \tau(\partial^* E, 1, \xi)$ where ξ orients $T_x(\mathcal{H}^{n-1} \lfloor \partial E)$ as the boundary of $P^+(x)$. In particular, $\mu_{\partial[E]} = \mathcal{H}^{n-1} \lfloor \partial^* E$ has unit density almost everywhere.

Let T be a boundary motion. Using the Clearing-Out Lemma, we will show that $\operatorname{spt} T$ is rectifiable and in fact $\mu_T = \mathcal{H}^n \lfloor \operatorname{spt} T$. Then we will use the Local Regularity Theorem to show that for μ_T-a.e. (x, t), the existence of the space-time tangent plane gives smoothness near (x, t).

For small r, a cylinder $B_r(x) \times [t - r, t + r]$ has an intrinsic time scale much larger than the spatial radius. We will need to bound $\int_{t-r}^{t+r} \int_{B_r} |\vec{H}| \, d\mu_t \, dt$ over such cylinders in order to control the mass growth. If we knew in advance that the motion were smooth almost everywhere, then since $|\dot{x}| = |\vec{H}|$ we would have

$$|D\tau| = \frac{1}{\sqrt{H^2 + 1}},$$

where $\tau =$ time, $D = D^T$, and then

$$(\dagger) \qquad \int_{t-r}^{t+r} \int_{B_r} |\vec{H}| \, d\mu_t \, dt = \int_{B_r \times [t-r, t+r]} |\vec{H}| |D\tau| \, d\mu_T \leq \mu_T(B_r \times [t - r, t + r])$$

so that $\iint |\vec{H}| \, d\mu_t \, dt$ is estimated in terms of the local mass of T. However, in the case at hand, we must use the Lebesgue-Besicovitch differentiation theorem and the global bound $\iint |\vec{H}| \, d\mu_t \, dt < \infty$.

By inserting $\phi = 1$ in 6.3 and using Hölder's inequality, we find

$$\int_0^L \int |\vec{H}| \, d\mu_t \, dt < \infty,$$

where L is the extinction time. Define Radon measures σ, ν by

$$\sigma(\psi) = \iint \psi \, d\mu_t \, dt, \qquad \nu(\psi) = \iint \psi |\vec{H}| \, d\mu_t \, dt,$$

for $\psi \in C_c^0(\mathbf{R}^n \times [0,\infty), \mathbf{R}^+)$. Evidently ν is absolutely continuous with respect to σ. Note also that by the support lemma 11.6(ii), σ and μ_T have the same null-sets. By DeGiorgi's theorem, $\mu_T = \mathcal{H}^n \lfloor X$, where X is the reduced boundary of E relative to $\mathbf{R}^n \times [0,\infty)$, and $\sigma = \sigma \lfloor X$, $\nu = \nu \lfloor X$. We obtain the following lemma by the Lebesgue-Besicovitch differentiation theorem [S, 4.7].

12.3. First H Lemma. Let T be a boundary motion with $\mu_T = \mathcal{H}^n \lfloor X$. Then for $\mathcal{H}^n$-a.e. $(x,t) \in X$,

$$\frac{d\nu}{d\sigma}(x,t) \equiv \lim_{r \downarrow 0} \frac{\displaystyle\iint_{B_r^{n+1}(x,t)} |\vec{H}|\, d\mu_s ds}{\displaystyle\iint_{B_r^{n+1}(x,t)} d\mu_s ds} = |\vec{H}(x,t)|. \quad \square$$

For points not covered by the above statement, we have

12.4 Second H Lemma. Let T be a boundary motion with $\mu_T = \mathcal{H}^n \lfloor X$. Then for $\mathcal{H}^n$-a.e. $(x,t) \in \mathbf{R}^n \times [0,\infty) \setminus X$,

$$\Theta^n(\nu,(x,t)) \equiv \lim_{r \downarrow 0} \frac{1}{r^n} \int_{t-r}^{t+r} \int_{B_r^n(x)} |\vec{H}|\, d\mu_s ds = 0. \quad \square$$

Proof. According to [S, 3.5], if ν is a Borel-regular measure and $X \subseteq \mathbf{R}^n$ is ν-measurable with $\nu(X) < \infty$, then $\Theta^k(\nu \lfloor X, x) = 0$ for $\mathcal{H}^k$-a.e. $x \in \mathbf{R}^n \setminus X$.

Replace k by n, n by $n+1$. As noted above, $\nu = \nu \lfloor X$. Evidently X is ν-measurable and $\nu(X) < \infty$. Thus $\Theta^n(\nu,(x,t)) = 0$ for $\mathcal{H}^n$-a.e. $x \in \mathbf{R}^n \times [0,\infty) \setminus X$. This is the lemma. Q.E.D.

Let us henceforth write $B_r(x)$ for $B_r^n(x)$.

12.5 Long-Time Clearing-Out Lemma. There exists $\eta_2 > 0$, $r_0 > 0$ depending on n, k such that for any boundary motion T, $(x_0,t_0) \in \mathbf{R}^n \times [0,\infty)$, and $r \in (0, r_0]$, if

(i) $\displaystyle\int_{t_0-r}^{t_0+r} \int_{B_r(x_0)} |\vec{H}|\, d\mu_t dt \le \eta_2 r^n$

(ii) $\displaystyle\int_{t_0-r}^{t_0-r/2} \mu_t(B_r(x_0))\, dt \le \eta_2 r^n$

then

$$\mu_T(B_{r/4}(x_0) \times [t_0 - r/4, t_0 + r]) = 0. \quad \square$$

12.6 Remarks. (a) Note that the time interval $[t_0 - r, t_0 + r]$ is relatively long. Actually, any longer time interval works also.

(b) The quantity in (i) is the volume crossed by T_t — once we have established a certain amount of regularity. Thus to reach the center of a ball, a surface must either have a large area (on average), cross a large volume, or do it right away.

