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Risk Theory and Heavy-Tailed Lévy Processes

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to my parents

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Contents

Abstract

The subject of risk theory is concerned with random fluctuations for the financial (liability) results of an insurance company. As a measure of risk we consider the ruin probability of a classical risk process perturbed by an α–stable Lévy motion with index $1 < \alpha < 2$ and skewness parameter $\beta = -1$. The restriction on the parameters implies that the process under consideration is a spectrally negative Lévy process with finite mean function. Using general theory, the Laplace transform of the hitting time can be expressed in terms of the logarithm of the characteristic function of the process. Inversion of the Laplace transform yields an explicit convolution formula for the infinite time ruin probability. This formula based on the Mittag–Leffler function generalizes the Pollaczek–Khintchine formula. We investigate the asymptotic behaviour of the ruin probability as the initial risk reserve becomes large. In doing so we have to take into account how the integrated tail distribution of the claim sizes is related to the stable component with respect to the tail decay.

Assuming that the claim size distribution of a risk process has infinite variance (but finite mean) we use a functional central limit theorem to show that an α–stable Lévy motion ($1 < \alpha < 2$) with linear drift can be constructed as an approximation to the risk process. We show that the ruin probabilities in finite time as well as in infinite time converge. For the latter convergence we rely on results from queuing theory. Our main result says that the equilibrium waiting time in a $GI/G/1$ queuing system has approximately Mittag–Leffler distribution. Numerical examples give an idea of the quality of the approximations.

Kurzfassung

Gegenstand der Risikotheorie sind die zufälligen Schwankungen der Kapitalreserven einer Versicherungsgesellschaft. Als Risikomass betrachten wir die Ruinwahrscheinlichkeit eines klassischen Risikoprozesses, welcher durch eine α–stabile Bewegung mit Index $1 < \alpha < 2$ und Schiefeparameter $\beta = -1$ gestört wird. Die Wahl der Parameter impliziert, dass der betrachtete Prozess ein stochastischer Prozess mit endlicher Erwartungswertfunktion und unabhängigen und stationären Zuwächsen ist, der überdies nur Sprünge nach unten aufweist. Die allgemeine Theorie besagt, dass für solche Prozesse die Laplace Transformierte der Eintrittswahrscheinlichkeiten durch den Logarithmus der charakteristischen Funktion des Prozesses ausgedrückt werden kann. Die Inversion der Laplace Transformierten liefert eine explizite Formel für die Ruinwahrscheinlichkeit in unendlicher Zeit. Diese Formel verallgemeinert die Pollaczek–Khintchine Formel und enthält die Mittag–Leffler Funktion. Wir untersuchen das asymptotische Verhalten der Ruinwahrscheinlichkeit für grosses Anfangskapital. Dabei unterscheiden wir, in welcher Relation die integrierte Schadenverteilung zur Verteilung der stabilen Komponente bezüglich des langreichweitigen Verhaltens steht.

Unter der Annahme, dass die Schadenverteilung eines Risikoprozesses unendliche Varianz (aber endlichen Erwartungswert) hat, zeigen wir mit Hilfe eines funktionalen Zentralen Grenzwertsatzes, dass eine α–stabile Bewegung mit linearem Drift als Approximation des Risikoprozesses dienen kann. Wir zeigen, dass die Ruinwahrscheinlichkeiten sowohl in endlicher wie auch in unendlicher Zeit konvergieren. Für die letztere Konvergenz machen wir einen Exkurs in die Theorie der Warteschlangen. Das Hauptresultat besagt, dass die Verteilungen der Wartezeiten in einer Folge von Einbedienersystemen schwach gegen die Mittag–Leffler Verteilung konvergieren. Numerische Beispiele geben Aufschluss über die Qualität der Approximation.

Chapter 1

Introduction

Collective risk theory has developed over a long period of time but mainly has its roots at the beginning of the 20th century. It was mostly inaugurated by Filip Lundberg who introduced in his thesis [57] the collective risk model for insurance claim data. Harald Cramér [18] then incorporated Lundberg's ideas and established the relationship between risk theory and general stochastic processes. In recent years the basic probabilistic models originating from the Cramér–Lundberg setup have been generalized in various directions. Martingale theory for instance leads to more "realistic" models allowing for economic factors such as inflation, interest etc. to be introduced. The observed increase over the recent past in both frequency as well as size of catastrophic claims (natural as well as man–made) has triggered a surge in research into the modelling of such events. From a mathematical point of view, the keywords in this context are *extremal events, heavy–tailed distributions,* (*divergent*) *moments, hitting times,* etc. Motivated by the growing need for modelling extremal events we consider so–called *α–stable processes* in the area of risk theory. An α–stable process is a random element whose finite–dimensional distributions are α–stable. Stable distributions share many properties with the Gaussian law. For instance, stable distributions play an important rôle in the context of the central limit theorem: only stable laws appear as the weak limit of normalized sums of independent and identically distributed (iid) random variables making them useful in modelling the contribution of many small random effects. For instance, one can assume that a random variable describing the daily price change of a stock is the sum of a great many smaller fluctuations, most of them approximately independent and identically distributed. Thus the original variable would tend to have a stable distribution. Despite this close kinship with the normal law, the family of stable distributions has attracted only a moderate amount of interest for a long time. The main reason for this undoubtedly is the complicated structure of the marginal densities (apart from a few cases) and the coupled difficulties in statistical estimation. However, the situation changed in the 1960's after the appearance of a series of papers by Mandelbrot who sketched the use of stable laws to model data suspected to have heavy–tailed distributions, especially certain economic variables such as stock price changes. Stable distributions seem to be an appropriate tool whenever standard "light–tailed" conditions are violated. The tail–behaviour of non–normal stable laws differs from the one of the Gaussian distribution in a significant way. The tails of non–Gaussian stable distributions decay like a power function. This implies that a stable random variable has much more variability than a normal one; it is much more likely to take values far away from the median. Stable laws have been used to model various phenomena such as gravitational fields of stars, temperature distributions in nuclear reactors and annual rainfall. The rôle of stable laws is also fundamental in such areas as sociology, biology, insurance risk and more recently in mathematical finance, see for instance Embrechts, Klüppelberg and Mikosch [31] or Mittnik and Rachev [59], [60] and references therein.

The outline of this thesis is as follows. The purpose of Section 2.1 in the chapter of preliminaries is to present a concise survey of the theory of Lévy processes. Lévy processes can be thought of as continuous time analogues of random walks. Historically, the theory goes back to the late 20's and mainly originated from Lévy and Khintchine. An interesting class of Lévy processes are the so–called spectrally negative Lévy processes. Because of the absence of positive jumps they were extensively studied in applied probability as models for queuing, insurance risk and dam theory. They are also of interest from a theoretical point of view because they are the processes for which fluctuation theory can be developed in detail. A classical risk process defined in Section 2.2 belongs to the class of spectrally negative Lévy processes. We give a brief review of the classical risk theory. The traditional Cramér–Lundberg estimate for "small" claims yields bounds for the ruin probability which are of exponential type. However, insurance data often involve data that makes the assumption of "small" claims not very reasonable. We conclude that section with the analysis of heavy–tailed insurance risk models. In Section 2.3 we present some basic properties of stable distributions. Special emphasis is given to the tail behaviour and to the fact that stable laws arise as weak limits of partial sums.

In Chapter 3 we extend the classical model of collective risk theory by adding an α–stable Lévy motion. The stable component can be viewed as describing perturbations around the underlying risk process. Contrary to a Brownian motion a non–Gaussian stable Lévy motion exhibits jumps and hence any hitting time problem (like ruin estimation) becomes more difficult. A suitable choice of the parameters guarantees that the extended risk process still belongs to the class of spectrally negative Lévy processes. From a modelling point of view, the additional downward jumps can be interpreted as certain extra random payments. By means of Laplace inversion we establish an explicit formula for the infinite–time ruin probability. In that formula the perturbation is represented by the Mittag–Leffler function. Section 3.2 is devoted to the asymptotic behaviour of the ruin probability as the initial risk reserve becomes large.

Weak approximations are studied in Chapter 4. We give a parallel to the classical Brownian diffusion approximation for risk processes. Our approach is especially relevant whenever the claim size distribution has infinite variance. The resulting limiting process is an α–stable Lévy motion with drift. We then show that the finite–time ruin probabilities as well as the ruin probabilities with infinite time horizon converge. For the latter proof of convergence we make an excursion into queuing theory. We present some numerical examples in order to investigate the quality of our approximations.

Chapter 2

Preliminaries

2.1 Lévy Processes

Our aim in this section is to state some facts about Lévy processes which will be used in the sequel. Particular emphasis is given to the subclass of spectrally negative Lévy processes. Here a much greater degree of explicitness is available in the results than in the general case, while at the same time it is this case which is important in applications, for instance in the theory of queues, dams, insurance risk etc. The material covered in this section is classical and thus the treatment is expository and only few proofs are given. We have used the papers by Bingham [12], Gusak and Korolyuk [45] and Shtatland [73]. A recent survey of the theory of Lévy processes can also be found in Bertoin [8].

Throughout this thesis let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a probability space on which all stochastic quantities are defined. We assume that all P–null sets of $\mathcal{F}$ are added to $\mathcal{F}_0$. Our starting point is the concept of infinite divisibility of a probability distribution or of a random variable. For a distribution function F on $[0, \infty)$ we define the n–fold convolution F^{*n}

as

$$F^{*n}(x) = \int_0^x F^{*(n-1)}(x-u)\,dF(u)\ , \qquad n \geq 1\ ,$$

where $F^{*0}(x) = 1$ for $x \geq 0$ and 0 elsewhere.

Definition 2.1 *A probability distribution F or a real–valued random variable X is said to be infinitely divisible, if for each $n \in \mathbb{N}$ there is a probability distribution F_n such that $F = F_n^{*n}$ or equivalently $X \stackrel{d}{=} X_1 + \cdots + X_n$, where the X_i have distribution F_n, $i = 1, \ldots, n$.*

Here "$\stackrel{d}{=}$" means equality in law. Assume now that Y is an infinitely divisible random variable. Then there is a unique continuous function ξ such that $\xi(0) = 0$ and

$$E\left[e^{sY}\right] = e^{\xi(s)}\ , \qquad \mathrm{Re}\{s\} = 0\,.$$

According to the Lévy–Khintchine formula, a function ξ is the logarithm of the characteristic function (chf) of an infinitely divisible distribution on the real line if and only if it may be written in the form

$$\xi(s) = cs + \frac{\sigma_0^2 s^2}{2} + \int_{-\infty}^{\infty} \left(e^{sx} - 1 - \frac{sx}{1+x^2}\right) \Pi(dx)\ , \qquad (2.1)$$

where $\mathrm{Re}\{s\} = 0$, $c \in \mathbb{R}$ and $\sigma_0 \geq 0$. Here Π is a measure on $\mathbb{R}\backslash\{0\}$ which satisfies

$$\int_{-\infty}^{\infty} (1 \wedge x^2)\,\Pi(dx) < \infty\,, \qquad (2.2)$$

where $a \wedge b = \min(a,b)$. The measure Π is called the Lévy measure. It is then possible to construct a strong Markov process $X = (X(t) : t \geq 0)$ with stationary independent increments such that $X(0) = 0$ and

$$E\left[e^{sX(t)}\right] = e^{t\xi(s)}\ . \qquad (2.3)$$

We shall call such a process a Lévy process. The quantity $\xi(s)$ is called the characteristic exponent. The process X is càdlàg, i.e. the sample path functions belong to the space $\mathbb{D} = \mathbb{D}[0,\infty)$ of real–valued functions on $[0,\infty)$, right–continuous on $[0,\infty)$ with left limits on $(0,\infty)$. For $\mathbb{D}$, Stone [77] extends the Skorokhod J_1–topology for càdlàg functions on compact intervals under which $x_n \to x$ whenever there exists a sequence of continuous bijections λ_n from $[0,\infty)$ onto itself such that (1) and (2) hold for each $T > 0$:

(1) $\quad \sup\{|\lambda_n(t) - t| : t \in [0,T]\} \to 0\ , \qquad n \to \infty\,.$

(2) $\quad \sup\{|x_n(t) - x(\lambda_n(t))| : t \in [0,T]\} \to 0\ , \qquad n \to \infty\,.$

The space $\mathbb{D}$ endowed with the extended J_1–topology is a complete separable metric space; see Stone [77].

The terms which appear in the Lévy–Khintchine formula (2.1) have a probabilistic interpretation. The constant c corresponds to the drift, σ_0 to the Brownian component and the Lévy measure Π represents the jump component of the process X. For instance, standard Brownian motion is obtained if we choose $c = 0$ and $\Pi = 0$. The Lévy measure of a homogeneous Poisson process with intensity $\lambda > 0$ is given by $\Pi(dx) = \lambda\varepsilon_1(dx)$. Throughout, ε_y denotes the Dirac measure at $y \in \mathbb{R}$. The measure $\Pi(dx) = \lambda F(dx)$ corresponds to the Lévy measure of a compound Poisson process $\sum_{k=1}^{N(t)} Y_k$, where $(Y_k : k \in \mathbb{N})$ is a sequence of iid random variables with common distribution function F. Besides Poisson processes and Brownian motion, stable processes form another important class of Lévy processes. For every $\alpha \in (0,2]$, a Lévy process X with characteristic exponent ξ is called a stable process with index α if the Lévy measure Π is given by

$$\Pi(dx) = \frac{p}{x^{1+\alpha}}\mathbb{I}_{(0,\infty)}(x)\,dx + \frac{q}{|x|^{1+\alpha}}\mathbb{I}_{(-\infty,0)}(x)\,dx\,, \qquad (2.4)$$

where $p+q > 0$ and $\mathbb{I}_A$ denotes the indicator function of the set A. Due to the integrability condition (2.2), one must have $\int_0^1 x^2/x^{1+\alpha}\,dx < \infty$, which explains the restriction on the range of the index α. Stable processes appear in particular in limit theorems and will in this context be studied in Chapter 4. The skewness parameter β of a stable law is defined as $\beta = (p-q)/(p+q)$. If for instance $\beta = -1$, i.e. $p = 0$, the Lévy measure Π in (2.4) attributes no mass to the positive half line and consequently there are no positive jumps.