(c) Once we have proven almost everywhere regularity of $\operatorname{spt} T$, we can use (†) and the proof of 12.7 to replace hypotheses (i) and (ii) by the single hypothesis

$$\mu_T(B_r(x_0) \times [t_0 - r, t_0 + r]) \le \eta_2 r^n.$$

Proof of 12.5. By (ii), there exists $t_1 \in [t_0 - r, t_0 - r/2]$ with

$$\mu_{t_1}(B_r) \le 2\eta_2 r^{n-1}.$$

Choose $\phi \in C_c^2(B_r(x_0), \mathbf{R}^+)$ with $\phi = 1$ on $B_{r/2}(x_0)$, $0 \le \phi \le 1$, $|\nabla\phi| \le 3/r$. Then for $t \in [t_0 - r/2, t_0 + r]$,

$$\begin{aligned}
\mu_t(B_{r/2}(x_0)) &\le \mu_t(\phi) \\
&\le \mu_{t_1}(\phi) + \int_{t_1}^{t_0+r} \int_{B_r(x_0)} -\phi H^2 + \nabla\phi \cdot S^\perp \cdot \vec{H}\, d\mu_t dt \\
&\le 2\eta_2 r^{n-1} + \max|\nabla\phi| \int_{t_0-r}^{t_0+r} \int_{B_r(x_0)} |\vec{H}|\, d\mu_t dt \\
&\le 5\eta_2 r^{n-1} \qquad \text{by (i)}.
\end{aligned}$$

Now choose η_2 according to

$$5\eta_2 = \left(\frac{1}{2}\right)^{n-1} \omega_{n-1}\eta_1.$$

Then by the Clearing-Out Lemma 12.2,

$$\mu_t(B_{r/4}(x_0)) = 0 \qquad \text{for } t \in [t_0 - r/2 + c_3 r^2/4, t_0 + r].$$

Choose r_0 so that $c_3 r^2/4 \le r/4$ for $r \le r_0$. Then

$$\mu_t(B_{r/4}(x_0)) = 0 \qquad \text{for } t \in [t_0 - r/4, t_0 + r],$$

from which the lemma follows by the Support Lemma 11.6(iii). Q.E.D.

12.7 Rectifiability of Support. Let T be a boundary motion. Then $\operatorname{spt} T$ is rectifiable and

$$\mu_T = \mathcal{H}^n \lfloor \operatorname{spt} T. \quad \square$$

Evans and Spruck have proven a related result in [ES3], namely rectifiability of the boundary of the level-set flow for a.e. $t \geq 0$.

Proof. Define X by

$$X = \{(x,t) : \Theta^n(\mu_T, (x,t)) = 1\}.$$

Then $\mu_T = \mathcal{H}^n \lfloor X$ by Lemma 1.3 and DeGiorgi's theorem on sets of finite perimeter. Define

$$X_* = \{(x,t) : \Theta^n(\mu_T, (x,t)) \text{ does not exist, or lies in } (0,1) \cup (1,\infty)\},$$
$$X_0 = \{(x,t) \in \operatorname{spt} T : \Theta^n(\mu_T, (x,t)) = 0\}.$$

Then $\operatorname{spt} T = X \cup X_* \cup X_0$. By 1.3, $\mathcal{H}^n(X_*) = 0$. For any fixed $(x,t) \in X_0$, there exists $r_2 \in (0, r_0]$ such that for all $r \leq r_2$,

$$\begin{aligned}\int_{t-r}^{t+r} \mu_t(B_r(x))\,dt &= \int_{B_r(x)\times[t-r,t+r]} |D\tau| d\mu_T \qquad \text{by the Slice Lemma}\\ &\leq \mu_T(B_r(x) \times [t-r, t+r])\\ &\leq \eta_2 r^n.\end{aligned}$$

By Lemma 12.4, for $\mathcal{H}^n$-a.e. $(x,t) \in X_0$ we can further decrease r so that in addition

$$\int_{t-r}^{t+r} \int_{B_r(x)} |\vec{H}|\, d\mu_s ds \leq \eta_2 r^n.$$

Then by the Long-Term Clearing-Out Lemma 12.5, $(x,t) \notin \operatorname{spt} T$. Since $X_0 \subseteq \operatorname{spt} T$, this shows $\mathcal{H}^n(X_0) = 0$. Thus $\mu_T = \mathcal{H}^n \lfloor X = \mathcal{H}^n \lfloor \operatorname{spt} T$. Q.E.D.

Remark. Once we have proven regularity almost everywhere of $\operatorname{spt} T$, we see that actually $X_0 = \emptyset$, by remark 12.6(c).

Let P be an $(n-1)$-plane in $\mathbf{R}^n$, and let P^+, P^- be the two components of $\mathbf{R}^n \setminus P$. Define

$$P_a = \{x \in \mathbf{R}^n : \operatorname{dist}(x, P) \leq a\}, \qquad P_a^+ = P^+ \setminus P_a, \qquad P_a^- = P^- \setminus P_a.$$

We turn our attention to a parabolic cylinder where $\operatorname{spt} T$ remains close to $P \times$ time and $\iint |H| \, d\mu_t \, dt$ is small. (Later these conditions can be achieved by blowing up parabolically by some finite amount.)

12.8 Local Regularity of T. There exist $r_1 > 0$, $\delta_0 > 0$ depending only on n, such that for any boundary motion T, $(x_0, t_0) \in \mathbf{R}^n \times (0, \infty)$, $r \in (0, \min(r_1, t_0)]$, and plane $P \ni x_0$, if

$$\begin{aligned}
&\text{(i)} && \int_{t_0-r}^{t_0-r/2} \mu_t(B_r(x_0)) \, dt \le (1+\delta_0)\omega_{n-1} \left(\frac{r}{2}\right) r^{n-1} \\
&\text{(ii)} && \int_{t_0-r}^{t_0-r/2} \mu_t(B_{r/2}(x_0)) \, dt \le (1+\delta_0)\omega_{n-1} \left(\frac{r}{2}\right)^n \\
&\text{(iii)} && \int_{t_0-r}^{t_0-r/2} \mu_t(B_r(x_0) \setminus P_{\delta_0 r}) \, dt \le \delta_0 r^n \\
&\text{(iv)} && \int_{t_0-r}^{t_0+r} \int_{B_r(x_0)} |\vec{H}| \, d\mu_t dt \le \delta_0 r^n \\
&\text{(v)} && \mathcal{L}^{n+1}(E \cap F^+) \ge \frac{7}{8} \mathcal{L}^{n+1}(F^+) \\
&\text{(vi)} && \mathcal{L}^{n+1}(E \cap F^-) \le \frac{1}{8} \mathcal{L}^{n+1}(F^-),
\end{aligned}$$

where $F^\pm = (B_r(x_0) \cap P^\pm_{\delta_0 r}) \times [t_0 - r, t_0 + r]$, then

$$\operatorname{spt} T \bigcap (B_{r/4}(x_0) \times [t_0 - r/4, t_0 + r])$$

is a C^∞ manifold. $\square$

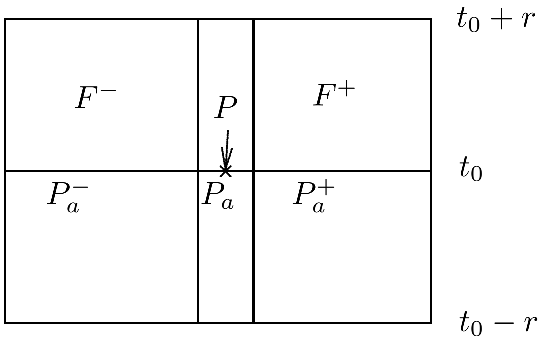

Figure 12. Regions.

Proof. The proof is a straightforward application of Brakke's Local Regularity Theorem and the Long-Term Clearing-Out Lemma.

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1. Let us first specify the constants. Define $r_1 = 1/(2c_1)$. Next, choose $\varepsilon > 0$ so that

$$
\begin{gathered}
\frac{1}{(1-\varepsilon)^{n-1}} < \frac{3}{2}, \qquad c_3\varepsilon^2 r_1 < 1, \qquad \frac{6\varepsilon^2}{(1-\varepsilon)^2}\omega_{n-1} \le \eta_0, \\
\left(\frac{3\varepsilon}{1-\varepsilon}\right)^2 + \left(\frac{1-\sqrt{\varepsilon}}{2}\right)^2 \le \left(\frac{1}{2}\right)^2, \qquad (1-\sqrt{\varepsilon})^{n-1} > \frac{1}{2}, \qquad \varepsilon < \frac{1}{10}.
\end{gathered} \tag{1}
$$

Next, take $\delta_0 \equiv \delta < \varepsilon$ so small that

$$
\frac{1}{(1-\varepsilon)^{n-1}}\left((1+\delta)\omega_{n-1} + \frac{2^{n+1}\delta}{\varepsilon}\right) < \frac{3}{2}\omega_{n-1}, \qquad \delta\left(4+\frac{3}{\varepsilon}\right) \le \eta_1\left(\frac{\varepsilon}{2}\right)^{n-2}. \tag{2}
$$

Define $R = (1-\varepsilon)r$. We will refer to (1) and (2) as needed.

2. Assume for convenience that $(x_0, t_0) = (0, 0)$ (with starting time before zero). Our aim is to apply Brakke's Local Regularity Theorem to B_R. Let us work on $\mu_t(B_R)$ first.

By (i), there is $t_1 \in [-r, -r/2]$ such that

$$
\mu_{t_1}(B_r) \le (1+\delta)\omega_{n-1}r^{n-1}.
$$

Let $\phi \in C_c^2(B_r, \mathbf{R}^+)$ be a function such that $\phi \equiv 1$ on B_R, $0 \le \phi \le 1$ and $|\nabla\phi| \le 2/(\varepsilon r)$. Then for all $t \in [-r/2, r]$,

$$
\begin{aligned}
\mu_t(B_R) &\le \mu_t(\phi) \\
&\le (1+\delta)\omega_{n-1}r^{n-1} + \max|\nabla\phi| \int_{-r}^{r}\int_{B_r} |\vec{H}|\, d\mu_t dt \\
&\le (1+\delta)\omega_{n-1}r^{n-1} + \frac{2}{\varepsilon r}\delta r^n && \text{by (iv)} \\
&\le \frac{(1+\delta)\omega_{n-1} + 2\delta/\varepsilon}{(1-\varepsilon)^{n-1}} R^{n-1} \\
&< \frac{3}{2}\omega_{n-1}R^{n-1} && \text{by (2).}
\end{aligned} \tag{3}
$$

3. Similarly, starting from (ii) we obtain for all $t \in [-r/2, r]$,

$$
\begin{aligned}
\mu_t(B_{R/2}) &\le \frac{(1+\delta)\omega_{n-1} + 2^{n+1}\delta/\varepsilon}{(1-\varepsilon)^{n-1}}\left(\frac{R}{2}\right)^{n-1} \\
&< \frac{3}{2}\omega_{n-1}\left(\frac{R}{2}\right)^{n-1}.
\end{aligned} \tag{4}
$$

4. Our next goal is to clear out the region not near P. Let $B_{\varepsilon r}(x) \subseteq B_r$ be any ball disjoint from $P_{\delta r}$. By (iii), there is $t_1 \in [-r, -3r/4]$ with

$$\mu_{t_1}(B_{\varepsilon r}) \leq 4\delta r^{n-1}.$$

As above, we find for $t \in [-3r/4, r]$,

$$\begin{aligned} \mu_t(B_{\varepsilon r/2}) &\leq 4\delta r^{n-1} + \max|\nabla\phi| \int_{-r}^{r}\int_{B_r} |\vec{H}|\, d\mu_t dt \\ &\leq 4\delta r^{n-1} + \frac{3}{\varepsilon r}\delta r^n \qquad \text{by (iv)} \\ &\leq \eta_2\left(\frac{\varepsilon r}{2}\right)^{n-1} \qquad \text{by choice of } \delta \text{ in (2).} \end{aligned}$$