We next formulate the Wiener–Hopf factorization due to Pecherskii and Rogozin [63]. Let $T = T(\sigma)$ be an exponentially distributed random variable with parameter $\sigma > 0$ independent of the Lévy process X. Elementary calculation shows that $\sigma/(\sigma - \xi(s)) = E[\, e^{sX(T)}\,]$.

Theorem 2.2 *The function $\sigma/(\sigma - \xi(s))$ may be factorised as*

$$\frac{\sigma}{\sigma - \xi(s)} = \Psi_\sigma^-(s)\Psi_\sigma^+(s) \ , \tag{2.5}$$

where

(*i*) *$\Psi_\sigma^-(s)$ is analytic in the half–plane $\mathrm{Re}\{s\} > 0$, continuous and non–vanishing in $\mathrm{Re}\{s\} \geq 0$, and is the Laplace transform of an infinitely divisible probability distribution ζ^- on $(-\infty, 0)$.*

(*ii*) *$\Psi_\sigma^+(s)$ is analytic in the half–plane $\mathrm{Re}\{s\} < 0$, continuous and non–vanishing in $\mathrm{Re}\{s\} \leq 0$, and is the Laplace transform of an infinitely divisible probability distribution ζ^+ on $(0, \infty)$.* □

Such a factorization is unique. It is called the Wiener–Hopf factorization of X. The functions $\Psi_\sigma^-(s)$ and $\Psi_\sigma^+(s)$ are called the left and right Wiener–Hopf factors of X, respectively.

Together with X we consider the processes fX, gX and hX, obtained from X in the following way.

$$fX(x) = \inf\{u \geq 0 \,|\, X(u) > x\} \qquad \text{(first passage functional)}\,, \tag{2.6}$$

$$gX(t) = \sup_{0 \leq s \leq t} X(s) \qquad \text{(supremum functional)}\,, \tag{2.7}$$

$$hX(t) = \inf_{0 \leq s \leq t} X(s) \qquad \text{(infimum functional)}\,. \tag{2.8}$$

Recall that T is exponentially distributed with parameter $\sigma > 0$. The following identities are referred to as Spitzer–Rogozin identities. For a proof, see Bingham [12, Theorem 1] or Gusak and Korolyuk [45].

Theorem 2.3 *Let ζ^-, ζ^+ be defined as in Theorem 2.2. Then for $x > 0$*

$$P[\zeta^- \leq -x] = P[hX(T) \leq -x] = \sigma \int_0^\infty e^{-\sigma t} P[hX(t) \leq -x]\, dt \ ,$$
$$P[\zeta^+ \leq x] = \ P[gX(T) \leq x] \ = \sigma \int_0^\infty e^{-\sigma t} P[gX(t) \leq x]\, dt \ .$$

□

The Wiener–Hopf factorization in principle allows us to express the double Laplace transforms of $\sup_{0 \leq u \leq t} X(u)$ and $\inf_{0 \leq u \leq t} X(u)$ in terms of $\Psi_\sigma^-(s)$, $\Psi_\sigma^+(s)$. However, we emphasise that the Wiener–Hopf factors in general can not be calculated explicitly, except for the class of Lévy processes with no positive (no negative) jumps. In that case fluctuation theory can be developed explicitly to its full extent.

For our purposes, we turn to the case where X is spectrally negative, i.e. the Lévy measure Π gives no mass to $(0, \infty)$. Then X has no positive jumps and has a continuous maximum, that is, the first passage across a positive level takes place continuously, and one has

$$X(fX(x)) = gX(fX(x)) = x \ , \qquad x \geq 0 \, .$$

We start with the important observation that the exponential moments of X are finite although $X(t)$ may take values of both signs, i.e.

$$E\left[e^{\lambda X(t)} \right] < \infty$$

for all $\lambda > 0$, see Bertoin [8, p. 188]. The chf $E[\, e^{sX(t)}\,]$, $\mathrm{Re}\{s\} = 0$, can therefore be extended to define an analytic function in the complex right half–plane $\{z \in \mathbb{C} \,|\, \mathrm{Re}\{z\} \geq 0\}$. For $s \in \mathbb{C}$ with $\mathrm{Re}\{s\} \geq 0$ the quantity $\xi(s)$ in (2.1) is called Laplace exponent.

If we consider the first–passage times across two positive levels x and $x+u$ $(u > 0)$ it follows from the strong Markov property that $fX(x+u) - fX(x)$ is independent of $fX(x)$ and that $fX(x+u) - fX(x) \stackrel{d}{=} fX(u)$. The non–decreasing process fX is thus homogeneous and additive. We write η for the exponent of fX:

$$E\left[e^{-s(fX)(x)} \right] = e^{-x\eta(s)} \ , \qquad s, x \geq 0 \, . \tag{2.9}$$

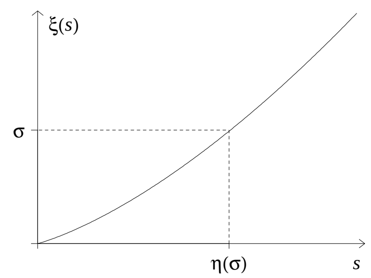

Figure 2.1: *Relation between the exponents ξ and η.*

The following proposition is due to Zolotarev [87] and relates the exponents ξ and η. A nice proof via martingale techniques is to be found in Bingham [12, Proposition 2, p. 721].

Proposition 2.4 *The exponents ξ, η of X, fX are related through*

$$\xi(\eta(\sigma)) = \sigma \ , \qquad \sigma \geq 0 \, . \tag{2.10}$$

□

Proposition 2.5 *Let $T = T(\sigma)$ be an exponential random variable with parameter $\sigma > 0$. Then $gX(T)$ has an exponential distribution with parameter $\eta(\sigma)$.*

Proof. Observe that $\{gX(T) > x\} = \{fX(x) \leq T\}$. We first condition on $\{T = t\}$ and then integrate by parts, yielding

$$\begin{aligned} P[gX(T) > x] &= P[fX(x) \leq T] \\ &= \int_0^\infty \sigma e^{-\sigma t} P[fX(x) \leq t] \, dt \\ &= \int_0^\infty e^{-\sigma t} \, d_t P[fX(x) \leq t] \\ &= E\left[e^{-\sigma (fX)(x)} \right] = e^{-x\eta(\sigma)} \ . \end{aligned}$$

In the last equality we used (2.9). □

Combining Theorem 2.3 and Proposition 2.5 we conclude that ζ^+ is exponentially distributed with parameter $\eta(\sigma)$. Recall from Theorem 2.2 that $\Psi_\sigma^+(s) = E[\,e^{s\zeta^+}\,]$. The representation for $\Psi_\sigma^+(s)$ in the following theorem is thus obvious. The expression for $\Psi_\sigma^-(s)$ derives from (2.5).

Theorem 2.6 *Let X be a spectrally negative Lévy process. Then*

$$\Psi_\sigma^-(s) = \left(\frac{\sigma}{\sigma - \xi(s)}\right)\left(1 - \frac{s}{\eta(\sigma)}\right) , \tag{2.11}$$

$$\Psi_\sigma^+(s) = \frac{\eta(\sigma)}{\eta(\sigma) - s} . \tag{2.12}$$

□

The feature that X moves continuously to the right allows us to determine the distribution of the maximum, see Proposition 2.5. However, in many applications one is merely interested in the distribution of the infimum.

We pause for a moment to introduce some further notation and preliminary results which will enable us to prove the fundamental Proposition 2.7. For $x > 0$, denote by τ_x the first entry time of X into the interval $(-\infty, -x)$, i.e. $\tau_x = f(-X)(x)$ or

$$\tau_x = \inf\{t \geq 0 \,|\, X(t) < -x\} .$$

It was shown by Zolotarev [87] that (2.10) may be re–written in the alternative form

$$P[fX(y) \leq t] = -y\,\frac{\partial}{\partial y}\int_0^t P[X(u) \geq y]\,\frac{du}{u} , \tag{2.13}$$

where $t, y > 0$. Suppose that the Lévy measure Π is absolutely continuous. Then the distribution function of $X(t)$ is absolutely continuous for each $t > 0$; let the density be $f(t, \cdot\,)$. The distribution function of $fX(y)$ is then absolutely continuous for each $y > 0$. Writing $h(\,\cdot\,, y)$ for its density function, we conclude from (2.13) that

$$h(t, y) = \frac{y}{t}\,f(t, y) . \tag{2.14}$$

Define $X^+(t) = X(t)\mathbb{I}_{\{X(t)>0\}}$. By (2.9) and (2.14) we have that

$$
\begin{aligned}
\int_0^\infty e^{-\sigma t}\frac{1}{t}E[\,X^+(t)\,]\,dt &= \int_0^\infty e^{-\sigma t}\frac{1}{t}\left(\int_0^\infty y\,f(t,y)\,dy\right)\,dt \\
&= \int_0^\infty \int_0^\infty e^{-\sigma t}h(t,y)\,dt\,dy \\
&= \int_0^\infty E\left[e^{-\sigma(fX)(y)}\right]\,dy \\
&= \int_0^\infty e^{-y\eta(\sigma)}\,dy = \frac{1}{\eta(\sigma)}\ .
\end{aligned}
\tag{2.15}
$$

The following proposition is crucial.

Proposition 2.7 *Let $X = (X(t) : t \geq 0)$ be a spectrally negative Lévy process and $\tau_x = \inf\{t \geq 0 \,|\, X(t) < -x\}$, $x \geq 0$. We set $\gamma = E[\,X(1)\,]$.*

(*a*) *Define $f_\sigma(x) = 1 - E[e^{-\sigma\tau_x}]$. Then*

$$s\int_0^\infty e^{-sx}f_\sigma(x)\,dx = \Psi_\sigma^-(s) = \left(\frac{\sigma}{\sigma-\xi(s)}\right)\left(1-\frac{s}{\eta(\sigma)}\right)\ . \tag{2.16}$$

(*b*) $$s\int_0^\infty e^{-sx}P[\,\tau_x < \infty\,]\,dx = 1 - \frac{\gamma s}{\xi(s)}\ . \tag{2.17}$$

(*c*) $$P[hX(t) \leq -x] = P\left[\inf_{0\leq u\leq t} X(u) \leq -x\right] \tag{2.18}$$

$$= P[X(t) \leq -x] - \frac{d}{dx}\int_0^t \frac{1}{u}E[\,X^+(u)\,]P[X(t-u) \leq -x]\,du\ .$$

Proof. (*a*) Let $\Psi_\sigma^-(s)$, $\Psi_\sigma^+(s)$ denote the Wiener–Hopf factors of X and write $\psi_\sigma^-(s)$, $\psi_\sigma^+(s)$ for the left and right Wiener–Hopf factors of $-X$. Then, see Bingham [12, p. 709],

$$
\begin{aligned}
\psi_\sigma^-(s) &= \Psi_\sigma^+(-s)\,, \qquad \mathrm{Re}\{s\} \geq 0\,, \\
\psi_\sigma^+(s) &= \Psi_\sigma^-(-s)\,, \qquad \mathrm{Re}\{s\} \leq 0\,.
\end{aligned}
$$

For any Lévy process it holds that $s\int_0^\infty e^{-sx}(1 - E[\,^{-\sigma fX(x)}\,])\,dx = \Psi_\sigma^+(-s)$ for $\mathrm{Re}\{s\} > 0$; see Bingham [12, Theorem 1e)]. Since $\tau_x =$

$f(-X)(x)$ and because of the relationship of the Wiener–Hopf factors of X and $-X$, the statement follows.
(b) Straightforward computation shows that

$$\begin{aligned}
\int_0^\infty e^{-sx} P[\,\tau_x < \infty\,]\,dx &= \int_0^\infty e^{-sx} \lim_{\sigma\to 0} E\left[\,e^{-\sigma\tau_x}\,\right]\,dx \\
&= \int_0^\infty e^{-sx} \lim_{\sigma\to 0} (1 - f_\sigma(x))\,dx \\
&= \frac{1}{s} - \lim_{\sigma\to 0} \int_0^\infty e^{-sx} f_\sigma(x)\,dx \\
&= \frac{1}{s} - \lim_{\sigma\to 0} \left(\frac{\sigma}{\sigma - \xi(s)}\right)\left(\frac{1}{s} - \frac{1}{\eta(\sigma)}\right) \\
&= \frac{1}{s} - \frac{1}{\xi(s)} \lim_{\sigma\to 0} \frac{\sigma}{\eta(\sigma)}\ .
\end{aligned}$$

Using l'Hôpital's rule we find $\lim_{\sigma\to 0} \sigma/\eta(\sigma) = \lim_{\sigma\to 0} 1/\eta'(\sigma)$. Differentiating $\xi(\eta(\sigma)) = \sigma$ with respect to σ yields $1/\eta'(\sigma) = \xi'(\eta(\sigma))$ and thus $\lim_{\sigma\to 0} 1/\eta'(\sigma) = \xi'(0+) = E[\,X(1)\,] = \gamma$. This concludes the proof of (b).
(c) The key for proving the last statement is equation (2.11) in Theorem 2.6. On the one hand, by the definition of $\Psi_\sigma^-(s)$ and Proposition 2.5 we can write

$$\begin{aligned}
\Psi_\sigma^-(s) = E[\,e^{s\zeta^-}\,] &= \int_0^\infty e^{-sy}\,d_y P[\zeta^- \le -y] \\
&= \int_0^\infty e^{-sy}\,d_y \left(\sigma \int_0^\infty e^{-\sigma t} P[hX(t) \le -y]\,dt\right)\ .
\end{aligned} \tag{2.19}$$