By the Clearing-Out Lemma 12.2, we obtain

$$\mu_t(B_{\varepsilon r/4}(x)) = 0 \qquad \text{for } t \in [-3r/4 + c_3\varepsilon^2 r^2/4, r].$$

Considering all such balls, since $R = (1-\varepsilon)r$, $\delta < \varepsilon$, $r \leq r_1$, and $c_3\varepsilon^2 r_1/4 < 1/4$ by (1), we obtain

$$\mu_t(B_R \setminus P_{2\varepsilon r}) = 0 \qquad \text{for } t \in [-r/2, r]. \tag{5}$$

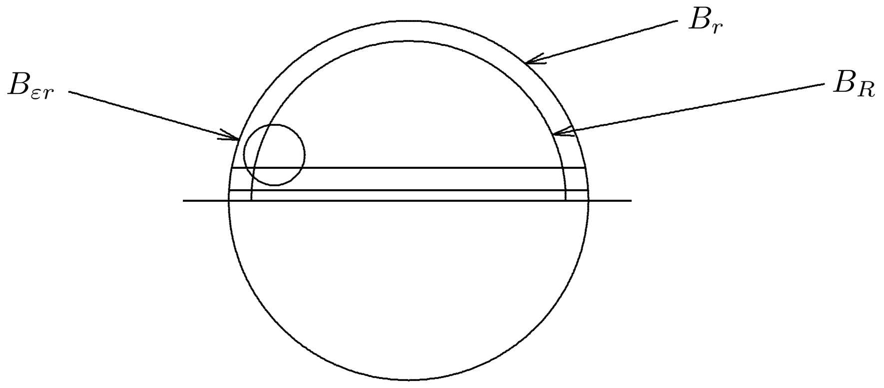

Figure 13. Picture of $B_R \setminus P_{2\varepsilon r}$.

5. Then for $t \in [-r/2, r]$,

$$\begin{aligned} \int_{B_R} |\text{dist}(x,P)|^2 d\mu_t &\leq (2\varepsilon r)^2 \mu_t(B_R) \qquad \text{by (5)} \\ &= \frac{6\varepsilon^2}{(1-\varepsilon)^2}\omega_{n-1}R^{n+1} \qquad \text{by (3)} \\ &\leq \eta_0 R^{n+1} \qquad \text{by choice of } \varepsilon \text{ in (2).} \end{aligned} \tag{6}$$

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6. Consider the open $(n-1)$-disk, and cylinder

$$D \equiv \operatorname{int} B^{n-1}_{(1-\sqrt{\varepsilon})R/2}(0) \subseteq P, \qquad C \equiv D \times [-3\varepsilon r, 3\varepsilon r] \subseteq \mathbf{R}^n.$$

Note that the condition

$$\left(\frac{R(1-\sqrt{\varepsilon})}{2}\right)^2 + \left(\frac{3\varepsilon R}{1-\varepsilon}\right)^2 \leq \left(\frac{R}{2}\right)^2$$

from (1) guarantees that $C \subseteq B_{R/2}$.

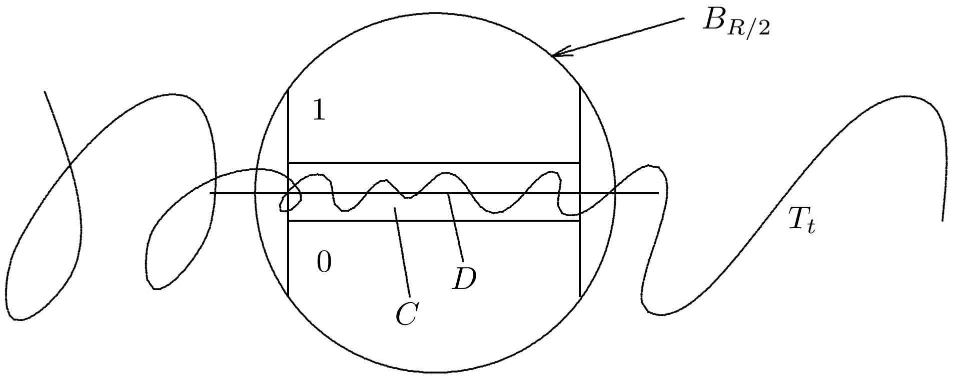

Figure 14. Picture of $T_t \lfloor C$.

It follows from (5) by 11.6(iii) and $\mu_{T_t} = \mu_t$ that

$$\text{(7)} \qquad \operatorname{spt} T_t \cap B_R \subseteq P_{2\varepsilon r}, \qquad \operatorname{spt} \partial[E] \cap (B_R \times [-r/2, r]) \subseteq P_{2\varepsilon r} \times [-r/2, r].$$

Hence $\operatorname{spt} \partial(T_t \lfloor C) \subseteq \partial D \times [-2\varepsilon r, 2\varepsilon r]$, so letting $\pi : \overline{C} \to \overline{D}$ be the projection map,

$$\partial(\pi_\#(T_t \lfloor C)) \lfloor D = 0.$$

By the Constancy Theorem [S, 26.27], $\pi_\#(T_t \lfloor C)$ is a constant integer multiple of $[D]$. By (7), (v), (vi) and the Constancy Theorem

$$E_t \supseteq B_R \cap P^+_{2\varepsilon r}, \qquad E_t \cap (B_R \cap P^-_{2\varepsilon r}) = \emptyset.$$

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Therefore the Constancy Theorem, the multiple cannot be zero, and we obtain a lower bound for the mass in $B_{R/2}$,

$$
\begin{aligned}
\mu_t(B_{R/2}) &\geq \mathbf{M}[T_t \lfloor C] \\
&\geq \mathbf{M}[\pi_\#(T_t \lfloor C)] \\
&\geq \mathbf{M}\,[[D]] && \text{by the above} \\
&= (1-\sqrt{\varepsilon})^{n-1}\omega_{n-1}\left(\frac{R}{2}\right)^{n-1} \\
&\geq \frac{1}{2}\omega_{n-1}\left(\frac{R}{2}\right)^{n-1} && \text{by (1)}
\end{aligned}
\tag{8}
$$

for all $t \in [-r/2, r]$.

8. We now apply Brakke's Local Regularity Theorem 12.1, using (3), (4), (6), (8), to conclude that

$$\Big(\bigcup \operatorname{spt}\mu_t \times \{t\}\Big) \bigcap \Big(B_{R/2} \times [-\frac{r}{2} + c_1 R^2, r]\Big)$$

is a smooth manifold. Note that $c_1 R^2 \leq c_1 r^2 \leq r/2$ by virtue of $r \leq r_1$, and $r/2 < R$. Using 11.6(iii) we find that

$$\operatorname{spt} T \bigcap \big(B_{r/4} \times [-\frac{r}{4}, r]\big)$$

is a smooth manifold. Q.E.D.

Now we present the main results of this section.

12.9 Almost Everywhere Regularity Theorem. Let T be a boundary motion. Then $\mathcal{H}^n(\operatorname{sing}(\operatorname{spt} T)) = 0$. □

Together with 11.4 this implies

12.10 Almost Everywhere Regularity for Nonfattening Level-Set. If $\{\Gamma_t\}_{t \geq 0}$ is a level-set motion that does not fatten up, then $\overline{\partial^*\{u > 0\}}$ is smooth $\mathcal{H}^n$-almost everywhere. □

However, it has not been proven that $\{u = 0\}$ is the closure of the reduced boundary of $\{u > 0\}$ or even its topological boundary. This seems tricky; see Appendix. On the other hand, the theory of slicing and the Clearing-Out Lemma of [ES3] for level-sets allows us to prove the following. We write Z_+ for $Z \cap (\mathbf{R}^n \times [0, \infty))$.

12.11 Almost Everywhere Regularity of Almost Every Level-Set. Let f be a Lipschitz function on $\mathbf{R}^n$ with compact level-sets, and let $u : \mathbf{R}^n \times [0, \infty) \to \mathbf{R}$ be the corresponding solution of equation §10($\star$). Then for a.e. $a \in \mathbf{R}$,

(i) $\{u = a\}_+ = (\overline{\partial^* \{u > a\}})_+$ and $\{u = a\}_+ = (\partial^* \{u > a\})_+$ $\mathcal{H}^n$-a.e.

(ii) $\mathcal{H}^n(\mathrm{sing}(\{u = a\})) = 0$.

(iii) For a.e. $t \geq 0$, $\mathcal{H}^{n-1}(\mathrm{sing}(\Gamma_t^a)) = 0$. □

Proof of 12.9. 1. Let X be the set of points $(x, t) \in \operatorname{spt} T$ such that

(a) $(x, t) \in \partial^* E$.

(b) $T_{(x,t)} T$ is not parallel to $\mathbf{R}^n \times \{0\}$.

(c) These two quantities exist and are equal:

$$\lim_{r \downarrow 0} \frac{\displaystyle\iint_{B_r^{n+1}(x,t)} |\vec{H}| \, d\mu_{t'} \, dt'}{\displaystyle\iint_{B_r^{n+1}(x,t)} d\mu_{t'} \, dt'} = |\vec{H}(x, t)|.$$

By DeGiorgi's theorem and the hypothesis that T is a boundary motion, we see that (a) holds μ_T-a.e. By 11.6(i), statement (b) holds μ_T-a.e. Lemma 12.3 says that (c) holds μ_T-a.e. We have seen in the rectifiability lemma 12.7 that $\mu_T = \mathcal{H}^n \lfloor \operatorname{spt} T$. Hence

$$X = \operatorname{spt} T \qquad \mathcal{H}^n\text{-a.e.}$$

Therefore it suffices to show that for $\mathcal{H}^n$-a.e. $(x, t) \in X$, there is a neighborhood $U \ni (x, t)$ such that $\operatorname{spt} T \cap U$ is smooth.

2. Define

$$X_A = \{(x, t) \in X \ : \ |D\tau(x, t)| \geq \frac{1}{A} \text{ and } |\vec{H}(x, t)| \leq A\},$$

where $D = D^T$, $\tau =$ time. Then $X = \bigcup_{A > 0} X_A$. Define the parabolic blowup

$$\rho_\lambda(x, t) = (\frac{x}{\lambda}, \frac{t}{\lambda^2}) = (y, s), \qquad T^\lambda = (\rho_\lambda)_\#(T), \qquad X^\lambda = \rho_\lambda(X).$$

Then T^λ is also a boundary motion.