Define $g(u) = E[\,X^+(u)\,]/u$. Recall that for the Laplace transform of g we have $\int_0^\infty e^{-\sigma u} g(u)\,du = 1/\eta(\sigma)$, see (2.15), and that $\sigma/(\sigma - \xi(s)) = E[\,e^{sX(T)}\,]$. We then obtain for the right–hand side of (2.11)

$$\begin{aligned}
\Psi_\sigma^-(s) &= \left(\frac{\sigma}{\sigma - \xi(s)}\right) - \frac{s}{\eta(\sigma)}\left(\frac{\sigma}{\sigma - \xi(s)}\right) \\
&= \int_0^\infty e^{-sy}\,d_y P[X(T) \le -y] - \frac{s}{\eta(\sigma)} \int_0^\infty e^{-sy}\,d_y P[X(T) \le -y] \\
&= \int_0^\infty e^{-sy}\,d_y \left(\sigma \int_0^\infty e^{-\sigma t} P[X(t) \le -y]\,dt\right)
\end{aligned}$$

$$
\begin{aligned}
&- s \left(\int_0^\infty e^{-\sigma t} g(t)\, dt \right) \int_0^\infty e^{-sy}\, d_y \left(\sigma \int_0^\infty e^{-\sigma t} P[X(t) \le -y]\, dt \right) \\
&= \int_0^\infty e^{-sy}\, d_y \left(\sigma \int_0^\infty e^{-\sigma t} P[X(t) \le -y]\, dt \right) \\
&\quad - s \int_0^\infty e^{-sy}\, d_y \left(\sigma \int_0^\infty e^{-\sigma t} \int_0^t g(u) P[X(t-u) \le -y]\, du\, dt \right) \\
&= \int_0^\infty e^{-sy}\, d_y \left(\sigma \int_0^\infty e^{-\sigma t} P[X(t) \le -y]\, dt \right) \\
&\quad - \int_0^\infty e^{-sy}\, d_y \left(\sigma \int_0^\infty e^{-\sigma t} \frac{d}{dy} \int_0^t g(u) P[X(t-u) \le -y]\, du\, dt \right) .
\end{aligned}
$$

A comparison of the last equality with (2.19) concludes the proof by means of uniqueness of Laplace transforms. □

2.2 The Classical Risk Process

The basic theory of risk starts at the beginning of this century with the pioneering work of Filip Lundberg in his 1903 thesis. Harald Cramér then incorporated Lundberg's ideas in the survey paper of 1955. At the base of the resulting model, in the following referred to as Cramér–Lundberg model, is the surplus process $R = (R(t) : t \ge 0)$ defined as follows:

$$
R(t) = x + ct - \sum_{k=1}^{N(t)} Y_k \,, \qquad \left(\sum_{k=1}^{0} Y_k \stackrel{\text{def}}{=} 0 \right) \,, \tag{2.20}
$$

where $x \ge 0$ denotes the initial risk reserve and c stands for the premium income rate. The process $N = (N(t) : t \ge 0)$ is a homogeneous Poisson process with intensity λ, $(Y_k : k \in \mathbb{N})$ is a sequence of iid random variables independent of N with distribution function F on $[0, \infty)$ and finite mean μ. We assume that the net profit condition $c - \lambda\mu > 0$ holds. In terms of a positive constant θ, the so–called relative safety loading, the premium rate c can be expressed as $c = (1+\theta)\lambda\mu$. For some simulations of R, see Figure 2.2. Of course, a classical risk process

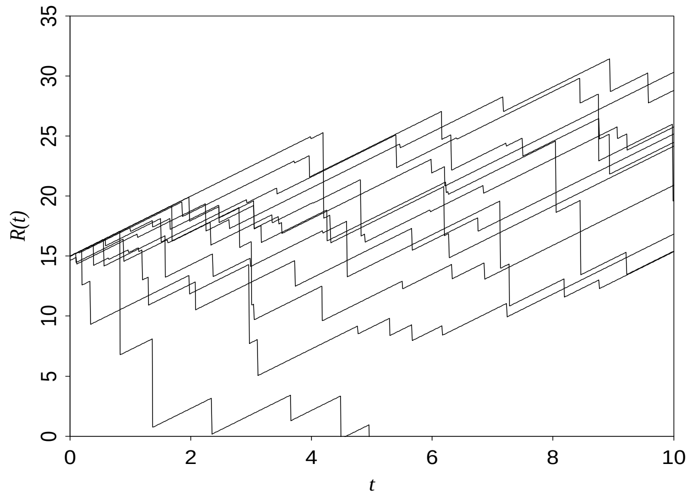

Figure 2.2: *Simulations of a classical risk process R with initial capital $x = 15$, premium rate $c = 2.5$, intensity $\lambda = 1$ and exponentially distributed claims with mean $\mu = 2$.*

as described in (2.20) is not comprehensive and does not take into account, for example, the nonlinear increase of the capital due to possible investment or inflation and dividend payments to stock holders. Nevertheless, processes of the form (2.20) are suitable approximations to the real situation and can be considered as a base for more detailed examinations of different aspects of the activity of insurance companies.
A central issue in actuarial risk theory is the possible ruin of the insurer, defined as the event when the cash balance falls to zero. There are different variants of this concept, depending on whether the time horizon is finite or not:

$$\Psi(x, t_0) = P[R(t) < 0 \text{ for some } 0 \leq t \leq t_0 \,|\, R(0) = x] \;, \tag{2.21}$$

$$\Psi(x) = P[R(t) < 0 \text{ for some } t \geq 0 \,|\, R(0) = x] \;. \tag{2.22}$$

Define the integrated tail distribution F_I as

$$F_I(x) = \frac{1}{\mu} \int_0^x \overline{F}(y)\, dy \ , \tag{2.23}$$

where $\overline{F}(x) = 1 - F(x)$ denotes the tail of the distribution F. The Laplace–Stieltjes transform $\hat{f}_I(s)$ of F_I is given by

$$\hat{f}_I(s) = \int_0^\infty e^{-su}\, dF_I(u) = \frac{1}{\mu} \int_0^\infty e^{-su} \overline{F}(u)\, du \ . \tag{2.24}$$

Note that $\mu s \hat{f}_I(s) = 1 - \hat{f}(s)$, where $\hat{f}(s)$ denotes the Laplace transform of F. In the Cramér–Lundberg model, $\Psi(x)$ can be calculated explicitly in terms of F_I :

$$1 - \Psi(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n F_I^{*n}(x) \ , \tag{2.25}$$

where $\rho = \lambda\mu/c = 1/(1+\theta) < 1$. Formula (2.25) is referred to as Pollaczek–Khintchine formula . It is a key tool to estimate ruin probabilities in the heavy–tailed case; see Section 2.2.2. One approach to prove (2.25) is to employ Wiener–Hopf techniques. Indeed, the process R is a spectrally negative Lévy process and thus equation (2.17) can be used to solve for $P[\tau_x < \infty] = \Psi(x)$. For a detailed discussion of this approach, see Chapter 3. Alternatively, by a probabilistic reasoning one may derive the following renewal equation for $\Psi(x)$, see for instance Embrechts, Klüppelberg and Mikosch [31, Section 1.2],

$$\Psi(x) = \rho \overline{F}_I(x) + \int_0^x \Psi(x-y)\, d(\rho F_I(y)) \ . \tag{2.26}$$

Using Laplace transforms, one obtains (2.25). Notice that the renewal equation (2.26) is defective since $\rho < 1$.
The latter reference gives a comprehensive overview on risk theory, annotated by an extensive bibliography where the original results are to be found.

2.2.1 Ruin Theory for Light-Tailed Claim Size Distributions

Calculating $\Psi(x)$ in closed form or asymptotically as $x \to \infty$ essentially depends on the behaviour of $\hat{f}_I(-s)$ near the right boundary point s_∞ of its convergence region. We make the following definition.

Definition 2.8 *A distribution function G is said to be light–tailed, if there exists $s_\infty > 0$ such that $\hat{g}(-s) \uparrow \infty$ as $s \uparrow s_\infty$ (we allow for the possibility of $s_\infty = \infty$).*

The important part of Definition 2.8 is that $\hat{g}(-s) < \infty$ for some $s > 0$. This means that the tail of G decreases at least exponentially fast. Consequently, the lognormal and the Pareto distributions for instance are excluded. The value s_∞ is called the abscissa of convergence of $\hat{g}(-s)$. The assumption "$\hat{g}(-s) \uparrow \infty$ as $s \uparrow s_\infty$" is convenient, since the Cramér–Lundberg theorem essentially requires a constant $R > 0$, the so–called Lundberg coefficient, such that $\hat{f}_I(-R) = 1 + \theta$:

Theorem 2.9 (Cramér–Lundberg theorem)
Assume that the integrated tail distribution F_I is light–tailed, i.e. that there is a constant $R > 0$ such that $\hat{f}_I(-R) = 1 + \theta = 1/\rho$. Then

$$\Psi(x) \sim Ce^{-Rx} \,, \qquad x \to \infty \,,$$

where $1/C = \dfrac{R}{\theta\mu} \displaystyle\int_0^\infty y e^{Ry} \overline{F}(y)\, dy.$

By $f(x) \sim g(x)$, $x \to \infty$, we mean that $\lim_{x\to\infty} f(x)/g(x) = 1$.

Sketch of the proof. The asymptotic behaviour of the solution of a renewal equation is described by the key renewal theorem, see Feller [33, p. 363]. In order to make use of that theorem, we first have to "remove" the defect. To this end we multiply (2.26) on both sides with e^{Rx}, yielding

$$e^{Rx}\Psi(x) = e^{Rx}\rho\overline{F}_I(x) + \int_0^x e^{R(x-y)}\Psi(x-y)e^{Ry}\, d(\rho F_I(y)) \,.$$

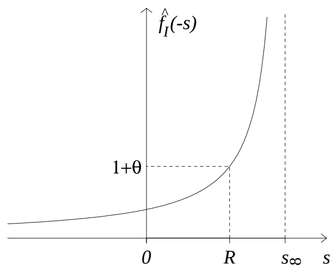

Figure 2.3: *Visualisation of the existence of the Lundberg coefficient.*

Set $dF_I^{(R)}(y) = e^{Ry}\, d(\rho F_I(y))$. It follows that

$$\int_0^\infty dF_I^{(R)}(y) = \rho \int_0^\infty e^{Ry}\, dF_I(y) = \rho \hat{f}_I(-R) = 1\,,$$

so the distribution $F_I^{(R)}(y)$ is non–defective. The function $e^{Rx}\rho\overline{F}_I(x)$ is seen to be directly Riemann integrable. Introduce the notation $\mu^* = \int_0^\infty y\, dF_I^{(R)}(y)$. From the key renewal theorem we then conclude that

$$\begin{aligned}\lim_{x\to\infty} e^{Rx}\Psi(x) &= \frac{1}{\mu^*}\int_0^\infty e^{Ry}\rho\overline{F}_I(y)\,dy\\ &= \left(\frac{R}{\theta\mu}\int_0^\infty y e^{Ry}\overline{F}(y)\,dy\right)^{-1}\,. \qquad \Box\end{aligned}$$

The intuitive picture in the light–tailed case is that ruin occurs as a consequence of a build–up over a period, see Asmussen and Klüppelberg [4]. The trajectory causing ruin in Figure 2.2 illustrates well this scenario. For distributions like Pareto, lognormal and Weibull we have $\hat{f}_I(-s) = \infty$ for all $s > 0$, i.e. $s_\infty = 0$ and so the condition on light–tailedness is violated. All these distributions share the property that their tail function decreases slower than any exponential function and we might expect that ruin occurs as a consequence of one big claim. The Cramér–Lundberg Theorem 2.9 is not applicable and so other methods are necessary to estimate $\Psi(x)$, see Section 2.2.2. In Chapter 4 we propose an approximation for $\Psi(x)$ when the claim size distribution has infinite variance, for instance Pareto with shape parameter $1 < \alpha < 2$. The latter distribution typically arises when fitting "heavy–tailed" claims in non–life insurance; see for instance Ramlau–Hansen [68], [69].

2.2.2 Ruin Theory for Heavy - Tailed Claim Size Distributions

In this section we investigate the asymptotic behaviour of $\Psi(x)$, $x \to \infty$ when the condition on light–tailedness of F_I is violated. Notice that $\phi(x) = 1-\Psi(x)$ given in (2.25) is the distribution function of the random sum $S_N = X_1 + \cdots + X_N$, where $(X_i : i \in \mathbb{N})$ is a sequence of iid random variables with common distribution function F_I and N is geometrically distributed with parameter $1-\rho$ independent of $(X_i : i \in \mathbb{N})$. Now, if F_I is "long–tailed", large observations of X_i may occur with high probability and it is not unreasonable to conjecture that the random sum S_N may be governed by just one summand. For that reason it might be possible to relate the tail behaviour of ϕ to that of F_I. It turns out that the proper class for this purpose is the class $\mathcal{S}$ of subexponential distributions, see Definition 2.11 below. The concept of large claims can now be made precise:

Definition 2.10 *We talk about large or heavy–tailed claims if $F_I \in \mathcal{S}$.*

The class $\mathcal{S}$ itself is defined as follows.

Definition 2.11 *A distribution G on $[0,\infty)$ with unbounded support belongs to the class $\mathcal{S}$ of subexponential distributions if*

$$\lim_{x\to\infty} \frac{1-G^{*2}(x)}{1-G(x)} = 2 \ .$$

REMARKS.

1. The class $\mathcal{S}$ was introduced by Chistyakov [15] and has been extensively studied among others by Embrechts and Goldie [26], [27].

2. The name subexponential stems from the following property: if $G \in \mathcal{S}$, then the right tail of G decreases slower than any exponential, i.e. $\lim_{x\to\infty} e^{\varepsilon x}\,\overline{G}(x) = \infty$, for all $\varepsilon > 0$.

3. One can show that for $G \in \mathcal{S}$ and $n \geq 2$,
$$P[S_n > x] \sim P[M_n > x]\ , \qquad x \to \infty\,.$$
Here $M_n = \max(X_1, \ldots, X_n)$, where $(X_i : i \in \mathbb{N})$ are iid with distribution function G. An intuitive interpretation of $\mathcal{S}$ is then that the sum $S_n = X_1 + \cdots + X_n$ is large because of one large value X_i, $1 \leq i \leq n$.

4. Sufficient conditions for subexponentiality can be found in Embrechts, Klüppelberg and Mikosch [31, Appendix A3.2]. □

Proposition 2.13 below shows that $\mathcal{S}$ contains the class of distribution functions with regularly varying tails.