Let us check how properties (a)-(c) transform under ρ_λ. For all $(x,t) \in X_A$,

$$|\vec{H}_{T^\lambda_s}(y,s)| = \lambda|\vec{H}_{T_t}(x,t)| \leq \lambda A.$$

Applying the Lebesgue-Besicovitch differentiation theorem to X^λ (as in 12.3), for μ_T-a.e. $(x,t) \in X_A$,

$$\lim_{r\downarrow 0} \frac{\displaystyle\iint_{B_r^{n+1}(y,s)} |\vec{H}_{T^\lambda_{s'}}|\, d\mu_{s'} ds'}{\displaystyle\iint_{B_r^{n+1}(y,s)} d\mu_{s'} ds'} = |\vec{H}_{T^\lambda_s}(y,s)| \leq \lambda A. \tag{1}$$

Conceivably $T_{(y,s)}T^\lambda$ does not exist, even where $T_{(x,s)}T$ does exist. But by the rectifiability of T, for μ_T-a.e. $(x,t) \in X_A$, $T_{(y,s)}T^\lambda$ does exist and equals $(\rho_\lambda)_\#(T_{(x,t)}T)$. Then for μ_T-a.e. $(x,t) \in X_A$,

$$\begin{aligned} |D^{T^\lambda}\tau(y,s)| &= \frac{|D^T\tau(x,t)|}{\sqrt{|D^T\tau(x,t)|^2 + \lambda^2(1 - |D^T\tau(x,t)|^2)}} \\ &\geq \frac{1}{\sqrt{1+\lambda^2(A^2-1)}}. \end{aligned} \tag{2}$$

Let δ_0 be as in 12.8. If we select $\lambda_0 = \lambda_0(A, \delta_0)$ sufficiently small, then (1) and (2) imply that for $\lambda \leq \lambda_0$ and μ_T-a.e. $(x,t) \in X_A$,

$$|D^{T^\lambda}\tau(y,s)| \geq 1 - \frac{\delta_0^2}{4}, \tag{3}$$

$$\lim_{r\downarrow 0} \frac{\displaystyle\iint_{B_r^{n+1}(y,s)} |\vec{H}_{T^\lambda_{s'}}|\, d\mu_{s'} ds'}{\displaystyle\iint_{B_r^{n+1}(y,s)} d\mu_{s'} ds'} \leq \frac{\delta_0}{4(\sqrt{2})^n\omega_n}. \tag{4}$$

Furthermore, for μ_T-a.e. $(x,t) \in X_A$, properties (a), (b), and (c) remain valid for T^λ at (y,s).

3. We wish to apply the Local Regularity Lemma 12.8 at a point (y,s) satisfying the properties above. For simplicity assume $(y,s) = (0,0)$ (with negative starting time). Define $Q = T_{(0,0)}T^\lambda$, $P = Q \cap (\mathbf{R}^n \times \{0\})$. Inequality (3) implies that

$$Q \cap (B_1 \times [-1,1]) \subseteq P_{\delta_0} \times [-1,1] \tag{5}$$

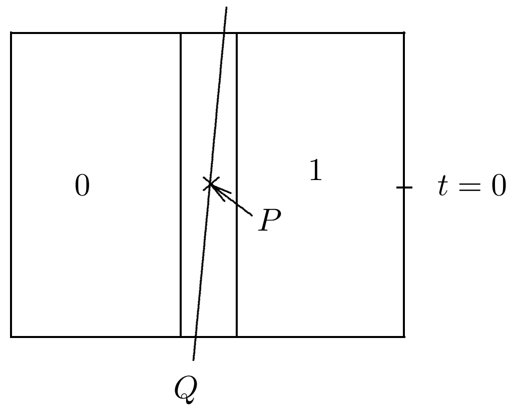

Figure 15. Parabolically Blown-up Tangent Plane.

According to (a) for T^λ at (0,0), as $r \downarrow 0$ the homothetic blowup $(\mu_{T^\lambda})_{(0,0),r}$ converges as a Radon measure to $\mathcal{H}^n \lfloor Q$. Let ψ be any test function in $C_c^0(B_1 \times [-1,1], \mathbf{R}^+)$. For small enough r, we have

$$\begin{aligned}
\iint \frac{1}{r^n}\psi\left(\frac{y}{r},\frac{s}{r}\right) d\mu_s^\lambda(y)\,ds &= \int \frac{1}{r^n}\psi\left(\frac{y}{r},\frac{s}{r}\right) |D^{T^\lambda}\tau(y,s)|\,d\mu_T(y,s) \quad \text{by co-area}\\
&\le \int \frac{1}{r^n}\psi\left(\frac{y}{r},\frac{s}{r}\right) d\mu_T(y,s)\\
&\approx \int_Q \psi\, d\mathcal{H}^n\\
&= \frac{1}{|D^Q\tau|}\int\int_{Q_{\tilde{s}}} \psi\, d\mathcal{H}^{n-1} d\tilde{s}\\
&\qquad \text{where } Q_{\tilde{s}} = Q \cap (\mathbf{R}^n \times \{\tilde{s}\})\\
&\le \frac{1}{1-\delta_0^2/4}\int\int_{Q_{\tilde{s}}} \psi\, d\mathcal{H}^{n-1} d\tilde{s} \qquad \text{by (3)}\\
&\le \left(1+\frac{\delta_0^2}{2}\right)\int\int_{Q_{\tilde{s}}} \psi\, d\mathcal{H}^{n-1} d\tilde{s}.
\end{aligned}$$

Take $\psi \in C_c^0(B_2 \times [-2,2])$ to approximate the characteristic function $\chi_{B_1\times[-1,-1/2]}$ from above. Then by the preceding,

$$\begin{aligned}
\frac{1}{r^n}\int_{-r}^{-r/2} \mu_s^\lambda(B_r)\,ds &\le \frac{1}{r^n}\iint \psi\left(\frac{y}{r},\frac{s}{r}\right) d\mu_s^\lambda(y)\,ds\\
&\le \left(1+\frac{\delta_0^2}{2}\right)\iint \psi\, d\mathcal{H}^{n-1} ds + o(1) \qquad (\text{as } r \downarrow 0)\\
&\le \frac{1}{2}(1+\delta_0)\omega_{n-1},
\end{aligned}$$

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for a good enough approximation and small enough r. This is hypothesis 12.8(i) of the Local Regularity Theorem. Similarly, by taking

$$\psi \approx \chi_{B_{1/2}\times[-1,-1/2]}, \qquad \psi \approx \chi_{(B_1\setminus P_{\delta_0})\times[-1,1]},$$

we obtain hypotheses 12.8(ii), (iii), for small enough r. Also, (5) above, together with statement (b) for T^λ, implies 12.8(v), (vi).

4. Now let us work on the remaining hypothesis, 12.8(iv). Using (4), we have

$$\begin{aligned}\int_{-r}^{r}\int_{B_r}|\vec{H}|\,d\mu_s^\lambda ds &\le \frac{\delta_0}{4(\sqrt{2})^n\omega_n}\iint_{B^{n+1}_{\sqrt{2}r}(0,0)} d\mu_s^\lambda ds && \text{by (4)}\\ &\le \frac{\delta_0}{4(\sqrt{2})^n\omega_n}\mu_{T^\lambda}(B^{n+1}_{\sqrt{2}r}(0,0)) && \text{by co-area}\\ &\le \frac{\delta_0}{4(\sqrt{2})^n\omega_n}2\omega_n(\sqrt{2}r)^n && \text{by (3)}\\ &= \delta_0 r^n,\end{aligned}$$

which is hypothesis 12.8(iv).

5. It follows that (0,0) is a smooth point. Since (0,0) represented $\rho_\lambda(x,t)$ for μ_T-a.e. $(x,t)\in X_A$ and $X=\bigcup_{A>0}X_A$, it follows that μ_T-a.e. $(x,t)\in X$ is smooth. Therefore $\mathcal{H}^n$-a.e. $(x,t)\in \operatorname{spt} T$ is smooth, proving 12.9. Q.E.D.

Proof of 12.11.

(i) Note that $\partial^*\{u>a\}_+\subseteq\partial\{u>a\}_+\subseteq\{u=a\}$. According to [ES1], $\operatorname{Lip} u(\cdot,\cdot)\le \operatorname{Lip} u(\cdot,0)<\infty$. Applying 1.21 with $U=\mathbf{R}^n\times(0,\infty)$ we obtain for a.e. $a\in\mathbf{R}$,

$$\tau(\{u=a\}_+,1,\xi_a)=\tau((\partial^*\{u>a\})_+,1,\xi_{\partial^*\{u>a\}_+}).$$

It follows that for a.e. $a\in\mathbf{R}$,

$$\mathcal{H}^n(\{u=a\}_+\setminus\partial^*\{u>a\}_+)=0. \tag{1}$$

Next we show

$$\{u=a\}_+=(\overline{\partial^*\{u>a\}})_+.$$

Suppose $(x,t)\in\{u=a\}_+$, so $x\in\Gamma_t^a$, $t>0$. According to the Clearing-Out Lemma of Evans-Spruck [ES3, 3.1] for the level-set flow, there exist constants $\beta>\alpha>0$, $\eta>0$ such that for all $r>0$,

$$\mathcal{H}^{n-1}(\Gamma_s^a\cap B_r(x))>\eta r^{n-1} \qquad \text{for } s\in[t-\beta r^2,t-\alpha r^2]\cap[0,\infty),$$

where $B_r = B_r^n$. Then, provided $0 < r^2 \le t/\beta$,

$$\mathcal{H}^n(\{u=a\} \cap (B_r^{n+1}(x) \times [t-\beta r^2, t-\alpha r^2])) \ge \int_{t-\beta r^2}^{t-\alpha r^2} \mathcal{H}^{n-1}(\Gamma_s^a \cap B_r(x))\, ds$$
$$\ge \eta r^{n-1}(\beta-\alpha) r^2$$

from which it follows by (1) that

$$\partial^*\{u>a\} \cap (B_r(x) \times [t-\beta r^2, t-\alpha r^2]) \ne \emptyset.$$

for arbitrarily small r. We conclude

$$(x,t) \in (\overline{\partial^*\{u>a\}})_+$$

for $(x,t) \in \{u=a\}_+$. Along with the first remark this shows $(\overline{\partial^*\{u>a\}})_+ = \{u=a\}_+$.

(ii) For all but countably many level-sets, $\mathcal{L}^{n+1}(\{u=a\}) = \mathcal{H}^n(\{f=a\}) = 0$. Therefore (ii) follows immediately from (i) and 12.10.

(iii) follows trivially from (ii). Q.E.D.

Appendix

Here are some open problems. I make no claim to originality here; it would not even be possible to formulate these questions without the foundational work of Federer, Almgren, Allard, Brakke, Simon, as well as Gage, Hamilton, Grayson, Huisken, Angenent and others. These problems are quite hard, and serve as distant, blinding lights.

Brakke Motion and Level-Set Flow

A. Better Partial Regularity. Is it possible to improve the result of §12 somewhat, say to $\mathcal{H}^{n-1}(\operatorname{sing}(\operatorname{spt} T)) = 0$ for all $t \geq 0$?

B. Motion of Edge. (Evans-Spruck) If a level-set fattens, is there a unit density boundary motion on $\partial\{u(\cdot, t) > 0\}$?

C. Equality in Brakke's Inequality. For the unit density boundary motion of §11, show that Brakke's inequality is actually an equality (for almost every time).

D. No Mass Drop. For the unit density boundary motion of §11, show $\mathbf{M}[\mu_t]$ is continuous in time. (This would imply $\mathcal{H}^{n-1}$-a.e. regularity for all times; see the final theorem of [B].)

E. Unit Density. (A very hard and old problem.) Can we show unit density for the motion of boundaries of sets? I.e.: If $(T, \{\mu_t\}_{t\geq 0})$ is an enhanced pair, with T_0 the boundary of an open set, is T a boundary motion?

Originally I thought that unit density might be preserved for any Brakke motion. The following simple example, the sum of two homothetic spoons, shows that this is false:

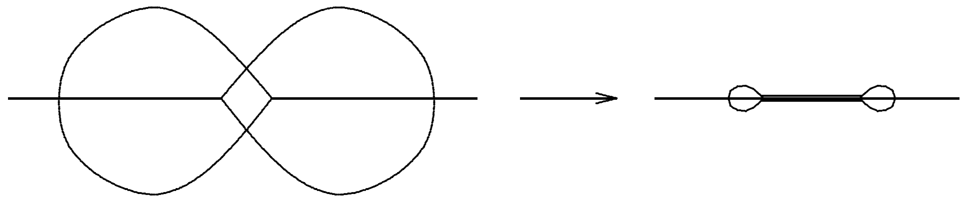

Figure 16. Double Density Develops.