Definition 2.12 *A positive measurable function h defined on $[0,\infty)$ is regularly varying with index δ if for all $t > 0$,*
$$\lim_{x\to\infty} \frac{h(tx)}{h(x)} = t^{\delta}\ . \tag{2.27}$$

Remark. Condition (2.27) can be relaxed in the sense that one only needs to specify that the limit exists and is positive, rather than that it has the functional form t^δ. Indeed, if we suppose in (2.27) that the limit exists for all $t > 0$ and equals $\chi(t)$, say, then it immediately follows that $\chi(ts) = \chi(t)\chi(s)$ and hence $\chi(t) = t^\delta$ for some $\delta \in \mathbb{R}$. □

We use the notation RV_δ for regularly varying functions with index $\delta \in \mathbb{R}$. The case $\delta = 0$ corresponds to the class of slowly varying functions, therefore $h \in RV_\delta$ can be written as $h(x) = x^\delta L(x)$, $L \in RV_0$.

Proposition 2.13 *If F_1 and F_2 are two distribution functions such that $\overline{F}_i(x) \sim x^{-\delta} L_i(x)$, $x \to \infty$, with $L_i \in RV_0$, then the convolution $G = F_1 * F_2$ has a regularly varying tail such that $\overline{G}(x) \sim x^{-\delta}(L_1(x) + L_2(x))$, $x \to \infty$.* □

For a proof, see Feller [33, p. 278]. By induction on n one gets $\overline{F^{*n}}(x) \sim nx^{-\delta}L(x)$ if $\overline{F}(x) \sim x^{-\delta}L(x)$, $x \to \infty$.

Surprisingly, if F_1, $F_2 \in \mathcal{S}$ then it does not necessarily follow that $F_1 * F_2 \in \mathcal{S}$, see Leslie [55]. However, the following closure properties under convolution operations hold (by $f(x) = o(g(x))$ we mean that $\lim_{x\to\infty} f(x)/g(x) = 0$):

Proposition 2.14 *Let $H = F_1 * F_2$ be the convolution of two distribution functions on $[0, \infty)$.*

(*a*) *If $F_2 \in \mathcal{S}$ and $\overline{F}_1(x) = o(\overline{F}_2(x))$ as $x \to \infty$, then $H \in \mathcal{S}$. Moreover $\overline{H}(x) \sim \overline{F}_2(x)$ as $x \to \infty$.*

(*b*) *If $H \in \mathcal{S}$ and $\overline{F}_1(x) = o(\overline{H}(x))$, then $F_2 \in \mathcal{S}$ and indeed $\overline{F}_2(x) \sim \overline{H}(x)$ as $x \to \infty$.* □

For a proof, see Embrechts, Goldie and Veraverbeke [28]. The next proposition can also be found in the same paper.

Proposition 2.15 *Suppose $\rho \in (0, 1)$ and H a proper distribution function on $[0, \infty)$. If $K(x) = (1 - \rho)\sum_{n=0}^{\infty} \rho^n H^{*n}(x)$, then the following assertions are equivalent:*

(*i*) $K \in \mathcal{S}$,

(*ii*) $H \in \mathcal{S}$,

(*iii*) $\dfrac{\overline{K}(x)}{\overline{H}(x)} \to \dfrac{\rho}{1-\rho}\ , \quad x \to \infty\ .$

□

Proposition 2.15 is the key tool to prove a Cramér–Lundberg–type theorem for claim size distribution functions with $F_I \in \mathcal{S}$; see for instance Embrechts, Klüppelberg and Mikosch [31, Theorem 1.3.8].

Theorem 2.16 *In the Cramér–Lundberg model with $F_I \in \mathcal{S}$ and safety loading $\theta > 0$ one has*

$$\Psi(x) \sim \frac{1}{\theta}\,\overline{F}_I(x)\ , \qquad x \to \infty\ . \tag{2.28}$$

□

For the rest of the present section we recall the relations between the tails and truncated moments of distributions with regularly varying tails. The main result is that if $\overline{F}(x)$ and $F(-x)$ vary regularly so do all truncated moments. Karamata's theorem will be used in the next chapter. We refer to the first chapter of Bingham, Goldie and Teugels [13] for details.

Theorem 2.17 (Karamata's Theorem)
Let $L \in RV_0$ be locally bounded on $[x_0, \infty)$ for some $x_0 \geq 0$. Then

(*a*) *for $\rho > -1$*

$$\int_{x_0}^{x} y^{\rho} L(y)\, dy \sim \frac{L(x)}{1+\rho}\, x^{1+\rho} \ , \qquad x \to \infty \, ,$$

(*b*) *for $\rho < -1$, or $\rho = -1$ and $\int_{x_0}^{\infty} y^{-1} L(y)\, dy < \infty$*

$$\int_{x}^{\infty} y^{\rho} L(y)\, dy \sim -\frac{L(x)}{1+\rho}\, x^{1+\rho} \ , \qquad x \to \infty \, .$$

Whenever $\rho \neq -1$ and the limit relations in either (a) or (b) hold for some positive function h, locally bounded on some interval $[x_0, \infty)$, $x_0 \geq 0$, then $h \in RV_{\rho}$. □

For later purposes we also need the following so–called Abelian and Tauberian theorem. It relates the behaviour of regularly varying functions at infinity to the behaviour of their Laplace transforms near zero; see Feller [33, p. 446]. We denote by $\Gamma(x)$ the Gamma function, i.e.

$$\Gamma(x) = \int_0^{\infty} e^{-u} u^{x-1}\, du \ , \qquad x \geq 0 \, .$$

Well–known properties of the Γ–function are

$$\Gamma(x+1) = x\Gamma(x) \ , \qquad \Gamma(n+1) = n\,! \ , \quad n \in \mathbb{N} \, ,$$
$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \ , \qquad \Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)} \ , \quad 0 < x < 1 \, .$$

For an increasing, right–continuous function U denote by $U(dx)$ the associated measure. Then the following Abelian and Tauberian theorem holds.

Theorem 2.18 (Karamata's Abelian and Tauberian theorem)
Let U be an increasing, right–continuous function on $[0,\infty)$. If $L \in RV_0$ and $\rho \geq 0$, then the following are equivalent:

$$(a) \qquad \hat{u}(s) = \int_0^\infty e^{-sy}\, U(dy) \sim s^{-\rho} L(1/s)\;, \qquad s \downarrow 0\,,$$

$$(b) \qquad U(y) \sim \frac{y^\rho L(y)}{\Gamma(1+\rho)}\;, \qquad y \to \infty\,.$$

□

The implication $(a) \Rightarrow (b)$ is called Tauberian, the converse $(b) \Rightarrow (a)$ Abelian. Another useful property concerns the case when the measure $U(dx)$ is absolutely continuous with a monotone density.

Theorem 2.19 (Monotone density theorem)
Suppose that $U(dx) = u(x)dx$, where $u : (0,\infty) \to (0,\infty)$ is monotone on (x_0,∞) say, for some $x_0 > 0$. If there is a positive real number ρ and a slowly varying function L such that

$$U(x) \sim x^\rho L(x)\;, \qquad x \to \infty\,,$$

then

$$u(x) \sim \rho x^{\rho-1} L(x)\;, \qquad x \to \infty\,.$$

Conversely, if $U(dx)$ is absolutely continuous with a density u that is regularly varying with index $\rho - 1$ for some $\rho > 0$, then U is regularly varying with index ρ and $U(x) \sim \rho^{-1} x u(x)$, $x \to \infty$. □

2.3 Stable Distributions

In this section we summarize the well–known theory of stable distributions. The complete theory of stable distributions has first been given in Gnedenko and Kolmogorov [40]. Most results can also be found in general books on probability, for example Feller [33]. For a comprehensive survey of one–dimensional stable distributions we refer to Zolotarev [88] and references therein.

Gaussian distributions and processes have long since been studied and their utility in stochastic modelling is well–accepted. However, they do not allow for large fluctuations and may sometimes be inadequate for modelling high variability. The main difference between the normal distribution and non–Gaussian stable distributions is the tail behaviour. The tails of the latter decrease like a power function. The rate of decay depends on a parameter α which takes values between 0 and 2. The smaller the value of α, the slower the decay and the heavier the tails. A first definition of a univariate stable distribution derives from the "stability" property which states that the family of stable distributions is preserved under convolution. More precisely, a random variable X is said to have a stable distribution if for any positive numbers a and b, there is a positive number c and a real number d such that

$$aX' + bX'' \stackrel{d}{=} cX + d\,,$$

where X' and X'' are independent copies of X. To give an analytical description of the class of stable laws we evaluate the integral in (2.1) with respect to the Lévy measure Π given in (2.4). We obtain (cf. Hall [47] or Bingham, Goldie and Teugels [13, Theorem 8.3.2]):

Proposition 2.20 *A random variable X has a stable distribution if and only if the logarithm of its characteristic function g can be represented in the form*

$$\log g(\theta) = \begin{cases} -\sigma^\alpha|\theta|^\alpha(1 - i\beta\,\mathrm{sign}(\theta)\tan\frac{\pi\alpha}{2}) + im\theta & \alpha \neq 1\,, \\ -\sigma|\theta|(1 + i\beta\frac{2}{\pi}\,\mathrm{sign}(\theta)\,\ln|\theta| + im\theta & \alpha = 1\,, \end{cases} \tag{2.29}$$

where $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\sigma \geq 0$, $m \in \mathbb{R}$ *and*

$$\mathrm{sign}(\theta) = \begin{cases} 1 & \theta > 0\,, \\ 0 & \theta = 0\,, \\ -1 & \theta < 0\,. \end{cases}$$

□

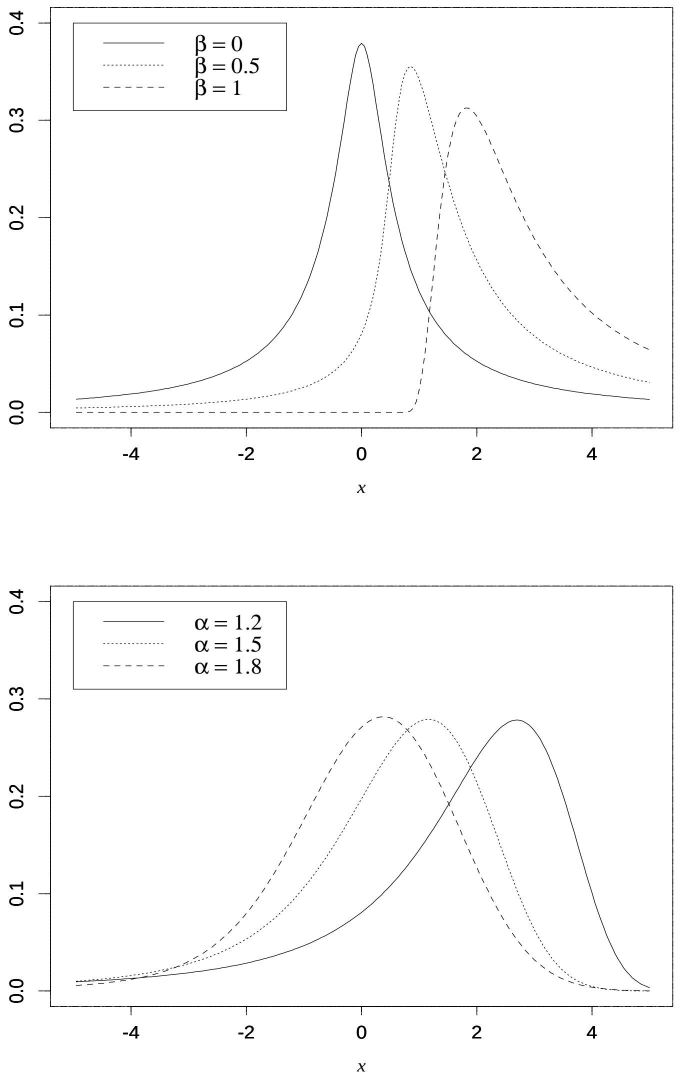

Figure 2.4: *Densities $g(x;\alpha,\beta)$ of stable distributions with $\alpha = 0.75$ and varying β (top) and densities of completely asymmetric stable distributions ($\beta = -1$) with varying α.*

REMARKS.

1. A univariate stable distribution is characterized by four parameters. These are the index of stability α, the scale parameter σ, the skewness parameter β and the shift parameter m. Hence, we denote stable distributions by $S_\alpha(\sigma,\beta,m)$ and write $X \sim S_\alpha(\sigma,\beta,m)$ to indicate that X has the stable law $S_\alpha(\sigma,\beta,m)$. Because σ and m merely determine location and scale we shall mostly consider stable distributions with $m = 0$ and $\sigma = 1$. Note that by doing so we are excluding the degenerate case $\sigma = 0$. We introduce the notation $G(\,\cdot\,;\alpha,\beta)$ respectively $g(\,\cdot\,;\alpha,\beta)$ for the distribution function and density function of a stable law with parameters α, β, $\sigma = 1$ and $m = 0$. From Proposition 2.21 it follows that

$$G(-x;\alpha,\beta) = 1 - G(x;\alpha,-\beta)\ .$$

2. Distributions with $|\beta| = 1$ are commonly called completely asymmetric or extremal stable distributions. In case $0 < \alpha < 1$ the stable laws with $|\beta| = 1$ are one–sided, i.e. the support is $[0,\infty)$ if $\beta = 1$ and $(-\infty,0]$ in case $\beta = -1$, see also Figure 2.4 (top).

3. When $\alpha = 2$ the chf becomes $E[e^{i\theta X}] = \exp\{-\sigma^2\theta^2 + im\theta\}$, hence X is normally distributed with mean m and variance $2\sigma^2$. In that case the parameter β is irrelevant.

   If $\alpha = 1$ and $\beta = m = 0$, we note that $E[e^{i\theta X}] = e^{-\sigma|\theta|}$. This corresponds to the Cauchy law.

   We recognize the Lévy distribution when setting $\alpha = 1/2$, $\beta = 1$ and $m = 0$, i.e. for $X \sim S_{1/2}(\sigma,1,0)$ we have $P[X \leq t] = 2(1 - \Phi(\sqrt{(\sigma/t)})$, where Φ denotes the normal distribution function. For standard Brownian motion B it is known that the first passage distributions $fB(\sqrt{\sigma})$ defined in (2.6) have Lévy distribution $S_{1/2}(\sigma,1,0)$; see for instance Karlin and Taylor [54, p. 363].

4. The densities of stable laws in general can only be expressed by complicated functions, see Hoffmann–Jørgensen [48]. Only in the above mentioned cases $\alpha = 1/2$, 1, 2 are the densities expressible via elementary functions. Most information and properties of stable distributions can be deduced from the fact that the chf have

quite a simple form. The explicitness of the Fourier transforms can to some extent compensate for the absence of elementary expressions for the densities when $\alpha \neq 1/2, 1, 2$.

5. Every stable distribution is unimodal, see Ibragimov and Linnik [49, Theorem 2.5.3].

6. There are many different forms of expression for the chf of stable laws. The logarithm of the chf g is sometimes written as

$$\log g(\theta) = -\sigma_B^\alpha |\theta|^\alpha \omega(\theta, \alpha, \beta_B) + im\theta\,,$$

where

$$\omega(\theta, \alpha, \beta_B) = \begin{cases} \exp\{-i\beta_B \operatorname{sign}(\theta)\, \pi K(\alpha)/2\} & \alpha \neq 1\,, \\ \dfrac{\pi}{2} + i\beta_B \log|\theta| \operatorname{sign}(\theta) & \alpha = 1\,, \end{cases}$$

$0 < \alpha \leq 2$, $-1 \leq \beta_B \leq 1$, $\sigma_B \geq 0$, $m \in \mathbb{R}$ and

$$K(\alpha) = \alpha - 1 + \operatorname{sign}(1-\alpha) = \begin{cases} \alpha & \alpha < 1\,, \\ \alpha - 2 & \alpha > 1\,. \end{cases} \tag{2.30}$$

The main parameter α is the same in both forms. If $\alpha \neq 1$ then

$$\beta_B = \frac{2}{\pi K(\alpha)} \arctan\left(\beta \tan \frac{\pi\alpha}{2}\right)\,. \tag{2.31}$$

The new scale parameter σ_B is related to σ as follows

$$\begin{aligned} \sigma_B &= \sigma \left(\cos\left(\beta_B \frac{\pi K(\alpha)}{2}\right)\right)^{-1/\alpha} \\ &= \sigma \left(1 + \beta^2 \tan^2\left(\frac{\pi\alpha}{2}\right)\right)^{1/(2\alpha)}\,. \end{aligned} \tag{2.32}$$

If we want to stress that the form (B) is being used, then the quantities and functions connected with it will be given the subscript "B". □

The following proposition readily follows from Proposition 2.20 and identifies σ as scale parameter.

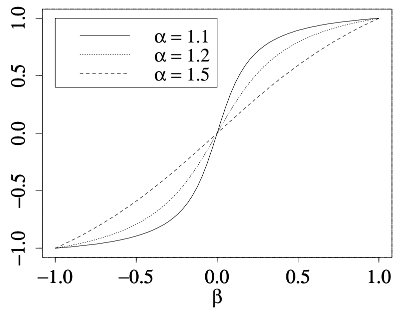

Figure 2.5: *The skewness parameter β_B versus β.*

Proposition 2.21 *Let $X \sim S_\alpha(\sigma, \beta, m)$ and let $a \in \mathbb{R}\backslash\{0\}$. Then for $\alpha \neq 1$*

$$aX \sim S_\alpha(|a|\sigma, \operatorname{sign}(a)\beta, am) \ .$$

□

As mentioned before the main difference between the normal distribution ($\alpha = 2$) and stable distributions with $\alpha < 2$ is that the tails of the latter decrease much slower. Recall that for the normal distribution one has

$$1 - \Phi(x) \sim \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} \ , \quad x \to \infty \, . \tag{2.33}$$

On the other hand, upper and lower tails of stable laws with $\alpha < 2$ decrease as $x^{-\alpha}$, see Proposition 2.22 below.

Proposition 2.22 *Let $X \sim S_\alpha(\sigma, \beta, m)$ with $0 < \alpha < 2$. Then*

$$\begin{aligned} P[\, X > x \,] &\sim C_{\alpha,\sigma} \left(\frac{1+\beta}{2} \right) x^{-\alpha} \ , \qquad x \to \infty \, , \\ P[\, X \leq -x \,] &\sim C_{\alpha,\sigma} \left(\frac{1-\beta}{2} \right) x^{-\alpha} \ , \qquad x \to \infty \, , \end{aligned} \tag{2.34}$$

where

$$C_{\alpha,\sigma} = \sigma^\alpha \left(\int_0^\infty x^{-\alpha} \sin(x)\, dx \right)^{-1}$$
$$= \begin{cases} \dfrac{\sigma^\alpha(1-\alpha)}{\Gamma(2-\alpha)\cos(\pi\alpha/2)} \,, & \alpha \neq 1 \,, \\ \dfrac{2\sigma}{\pi} \,, & \alpha = 1 \,. \end{cases} \tag{2.35}$$

□

For a proof, see Samorodnitsky and Taqqu [71, p. 16]. Since $E[\,|X|^r\,] = \int_0^\infty P[\,|X|^r > y\,]\,dy$ we have:

Corollary 2.23 *Let* $X \sim S_\alpha(\sigma,\beta,m)$ *with* $0 < \alpha < 2$*. Then*

$$E[\,|X|^r\,] < \infty \quad \textit{for} \quad 0 < r < \alpha \,,$$
$$E[\,|X|^r\,] = \infty \quad \textit{for} \quad \alpha \leq r \,.$$

□

Hence, when $\alpha \leq 1$ we have $E[\,|X|\,] = \infty$. In case $1 < \alpha \leq 2$, the following proposition holds.

Proposition 2.24 *When* $1 < \alpha \leq 2$*, the shift parameter* m *equals the mean.*

Proof. Let $X \sim S_\alpha(\sigma,\beta,m)$, $1 < \alpha \leq 2$. By Corollary 2.23 the random variable X has finite mean. Let X', X'' be two independent copies of X. We have $X - m \sim S_\alpha(\sigma,\beta,0)$, which implies

$$a(X' - m) + b(X'' - m) \stackrel{d}{=} c(X - m)$$

for any positive numbers a, b. Taking expectations on both sides completes the proof. □

The density $g(x;\alpha,1)$ decreases as $x \to -\infty$ more rapidly than e^{cx} for any $c > 0$. Therefore, an analytic extension of the chf to the complex plane with a cut along some half–line, say $\arg\{z\} = -3\pi/4$, is possible in the cases $\alpha < 2$ and $\beta \in \{-1, 1\}$, which correspond to extremal stable distributions. In particular, the Laplace transforms exist.

Proposition 2.25 *Let $X \sim S_\alpha(\sigma, 1, 0)$ and $0 < \alpha < 2$. Then*

$$E\Big[e^{-sX}\Big] = \exp\left\{-\frac{(\sigma s)^\alpha}{\cos(\pi\alpha/2)}\right\}, \qquad s \geq 0. \tag{2.36}$$

□

The proof can be found in Samorodnitsky and Taqqu [71, Proposition 1.2.12] or Zolotarev [88, Theorem 2.6.1]. The high variability of stable distributions is one of the reasons they play an important rôle in stochastic modelling. In the remainder of this section we consider stable distributions in the context of the central limit theorem. The main result states that only stable laws appear as non–degenerate weak limits of normalized sums of iid random variables. Let $(X_i : i \in \mathbb{N})$ be a sequence of iid random variables with common distribution function F.

Definition 2.26 *The distribution function F belongs to the domain of attraction of a non–degenerate distribution G if there exist norming constants $a_n > 0$, b_n such that*

$$\frac{S_n - b_n}{a_n} \Rightarrow G, \qquad n \to \infty,$$

where $S_n = X_1 + \cdots + X_n$ and " $\Rightarrow$ " denotes weak convergence.

Loosely speaking, this means that the distribution F in the domain of attraction of G has properties which are close to the ones of G. The following theorem is crucial, see for instance Ibragimov and Linnik [49, Theorem 2.1.1] or Bingham, Goldie and Teugels [13, p. 343].

Proposition 2.27 *Only stable distribution functions have non–empty domains of attractions.* □

We use the notation $F \in \mathcal{D}(\alpha, \beta)$ to indicate that F belongs to the domain of attraction of a stable law with parameters α and β. The following proposition can be found in Feller [33, p. 578].

Proposition 2.28 *A distribution $F \in \mathcal{D}(\alpha, \beta)$ possesses absolute moments of all orders $r < \alpha$. If $\alpha < 2$, no moments of order $r > \alpha$ exist.*

□

It is possible to check whether or not a distribution is in the domain of attraction of a stable law by examing only certain parts of the distribution, usually the tails because only these parts dictate the properties of the distribution. The following criterion can be found in Bingham, Goldie and Teugels [13, Theorem 8.3.1].

Theorem 2.29 (Domain of attraction theorem)

(i) *F belongs to the domain of attraction of a normal law ($\alpha = 2$) if and only if the truncated variance*

$$V(x) = \int_{-x}^{x} y^2 \, dF(y)$$

is slowly varying.

(ii) *F belongs to the domain of attraction of a non–normal stable law ($F \in \mathcal{D}(\alpha, \beta)$, $0 < \alpha < 2$) if and only if*

$$(1 - F(x) + F(-x)) \in RV_{-\alpha} \text{ and}$$

$$\frac{F(-x)}{1 - F(x) + F(-x)} \to \frac{1 - \beta}{2}, \qquad x \to \infty,$$

$$\frac{1 - F(x)}{1 - F(x) + F(-x)} \to \frac{1 + \beta}{2}, \qquad x \to \infty.$$

□

It follows that $a_n = n^{1/\alpha}\ell(n)$, where $\ell \in RV_0$. If F has finite mean μ, the natural centering $b_n = n\mu$ suffices. Now the mean is finite if $\alpha > 1$. If $\alpha < 1$, no centering is in fact needed – nor indeed if $\alpha = 1$ if F is symmetric.

Any stable law with index α belongs to its own domain of attraction with $a_n = n^{1/\alpha}$. This suggests the following definition.

Definition 2.30 *A distribution function F (or a random variable X) belongs to the normal domain of attraction of a stable law with index α if $a_n = n^{1/\alpha}$. In that case we write $F \in \mathcal{D}_{\mathcal{N}}(\alpha, \beta)$.*

Proposition 2.31 *$F \in \mathcal{D}_{\mathcal{N}}(\alpha, \beta)$ if and only if*

$$x^\alpha \overline{F}(x) \to C\left(\frac{1+\beta}{2}\right), \qquad x \to \infty,$$

$$x^\alpha F(-x) \to C\left(\frac{1-\beta}{2}\right), \qquad x \to \infty,$$

for some constant $C > 0$. □

For a proof, see Feller [33, p. 581].

Remark. The generalized central limit theorem (Proposition 2.27) is undoubtedly one of the main reasons why stable laws are attractive in stochastic modelling. However, DuMouchel [22] showed that this argument is not always persuasive, especially in the infinite variance case where the convergence of convolutions to their limiting stable distribution may be rather slow. More precisely, the nth convolution of a distribution $F \in \mathcal{D}(\alpha, \beta)$, $\alpha < 2$, differs (measured in the L^1–metric) from its limiting stable law $G \sim S_\alpha(\sigma, \beta, m)$ by a factor proportional to $n^{-(2-\alpha)/\alpha}$, $n \to \infty$, while in the finite variance case ($\alpha = 2$) the difference is of order $n^{-1/2}$. □

Chapter 3

Risk Processes Perturbed by α-stable Lévy Motion

Over the recent years we have seen a number of generalizations of the classical Cramér–Lundberg model, incorporating various new techniques in risk theory. Martingale theory, first introduced by Gerber [38], [39, Chapter 9] yields more "realistic" models allowing for economic factors to be introduced. Delbaen and Haezendonck [19] for instance investigate the influence of deterministic inflation and interest on the surplus process and study their effects on (bounds of) ruin probabilities. The only source of uncertainty in that model is the number and severity of claims. Paulsen [62] goes one step further and allows the economic factors to be stochastic. Semimartingales coupled with integro–differential equations lead in some cases to exact probabilities of ruin and in others to inequalities. Economic factors and their influence on ruin probabilities for the Brownian diffusion approximation of a classical risk process are discussed by Norberg [61]. Dufresne and Gerber [21] consider a process $R_B = (R_B(t) : t \geq 0)$ defined as

$$R_B(t) = x + ct - \sum_{k=1}^{N(t)} Y_k + \eta B(t) \ , \qquad (3.1)$$

where $\eta \in \mathbb{R}$ and $B = (B(t) : t \geq 0)$ denotes standard Brownian motion independent of the compound Poisson process $\sum_{k=1}^{N(t)} Y_k$. The diffusion component expresses either an additional uncertainty of the aggregate claims and/or of the premium income. By a renewal argument, the following extension of the Pollaczek–Khintchine formula can be deduced, see Dufresne and Gerber [21, (3.4)]:

$$1 - \Psi_B(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n \left(F_I^{*n} * H^{*(n+1)} \right)(x) \ , \qquad (3.2)$$

where $H(x) = 1 - \exp\{-2cx/\eta^2\}$ denotes the exponential distribution function with parameter $2c/\eta^2$. For light–tailed claim size distributions Dufresne and Gerber [21] employ martingale techniques to establish Lundberg–type estimates for $\Psi_B(x)$. Veraverbeke [81] investigates the asymptotic behaviour of $\Psi_B(x)$ as $x \to \infty$ in the heavy–tailed case. For claim size distributions with $F_I \in \mathcal{S}$ it is shown that $\Psi_B(x) \sim \overline{F_I}(x)/\theta$, see [81, Theorem 1]. Hence, for large initial risk reserves the influence of the Brownian perturbation component on the ruin probability fades away. For Lundberg–type inequalities of $\Psi_B(x)$ in the renewal– and Cox–process setup, see Furrer and Schmidli [37].