(D) and (E) are related, because mass loss for the undercurrent should occur at the same time that two pieces of the varifold double up.

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The following example suggests: For boundaries, multiple density can form in infinite time but not in finite time.

Example. (with Evans) There is a surface in $\mathbf{R}^3$ which looks like two parallel planes attached by a tube, that evolves to form a double density plane in infinite time.

F. Nonfattening. (Evans-Spruck, DeGiorgi [D]) If a level-set fattens, it fattens at $t = 0$. (In particular, a smooth set never fattens, even past singularities.)

G. Criterion for Nonfattening. When does a set X fatten? This is especially interesting if X is the support of a stationary varifold.

Example. ([I5]) A minimizing current (of finite mass) never fattens. On the other hand, a stationary, non-minimizing cone fattens, at least if $\operatorname{sing}(\operatorname{spt} T_0) = \{0\}$.

Role of minimization.

Minimizing currents are much smoother that merely stationary ones. Brakke's regularity theory is of the "stationary" character, that is, it only uses the Euler-Lagrange equation. Normally nothing further is possible for parabolic equations, since

(a) There is no minimization principle.

(b) Stationary objects for the elliptic equation solve the parabolic equation.

On the other hand, in certain low-dimensional examples of nonunique evolution, some of the flows decrease surface area faster and are more regular. These seem to be the ones that arise as limits of the minimizers of the elliptic regularization scheme.

The simplest example of this phenomenon is the cross in $\mathbf{R}^2$ (see the figure in 6.5). More generally, the boundary of the fattening stationary cone of [I5] is smooth after time $t = 0$.

H. Characterization of Limit. Characterize these limits of minimizers intrinsically.

I. Higher Regularity. Show that limits of minimizers for the mean curvature flow have very high regularity, analogous to the codimension seven regularity for the elliptic minimizers.

One can ask the same question for the time-step minimization method introduced in Luckhaus [Lu], Almgren-Wang [AW] and Almgren-Taylor-Wang [ATW].

Blowups.

It is possible to pass back and forth between translative and homothetic solitons by means of blowup and blowdown, along the lines of the elliptic regularization construction.

J. Correspondence. For what class of homothetic and translative solitons is this a one-to-one correspondence?

It is possible to blow up any Brakke motion parabolically at any point by Huisken's method to obtain a (possibly nonsmooth) homothetic soliton. If the Brakke motion was the limit of minimizers, the parabolic blowup can be obtained directly, via a diagonal subsequence, as the blowdown of a sequence of minimizing translative solitons P^i (with boundary).

K. Provenance of the Blowup. Can the parabolic blowup of a Brakke motion that is the limit of minimizers also be obtained by blowing down a single complete, minimizing translative soliton?

If true, this would give a general tool for studying the asymptotic behavior of singularities.

Positive Mean Curvature.

We conjecture

L. Classification of Positive Translative Solitons. A complete, minimizing translative soliton $T \subseteq \mathbf{R}^{n+1}$ $(n \leq 7)$ of positive mean curvature is either

(a) $\Sigma^k \times \mathbf{R}^{n-k-1}$ (possibly tilted) where Σ^k is a rotationally symmetric translative soliton in $\mathbf{R}^{k+1}$, $k \geq 2$, or

(b) $\Gamma \times \mathbf{R}^{n-2}$ (possibly tilted), $\Gamma =$ grim reaper.

If we allow nonnegative mean curvature, we also get planes.

M. Positive Curvature Blowups. Suppose the motion has positive mean curvature. Then only a blowdown of (a) above can occur as the blowup of a motion that is obtained as the limit of minimizers. That is, the only blowups of singularities are the sphere and cylinders.

Huisken has already proven this conjecture under the hypothesis that the limit is a smooth surface.

Generic Statements

N. Genericity of Nonnegative Mean Curvature. (Huisken) Show for all but countably many level sets, each blowup has curvature of one sign.

This can be proven by a topological argument under the strong additional hypothesis that each flow is smooth on a suitable small neighborhood leading up to the critical point.

With (M), this would show that for the mean curvature flow, most blowups are spheres or cylinders, up to $n = 7$. Neckpinches suggest that there will be a point singularity, but this is not quite true, as shown by the following example.

Marriage-Ring Example. A thin, rotationally symmetric torus in $\mathbf{R}^3$ will shrink to a circle of singularities.

O. Strongest Regularity Conjecture.

(a) For $n \leq 7$, any boundary motion in $\mathbf{R}^n$ can be perturbed slightly to a motion with a finite number of singularities in space-time (past a given time).

(b) For $n > 7$, any boundary motion can be perturbed to a motion with codimension seven singularities except for a finite number of transition times (past a given time).

Dimension Three

P. Singularities. In $\mathbf{R}^3$, show that any singularity of a boundary motion that arises a the limit of minimizers is either

(a) a point singularity, or

(b) topologically equivalent to the marriage-ring example.

This includes the conjecture that a singularity with cylinder as blowup is topologically a neckpinch, cusp, or peanut, unless it is the marriage-ring. The conjecture is due to Grayson, Hamilton, and Huisken. In particular, it would show that there are only a finite number of singularities (either a point or a marriage-ring) past a given time. The following speculation would strengthen this slightly.

Q. Finitude of Spheres. In $\mathbf{R}^3$, a shrinking sphere occurs as the parabolic blowup at only a finite number of points.

Using such results, it would be possible to estimate the number of singularities in space-time using Euler characteristic. See Ecker [Ec], Toro [T] for interesting work.

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