We now add instead of a Brownian component an α–stable Lévy motion to the classical risk process. We allow the α–stable Lévy motion only to have downward jumps. This can be achieved by choosing the skewness parameter β equal to -1, see (2.4). In Section 3.1 we derive an analogous convolution formula for the infinite–time ruin probability when the classical risk process is perturbed by α–stable Lévy motion. The convolution formula now contains the Mittag–Leffler function, a generalization of the exponential function to which it reduces in the Gaussian case ($\alpha = 2$). Section 3.2 is devoted to the asymptotic behaviour of the ruin probability as the initial capital becomes large.

3.1 Description of the Model and Main Result

Brownian motion assumes a central rôle in the modern theory of stochastic processes. It is basic to descriptions of financial markets, the construction of a large class of Markov processes called diffusions and it arises in approximations to many queuing models. However, analysis of financial and insurance data often involves oscillations that indicate the presence of heavy tails. Stable processes are an appropriate source of heavy tailed probabilistic models. The definition is as follows.

Definition 3.1 *A càdlàg process $Z_\alpha = (Z_\alpha(t) : t \geq 0)$ on a probability space $(\Omega, \mathcal{F}, P)$ is called a (standard) α–stable Lévy motion if*

(i) $Z_\alpha : [0,\infty) \times \Omega \to \mathbb{R}$.

(ii) $Z_\alpha(0, \cdot\,) = 0$ *a.s.*

(iii) Z_α *has independent increments.*

(iv) $Z_\alpha(t) - Z_\alpha(s) \sim S_\alpha((t-s)^{1/\alpha}, \beta, 0)$ *for any* $0 \leq s < t < \infty$ *and for some* $0 < \alpha \leq 2$, $|\beta| \leq 1$.

Implicit in the definition of the stochastic process Z_α is the assumption that each $Z_\alpha(t)$ is $\mathcal{F}$–measurable. However, Z_α is really a function of the pair of variables (t, ω) and so, for technical reasons, it is often convenient to have some joint measurability properties; see for instance Karatzas and Shreve [53, Definition 1.6]. Since Z_α is right–continuous, Z_α satisfies joint measurability conditions, see [53, Remark 1.14].
Observe that the process Z_α has stationary increments. It is Brownian motion when $\alpha = 2$. The α–stable Lévy motions are $1/\alpha$–self–similar (unless $\alpha = 1$, $\beta \neq 0$), i.e. for all $c > 0$,

$$Z_\alpha(ct) \stackrel{d}{=} c^{1/\alpha}\, Z_\alpha(t) \;, \qquad t \geq 0 \;. \tag{3.3}$$

A self–similar structure is one that "looks the same" on a small or a large scale. If one interprets t as "time" and Z_α as "space", then the equality in law $Z_\alpha(ct) \stackrel{d}{=} c^{1/\alpha}\, Z_\alpha(t)$ tells us that every change of time scale $t \mapsto ct$, $c > 0$, corresponds to a change of space scale $x \mapsto c^{1/\alpha}x$.

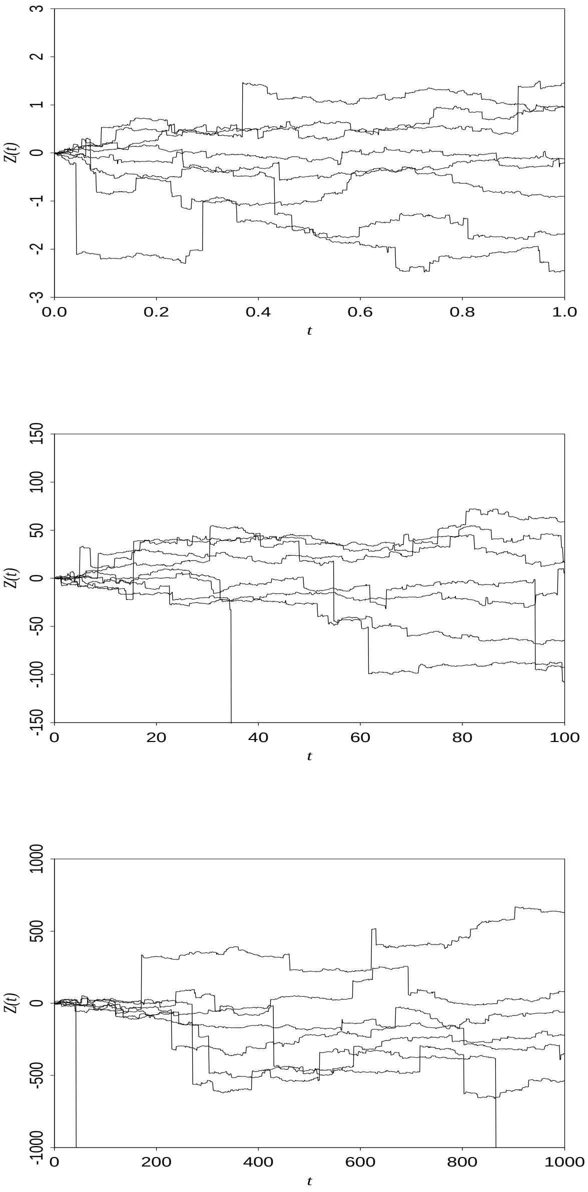

Figure 3.1: *Illustration of the scaling property of an α–stable Lévy motion ($\alpha = 1.2$, $\beta = 0$) for scaling factors $c = 1$ (top), $c = 100$ (middle) and $c = 1000$ (bottom).*

Self–similarity can be used for simulations. In order to simulate a sample path of Z_α on a time interval $[0, 100]$, say, we multiply a sample path on the interval $[0, 1]$ by $100^{1/\alpha}$ and re–scale the time axis by 100, see Figure 3.1.

We now consider a process $R_Z = (R_Z(t) : t \geq 0)$ given by $R_Z(t) = R(t) + \eta Z_\alpha(t)$, where R is defined within the Cramér–Lundberg model (2.20), i.e.

$$R_Z(t) = x + ct - \sum_{k=1}^{N(t)} Y_k + \eta Z_\alpha(t) \;, \qquad t \geq 0 \,. \tag{3.4}$$

Here η is a positive number and Z_α is α–stable Lévy motion with $1 < \alpha < 2$ and $\beta = -1$, independent of R. The condition $\beta = -1$ ensures that there are no upward jumps of Z_α and the condition $\alpha > 1$ is imposed in order to have a finite mean, see Proposition 2.23. The case $\alpha = 2$ is treated in Dufresne and Gerber [21]. If not stated otherwise explicitly, we make for the remainder of this thesis the following assumption.

Assumption 3.2 *The index of stability is*

$$1 < \alpha < 2 \,.$$

The above conditions on the parameters of R and Z_α guarantee that the process $(R_Z(t) - x : t \geq 0)$ is a spectrally negative Lévy process starting from 0 with positive drift. From a modelling point of view, one could view the downward jumps of Z_α as certain extra random payments either involving the income side or the claim payment side. Figure 3.2 depicts a simulated sample path of the process R_Z together with the underlying classical risk process R. In accordance with (2.22) we define the ruin probability in infinite time as

$$\Psi_Z(x) = P[\, R_Z(t) < 0 \text{ for some } t \geq 0 \,|\, R_Z(0) = x \,] \;. \tag{3.5}$$

Recall from Proposition 2.7 that for a spectrally negative Lévy process X we have

$$s \int_0^\infty e^{-sx} P[\, \tau_x < \infty \,] \, dx = 1 - \frac{\gamma s}{\xi(s)} \;, \tag{3.6}$$

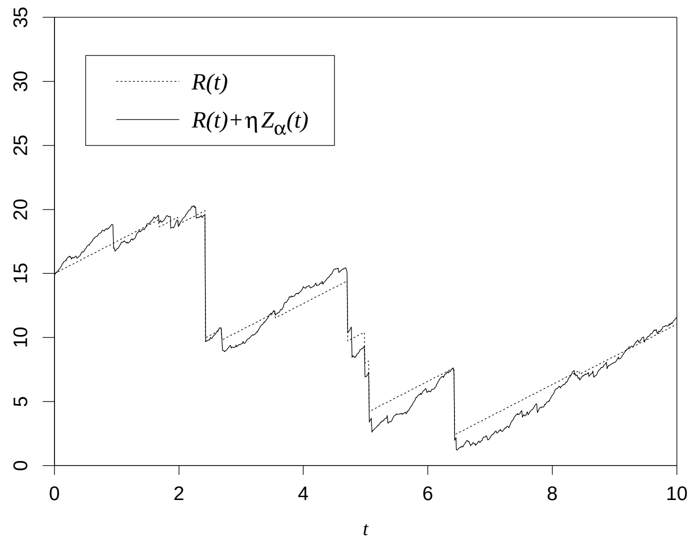

Figure 3.2: *Simulation of a classical risk process R (initial capital $x = 15$, premium rate $c = 2.5$, intensity $\lambda = 1$ and exponentially distributed claims with mean $\mu = 2$) and of its perturbed version $R_Z = R+\eta Z_\alpha$. Here $\alpha = 1.5$ and $\eta = 0.75$.*

where $\gamma = E[X(1)]$ and $\xi(s)$ is the Laplace exponent. We denote by τ_x the first entry time in the interval $(-\infty, -x)$, i.e. $\tau_x = \inf\{t \geq 0 \,|\, X(t) < -x\}$, $x \geq 0$. The idea now is to solve for $P[\tau_x < \infty] = \Psi_Z(x)$ in (3.6). Using Propositions 2.21 and 2.25 and the independence of R and Z_α, we get for the process $(R_Z(t) - x : t \geq 0)$

$$E\left[e^{s(R_Z(t)-x)}\right] = e^{t\xi(s)}\,, \qquad s \geq 0\,,$$

where

$$\xi(s) = cs - \lambda\mu s\hat{f}_I(s) + \kappa s^\alpha \quad \text{and} \quad \kappa = \kappa(\eta,\alpha) = \frac{\eta^\alpha}{\cos\left(\frac{\pi K(\alpha)}{2}\right)}\,. \tag{3.7}$$

Under the net profit condition, the quantity $\gamma = E[R_Z(1) - x] = c - \lambda\mu$ is strictly positive. Recall that $\rho = \lambda\mu/c$ and denote $\tilde{\alpha} = \alpha - 1$. Then we can write

$$\begin{aligned}\frac{\gamma s}{\xi(s)} &= \frac{\gamma}{c - \lambda\mu\hat{f}_I(s) + \kappa s^{\tilde{\alpha}}} \\ &= (1-\rho)\,\frac{c}{c + \kappa s^{\tilde{\alpha}} - \lambda\mu\hat{f}_I(s)} \\ &= (1-\rho)\,\frac{\dfrac{c}{c+\kappa s^{\tilde{\alpha}}}}{1 - \rho\hat{f}_I(s)\,\dfrac{c}{c+\kappa s^{\tilde{\alpha}}}} \\ &= (1-\rho)\,\frac{\dfrac{a}{a+s^{\tilde{\alpha}}}}{1 - \rho\hat{f}_I(s)\,\dfrac{a}{a+s^{\tilde{\alpha}}}}\,,\end{aligned}$$

where we set

$$a = \frac{c}{\kappa} = \frac{c\cos\left(\frac{\pi K(\alpha)}{2}\right)}{\eta^{\alpha}} \tag{3.8}$$

with $K(\alpha)$ defined in (2.30). The function $\hat{u}(s) = a/(a+s^{\tilde{\alpha}})$ possesses derivatives $\hat{u}^{(n)}$ of all orders and $(-1)^n\hat{u}^{(n)}(s) \geq 0$, $s > 0$. So $\hat{u}$ is by definition completely monotone. Moreover, $\hat{u}(0) = 1$ and so the function $\hat{u}$ is the Laplace transform of a probability distribution U on $\mathbb{R}$ (cf. Feller [33, p. 439]). The following proposition characterizes the law U in terms of the density of extremal stable laws ($\beta = 1$) or the so–called Mittag–Leffler function $E_\sigma(x) = \sum_{n=0}^{\infty} x^n/\Gamma(1+\sigma n)$, $\sigma > 0$. The same proposition also contains some additional properties of the law U.

Proposition 3.3 *Let $X \sim S_{\tilde{\alpha}}(1,1,0)$ with index $0 < \tilde{\alpha} < 1$ and denote by $G(\,\cdot\,;\tilde{\alpha},1)$, $g(\,\cdot\,;\tilde{\alpha},1)$ the distribution function and density function of X, respectively. Then for any $a > 0$ one has*

(*a*) $\hat{u}(s) = \dfrac{a}{a+s^{\tilde{\alpha}}} = \displaystyle\int_0^\infty e^{-sv}u(v)\,dv$, *where* $s \geq 0$ *and*

$$u(v) = \int_0^\infty \frac{g(v/(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}};\tilde{\alpha},1)}{(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}}\,a\,e^{-ay}\,dy\ .$$

(*b*) $U(x) = \int_0^x u(v)\,dv = 1 - \sum_{n=0}^{\infty} \frac{(-a)^n}{\Gamma(1+\tilde{\alpha}n)}\, x^{\tilde{\alpha}n} = 1 - E_{\tilde{\alpha}}(-ax^{\tilde{\alpha}})$,

where $E_\sigma(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(1+\sigma n)}$, $\sigma > 0$.

(*c*) $\overline{U} = 1 - U \in RV_{-\tilde{\alpha}}$. *More precisely*

$$\overline{U}(x) \sim \frac{1}{a\,\Gamma(1-\tilde{\alpha})}\, x^{-\tilde{\alpha}} \ , \quad x \to \infty .$$

(*d*) *The Mellin transform of* U *is given by*

$$M_U(\delta) = \int_0^\infty v^\delta u(v)\,dv = \frac{\Gamma(1+\delta/\tilde{\alpha})\,\Gamma(1-\delta/\tilde{\alpha})}{a^{\delta/\tilde{\alpha}}\,\Gamma(1-\delta)} \ ,$$

where $0 \le \delta < \tilde{\alpha}$. *In particular,* U *has infinite mean.*

(*e*) *The distribution* U *is infinitely divisible.*

(*f*) *The distribution* U *is geometrically infinitely divisible, i.e. for every* p, $0 < p < 1$, *there is a distribution function* H_p *with* $H_p(0) = 0$ *such that*

$$U(x) = \sum_{j=1}^{\infty} p(1-p)^{j-1} H_p^{*j}(x) \ , \quad x \ge 0 . \tag{3.9}$$

We call $U(x) = 1 - E_{\tilde{\alpha}}(-ax^{\tilde{\alpha}})$, $x \ge 0$, a Mittag–Leffler distribution.

REMARKS.

1. The Mittag–Leffler function $E_\sigma(x) = \sum_{n=0}^{\infty} x^n/\Gamma(1+\sigma n)$, $\sigma > 0$, is a generalization of the exponential to which it reduces when $\sigma = 1$.

2. As $\tilde{\alpha}$ decreases from 1 to 0, the tails of the distribution U become heavier, see Figure 3.3. □

For the proof of Proposition 3.3 we need the following lemma.

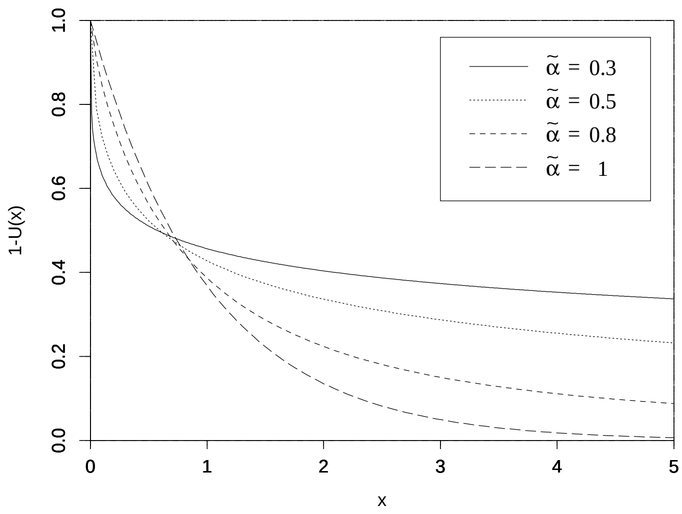

Figure 3.3: *Function $\overline{U}(x)$ for $a = 1$ and $\tilde{\alpha} \in \{0.3, 0.5, 0.8, 1\}$.*

Lemma 3.4 *Let $X \sim S_{\tilde{\alpha}}(1,1,0)$ with index $0 < \tilde{\alpha} < 1$ and denote by $g(\,\cdot\,;\tilde{\alpha},1)$ the density function of X. Then for any $a \geq 0$,*

$$\sum_{n=0}^{\infty} \frac{(-a\cos\frac{\pi\tilde{\alpha}}{2})^n}{\Gamma(1+\tilde{\alpha}n)}\, x^{\tilde{\alpha}n} = \int_0^\infty e^{-au} \left(\frac{x}{\tilde{\alpha}u} \frac{g(x/u^{1/\tilde{\alpha}};\tilde{\alpha},1)}{u^{1/\tilde{\alpha}}} \right) du\,. \tag{3.10}$$

Proof. Define a distribution function F_x as $F_x(u) = \overline{G}(x/u^{1/\tilde{\alpha}};\tilde{\alpha},1)$, $u > 0$. Notice that F_x is absolutely continuous with density $f_x(u) = x\,g(x/u^{1/\tilde{\alpha}};\tilde{\alpha},1)\,u^{-\alpha/\tilde{\alpha}}\,\tilde{\alpha}^{-1}$. Introduce the notation

$$b = a\cos\frac{\pi\tilde{\alpha}}{2}\ .$$

We show that

$$\int_0^\infty e^{-sx} \left(\sum_{n=0}^{\infty} \frac{(-b)^n}{\Gamma(1+\tilde{\alpha}n)}\, x^{\tilde{\alpha}n} \right) dx$$
$$= \int_0^\infty e^{-sx} \left(\int_0^\infty e^{-au} f_x(u)\, du \right) dx\,. \tag{3.11}$$

For the left–hand side of (3.11) we get

$$
\begin{aligned}
\int_0^\infty e^{-sx}\left(\sum_{n=0}^\infty \frac{(-b)^n}{\Gamma(1+\tilde\alpha n)}\,x^{\tilde\alpha n}\right)dx &= \sum_{n=0}^\infty \frac{(-b)^n}{\Gamma(1+\tilde\alpha n)}\int_0^\infty e^{-sx}x^{\tilde\alpha n}\,dx\\
&= \sum_{n=0}^\infty \frac{(-b)^n}{\Gamma(1+\tilde\alpha n)}\,\frac{\Gamma(1+\tilde\alpha n)}{s^{1+\tilde\alpha n}}\\
&= \frac{1}{s}\sum_{n=0}^\infty\left(\frac{-b}{s^{\tilde\alpha}}\right)^n\\
&= \frac{1}{s}\,\frac{s^{\tilde\alpha}}{b+s^{\tilde\alpha}} \qquad (3.12)\\
&= \frac{s^{\tilde\alpha-1}}{a\cos\frac{\pi\tilde\alpha}{2}+s^{\tilde\alpha}}\ .
\end{aligned}
$$

Notice that for the right–hand side of (3.11) we have

$$
\begin{aligned}
\int_0^\infty e^{-sx}\left(\int_0^\infty e^{-au}f_x(u)\,du\right)dx &= \int_0^\infty e^{-au}\,d_u\left(\int_0^\infty e^{-sx}F_x(u)\,dx\right)\\
&= \int_0^\infty e^{-au}\,d_uL(u)\ ,
\end{aligned}
$$

say. For the function L we then obtain by the definition of F_x

$$
\begin{aligned}
L(u) &= \int_0^\infty e^{-sx}\left(1-G\left(x/u^{1/\tilde\alpha};\tilde\alpha,1\right)\right)dx\\
&= \frac{1}{s}-\int_0^\infty e^{-sx}G\left(x/u^{1/\tilde\alpha};\tilde\alpha,1\right)dx\ .
\end{aligned}
$$

We first integrate by parts and then make the substitution $y=x/u^{1/\tilde\alpha}$, yielding

$$
\begin{aligned}
L(u) &= \frac{1}{s}\left(1-\int_0^\infty e^{-(su^{1/\tilde\alpha})y}g(y;\tilde\alpha,1)\,dy\right)\\
&= \frac{1}{s}\left(1-\exp\left\{-\frac{s^{\tilde\alpha}u}{\cos\frac{\pi\tilde\alpha}{2}}\right\}\right)\ . \qquad (3.13)
\end{aligned}
$$

For the last equality we used Proposition 2.25. Except of the norming factor $1/s$, the last term (3.13) represents the exponential distribution function with parameter $s^{\tilde{\alpha}}/\cos\frac{\pi\tilde{\alpha}}{2}$. Hence

$$\int_0^\infty e^{-au}\, d_u L(u) = \frac{s^{\tilde{\alpha}-1}}{a\cos\frac{\pi\tilde{\alpha}}{2} + s^{\tilde{\alpha}}} \ .$$

Since the Laplace transform uniquely determines the underlying distribution, the assertion follows. □

REMARK. There might be some confusion about the expression 'Mittag–Leffler distribution'. Some authors use the latter notation for a distribution whose Laplace transform is a Mittag–Leffler function, see Lemma 3.4. In our terminology, however, a Mittag–Leffler law is characterized by the distribution function $U(x) = 1 - E_{\tilde{\alpha}}(-ax^{\tilde{\alpha}})$, $x \geq 0$, for some $a > 0$ and $0 < \tilde{\alpha} < 1$. □

We have throughout tacitly assumed that the conditions for changing the order of integration are satisfied. Indeed, using Lebesgue's dominated convergence theorem it is not hard to verify that here and in the sequel these conditions hold. We are now able to prove Proposition 3.3.

PROOF OF PROPOSITION 3.3. (*a*) Consider $\hat{v}(s) = a/(a+s)$. Then $\hat{v}(s)$ is the Laplace transform of the exponential distribution, i.e. $\hat{v}(s) = \int_0^\infty e^{-sy} v(y)\, dy$, where $v(y) = ae^{-ay}$. Now

$$\hat{u}(s) = \frac{a}{a+s^{\tilde{\alpha}}} = \hat{v}(s^{\tilde{\alpha}}) = \int_0^\infty e^{-s^{\tilde{\alpha}} y} v(y)\, dy \ . \tag{3.14}$$

Recall from Proposition 2.25 that the Laplace transform of X exists and is given by $E[e^{-sX}] = \exp\{-s^{\tilde{\alpha}}/\cos\frac{\pi\tilde{\alpha}}{2}\}$. It follows that

$$\begin{aligned} e^{-s^{\tilde{\alpha}} y} &= \exp\left\{ -\frac{(s(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}})^{\tilde{\alpha}}}{\cos\frac{\pi\tilde{\alpha}}{2}} \right\} \\ &= \int_0^\infty \exp\left\{ -s(y\ \cos\tfrac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}\, u \right\}\, g(u;\tilde{\alpha},1)\, du \\ &= \int_0^\infty \exp\{-sv\}\, \frac{g(v/(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}};\tilde{\alpha},1)}{(y\ \cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}}\, dv \ . \end{aligned} \tag{3.15}$$

In the last equality we substituted $(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}} u = v$.

With (3.15) we obtain for (3.14)

$$
\begin{aligned}
\hat{u}(s) &= \int_0^\infty e^{-s^{\tilde{\alpha}} y} a\, e^{-ay}\, dy \\
&= \int_0^\infty \left(\int_0^\infty e^{-sv} \frac{g(v/(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}}\, dv \right) a\, e^{-ay}\, dy\ .
\end{aligned}
$$

Changing the order of integration yields

$$
\hat{u}(s) = \int_0^\infty e^{-sv} \left(\int_0^\infty \frac{g(v/(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}}\, a\, e^{-ay}\, dy \right) dv
$$

which proves (a).
(b) Straightforward calculation yields

$$
\begin{aligned}
U(x) = \int_0^x u(v)\, dv &= \int_0^x \left(\int_0^\infty \frac{g(v/(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}}\, a\, e^{-ay}\, dy \right) dv \\
&= \int_0^\infty a\, e^{-ay} \left(\int_0^x \frac{g(v/(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}}\, dv \right) dy\ .
\end{aligned}
$$

Making the change of variable $u(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}} = v$ we find

$$
\begin{aligned}
U(x) &= \int_0^\infty a\, e^{-ay} \left(\int_0^{x/(y\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}} g(u; \tilde{\alpha}, 1)\, du \right) dy \\
&= -\int_0^\infty \left(\frac{d}{dy} e^{-ay} \right) G\left(x/(y\ \cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}; \tilde{\alpha}, 1 \right) dy \\
&= 1 - \int_0^\infty e^{-ay} \Bigg(\frac{x(\cos\frac{\pi\tilde{\alpha}}{2})^{-1/\tilde{\alpha}}}{\tilde{\alpha} y} \\
&\qquad\qquad \times\ \frac{g(x(\cos\frac{\pi\tilde{\alpha}}{2})^{-1/\tilde{\alpha}}/y^{1/\tilde{\alpha}}; \tilde{\alpha}, 1)}{y^{1/\tilde{\alpha}}} \Bigg) dy \\
&= 1 - \sum_{n=0}^\infty \frac{(-a\cos\frac{\pi\tilde{\alpha}}{2})^n}{\Gamma(1+\tilde{\alpha}n)} \left(\frac{x}{(\cos\frac{\pi\tilde{\alpha}}{2})^{1/\tilde{\alpha}}} \right)^{\tilde{\alpha}n} \\
&= 1 - \sum_{n=0}^\infty \frac{(-a)^n}{\Gamma(1+\tilde{\alpha}n)}\, x^{\tilde{\alpha}n}\ .
\end{aligned}
$$

For the second–last equality we used Lemma 3.4.
(c) By (3.12) we have

$$s\int_0^\infty e^{-su}E_{\tilde\alpha}(-au^{\tilde\alpha})\,du = \frac{s^{\tilde\alpha}}{a}\,\frac{a}{a+s^{\tilde\alpha}}$$

and we conclude that

$$s\int_0^\infty e^{-su}E_{\tilde\alpha}(-au^{\tilde\alpha})\,du \sim \frac{s^{\tilde\alpha}}{a}\;, \quad s\downarrow 0\,.$$

An extension of Theorem 2.18 then yields

$$1-U(x) = E_{\tilde\alpha}(-ax^{\tilde\alpha}) \sim \frac{1}{a\,\Gamma(1-\tilde\alpha)}\,x^{-\tilde\alpha}\;, \quad x\to\infty\,.$$

(d) The proof of this statement follows from the Mellin transform of an extremal stable law. Let $0<\delta<\tilde\alpha$. Then, see Zolotarev [86, Theorem 3],

$$E\left[\,X^{\delta}\,\right] = \int_0^\infty y^{\delta} g(y;\tilde\alpha,1)\,dy = \sigma_B^{\delta}\,\frac{\Gamma(1-\delta/\tilde\alpha)}{\Gamma(1-\delta)}\;, \tag{3.16}$$

where σ_B is defined in (2.32). Again it is not hard to verify that the conditions for changing the order of integration are satisfied. We have that

$$\begin{aligned}
\int_0^\infty y^{\delta}u(y)\,dy &= \int_0^\infty y^{\delta}\left(\int_0^\infty \frac{g(y/(w\cos\frac{\pi\tilde\alpha}{2})^{1/\tilde\alpha};\tilde\alpha,1)}{(w\cos\frac{\pi\tilde\alpha}{2})^{1/\tilde\alpha}}\,a\,e^{-aw}\,dw\right)dy \\
&= \int_0^\infty ae^{-aw}\left(\int_0^\infty \frac{y^{\delta}g(y/(w\cos\frac{\pi\tilde\alpha}{2})^{1/\tilde\alpha};\tilde\alpha,1)}{(w\cos\frac{\pi\tilde\alpha}{2})^{1/\tilde\alpha}}\,dy\right)dw \\
&= \int_0^\infty ae^{-aw}(w\,\cos\tfrac{\pi\tilde\alpha}{2})^{\delta/\tilde\alpha}\left(\int_0^\infty x^{\delta}g(x;\tilde\alpha,1)\,dx\right)dw \\
&= \int_0^\infty ae^{-aw}w^{\delta/\tilde\alpha}\sigma_B^{-\delta}\left(\sigma_B^{\delta}\,\frac{\Gamma(1-\delta/\tilde\alpha)}{\Gamma(1-\delta)}\right)dw \\
&= \frac{\Gamma(1-\delta/\tilde\alpha)}{\Gamma(1-\delta)}\,\frac{\Gamma(1+\delta/\tilde\alpha)}{a^{\delta/\tilde\alpha}}\;.
\end{aligned}$$

(e) By Feller [33, Theorem 1, p. 450] a distribution $F(x)$, $x\geq 0$, is infinitely divisible if and only if its Laplace transform is of the form

$\hat{f}(s) = e^{-g(s)}$, $s \geq 0$, where $g(s)$ has a completely monotone derivative and $g(0) = 0$. Here, for $U(x)$ one has

$$\hat{u}(s) = \frac{a}{a + s^{\tilde{\alpha}}} = \exp\left\{-\log\frac{a + s^{\tilde{\alpha}}}{a}\right\}$$

and $g'(s) = s^{\tilde{\alpha}-1}\tilde{\alpha}/(a + s^{\tilde{\alpha}})$. Both $\tilde{\alpha}/(a + s^{\tilde{\alpha}})$ as well as $s^{\tilde{\alpha}-1}$ are seen to be completely monotone. Therefore, the product is completely monotone, see Feller [33, p. 441].

(f) Let $q = 1 - p$. The right–hand side of (3.9) is the distribution function of the random geometric sum $S_N = X_1 + \cdots + X_N$, where $(X_i : i \in \mathbb{N})$ is a sequence of iid random variables with common distribution H_p and N is geometrically distributed with parameter p. For the Laplace transform of S_N we obtain

$$\begin{aligned} E\left[e^{-\lambda S_N}\right] &= \sum_{j=1}^{\infty} E\left[e^{-\lambda S_N} \mid N = j\right] P[N = j] \\ &= \frac{p\hat{h}(\lambda)}{1 - q\hat{h}(\lambda)} , \end{aligned}$$

where $\hat{h}(\lambda) = E[e^{-\lambda X_i}]$. Set $H_p(x) = 1 - E_{\tilde{\alpha}}(-ax^{\tilde{\alpha}}/p)$, yielding $\hat{h}(\lambda) = a(a + p\lambda^{\tilde{\alpha}})^{-1}$. Once the dust has settled, we obtain

$$E\left[e^{-\lambda S_N}\right] = \frac{a}{a + \lambda^{\tilde{\alpha}}} = \hat{u}(\lambda) ,$$

which completes the proof. □

We now return to the calculation of the function $\Psi_Z(x)$. Recall that $\rho = \lambda\mu/c$. Hence

$$\begin{aligned} \int_0^{\infty} e^{-sx}\, d(1 - \Psi_Z(x)) &= 1 - s\int_0^{\infty} e^{-sx}\Psi_Z(x)\, dx \\ &= \frac{\gamma s}{\xi(s)} \\ &= (1 - \rho)\frac{\hat{u}(s)}{1 - \rho\hat{f}_I(s)\,\hat{u}(s)} \\ &= (1 - \rho)\hat{u}(s)\sum_{n=0}^{\infty}\left(\rho\hat{f}_I(s)\,\hat{u}(s)\right)^n . \end{aligned}$$

Inverting the last expression yields

$$1 - \Psi_Z(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n \big(F_I^{*n} * U^{*(n+1)}\big)(x) \ .$$

We summarize our result in the following theorem.

Theorem 3.5 *Consider a classical risk process perturbed by α–stable Lévy motion with $1 < \alpha < 2$ and skewness parameter $\beta = -1$,*

$$R_Z(t) = x + ct - \sum_{k=1}^{N(t)} Y_k + \eta Z_\alpha(t) \ , \qquad t \geq 0 \, ,$$

where $x > 0$, $\eta > 0$, $c = (1+\theta)\lambda\mu$. $(N(t) : t \geq 0)$ is a homogeneous Poisson process with intensity λ, $(Y_k : k \in \mathbb{N})$ is a sequence of iid random variables with distribution function F on $[0, \infty)$ and mean μ. Then the probability of ruin $\Psi_Z(x)$ defined in (3.5) satisfies

$$1 - \Psi_Z(x) = (1-\rho) \sum_{n=0}^{\infty} \rho^n \big(F_I^{*n} * U^{*(n+1)}\big)(x) \ , \tag{3.17}$$

where $\rho = \lambda\mu/c$, $F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)\,dy$ and $\overline{U}(x) = \sum_{n=0}^{\infty} \frac{(-a)^n}{\Gamma(1+\tilde{\alpha}n)}\, x^{\tilde{\alpha}n}$ with $\tilde{\alpha} = \alpha - 1$ and $a = c\cos(\pi K(\alpha)/2)/\eta^\alpha$, $K(\alpha) = \alpha - 2$.

□

REMARKS.

1. Formula (3.17) generalizes formula (3.4) of Dufresne and Gerber [21] to which it reduces when $\alpha = 2$. In that case $U(x) = 1 - e^{-ax}$ is the distribution function of the exponential law.

2. The survival probability $\phi(x) := 1 - \Psi_Z(x)$ in (3.17) satisfies the renewal equation $\phi = z + H * \phi$, where $z(x) = (1-\rho)U(x)$ and $H(x) = \rho(F_I * U)(x)$, i.e.

$$\phi(x) = (1-\rho)U(x) + \int_0^x \phi(x-y)(\rho f_I * u)(y)\,dy \ . \tag{3.18}$$

   The renewal equation (3.18) is defective since, under the net profit condition $c - \lambda\mu > 0$, $\lim_{x\to\infty} H(x) = \rho < 1$. The solution of a renewal equation $\phi = z + H * \phi$ has the form $\phi = M * z$, where $M(dx) = \sum_0^\infty H^{*n}(dx)$ denotes the renewal measure; see for example Feller [33, Theorem 1, p. 185]. Straightforward computation then shows that the solution of (3.18) is given by (3.17).

3. Intuitively we expect that the ruin probability is a decreasing function of α. The smaller the value of α, the more "dramatic" the stable Lévy motion behaves. Let $1 < \alpha_1 \leq \alpha_2 < 2$. It is tempting to conjecture that for fixed $x > 0$ the following inequality holds

$$\Psi_Z(\alpha_1, x) \geq \Psi_Z(\alpha_2, x) \ . \tag{3.19}$$

   However, the question is still open whether or not (3.19) is true.

4. It is worthwhile considering the important special case $\lambda = 0$. The process $(R_Z(t) : t \geq 0)$ then reduces to a stable Lévy motion with positive linear drift c. By virtue of (3.17), the probability that such a process ever attains the level 0 is given by

$$\begin{aligned} \Psi_Z(x) &= \sum_{n=0}^{\infty} \frac{(-a)^n}{\Gamma(1+\tilde{\alpha}n)} \, x^{\tilde{\alpha}n} \\ &= E_{\tilde{\alpha}}(-a x^{\tilde{\alpha}}) \ , \end{aligned} \tag{3.20}$$

   where $a = c/\kappa$ and κ is given in (3.7). Whereas drift–free stable processes and functionals related to them have been studied extensively in the literature, there is much less written on extremal stable processes with linear drift. It seemed not possible to compute explicitly the distribution of the infimum of such processes; see Port [65] for some background. Formula (3.20) is thus interesting from a theoretical point of view. It closes that gap and allows in a simple and closed form to calculate the distribution of the infimum. On the other hand the interest of relation (3.20) is also due to a practical problem. In the next chapter we study weak approximations in risk theory when the claim process allows for heavy–tailed claims. The resulting process is stable Lévy motion $(\beta = -1)$ with positive drift. We show that (3.20) can be interpreted as an approximation for the infinite–time ruin probability of a risk process with infinite claim size variance. □

A stable process Z_α $(1 < \alpha < 2)$ with linear drift and skewness parameter $|\beta| < 1$ is no longer extremal in the sense that there are only jumps in one direction. An explicit formula for the distribution of the infimum might not exist. The following unpublished result is due to Samorodnitsky [70].

Proposition 3.6 *Let Z_α be α–stable Lévy motion, $1 < \alpha < 2$, $|\beta| \le 1$. Then for positive numbers x, c and η one has*

$$P\left[\inf_{t\ge 0}(x + ct + \eta Z_\alpha(t)) < 0\right] \le C\, x^{-\tilde{\alpha}} \ ,$$

where $\tilde{\alpha} = \alpha - 1$ and C denotes some positive constant.

Proof. We have

$$\begin{aligned} &P[x + ct + \eta Z_\alpha(t) < 0 \text{ for some } t \ge 0\,] \\ &\qquad \le P[x + ct + \eta Z_\alpha(t) < 0 \text{ for some } 0 \le t \le x\,] \\ &\qquad\quad + \sum_{k=1}^{\infty} P\big[\,x + ct + \eta Z_\alpha(t) < 0\,,\, x2^{k-1} \le t \le x2^k\,\big] \\ &\qquad = P_1 + \sum_{k=1}^{\infty} P_2(k)\ , \end{aligned}$$

say. For the first summand one has

$$P_1 \le P\left[\sup_{0\le t\le x}(-\eta Z_\alpha(t)) > x\right] \le \vartheta^{-1} P[\,-Z_\alpha(x) > x/\eta\,]\ ,$$

see Lemma 4.15, where $\vartheta = P[\,Z_\alpha(t) < 0\,]$. Denote by $G(\,\cdot\,;\alpha,-\beta)$ the distribution function of the stable random variable $X \sim S_\alpha(1,-\beta,0)$. From the scaling property (3.3) and Proposition 2.22 we then obtain

$$\begin{aligned} P_1 &\le \vartheta^{-1}\, P\big[\,-Z_\alpha(1) > x^{\tilde{\alpha}/\alpha}/\eta\big] = \vartheta^{-1}\, \overline{G}\big(x^{\tilde{\alpha}/\alpha}/\eta\,;\alpha,-\beta\big) \\ &\le \vartheta^{-1} C_1\, x^{-\tilde{\alpha}} \end{aligned}$$

for some constant C_1. Similarly, for all $k \ge 1$,

$$P_2(k) \le P\big[x + cx2^{k-1} + \eta Z_\alpha(t) < 0 \text{ for some } 0 \le t \le x2^k\,\big]$$

$$
\begin{aligned}
&\le P\big[\sup\nolimits_{0\le t\le x2^k}(-\eta Z_\alpha(t)) > x(1+c2^{k-1})\big]\\
&\le \vartheta^{-1}P\big[-\eta Z_\alpha(x2^k) > x(1+c2^{k-1})\big]\\
&= \vartheta^{-1}P\big[-Z_\alpha(1) > x^{\tilde\alpha/\alpha}(1+c2^{k-1})/(\eta 2^{k/\alpha})\big]\\
&\le \vartheta^{-1}C_2\,\eta^\alpha\,\frac{2^k}{(1+c2^{k-1})^\alpha}\,x^{-\tilde\alpha}\;.
\end{aligned}
$$

Summing over k yields $\sum_{k=1}^\infty P_2(k) \le C_3\,x^{-\tilde\alpha}$ for some constant $C_3>0$ and, since $\alpha>1$, implying that the series is convergent. The assertion now follows. □

Port [65, Theorem 9] derives a stronger result; see Proposition 3.7 below. However, his proof is less elementary.

Proposition 3.7 *Let Z_α denote α–stable Lévy motion, $1<\alpha<2$, $-1\le\beta<1$. Then for positive numbers c and η*

$$
P\left[\inf_{t\ge 0}(x+ct+\eta Z_\alpha(t))<0\right] \sim \frac{A(\alpha,\beta)\eta^\alpha}{\alpha\tilde\alpha c}\,x^{-\tilde\alpha}\;,\quad x\to\infty\,, \tag{3.21}
$$

where $\tilde\alpha=\alpha-1$ and

$$
A(\alpha,\beta)=
$$

$$
\frac{\Gamma(1+\alpha)}{\pi}\sqrt{1+\beta^2\tan^2(\frac{\pi\alpha}{2})}\,\sin\left(\frac{\pi\alpha}{2}+\arctan\left(-\beta\tan(\frac{\pi\alpha}{2})\right)\right)\;.
$$

□

Remarks.

1. Note that the right–hand side of (3.21) is inversely proportional to the drift c.
2. In the special case $\beta=-1$ we have seen that

$$
P\left[\inf_{t\ge 0}(x+ct+\eta Z_\alpha(t))<0\right] = E_{\tilde\alpha}(-ax^{\tilde\alpha})\;,
$$

where $a=c\cos\left(\frac{\pi K(\alpha)}{2}\right)/\eta^\alpha$, see (3.20). From Proposition 3.3 (*c*) we know that

$$
E_{\tilde\alpha}(-ax^{\tilde\alpha}) \sim \frac{1}{a\Gamma(1-\tilde\alpha)}\,x^{-\tilde\alpha}\;,\quad x\to\infty\,. \tag{3.22}
$$

Routine computation shows that

$$A(\alpha,-1) = \frac{\alpha\tilde{\alpha}}{\cos\left(\frac{\pi K(\alpha)}{2}\right)\Gamma(1-\tilde{\alpha})}$$

and so the right–hand side of (3.21) reduces to $x^{-\tilde{\alpha}}\,(a\Gamma(1-\tilde{\alpha}))^{-1}$ in accordance with (3.22). □

A stable Lévy motion Z_α with $1 < \alpha < 2$, $\beta = 1$ and (positive) linear drift c is skip–free downwards and so the process can go to the left only in a continuous manner. The following result is added for the sake of completeness.

Proposition 3.8 *Let Z_α denote α–stable Lévy motion, $1 < \alpha < 2$, $\beta = 1$. Then for positive numbers x, c and η*

$$P\left[\inf_{t\geq 0}(x+ct+\eta Z_\alpha(t)) < 0\right] = \exp\left\{-a^{1/\tilde{\alpha}}x\right\}\ ,$$

where $\tilde{\alpha} = \alpha - 1$ and $a = c/\kappa$ with κ defined in (3.7).

Proof. Consider the process $\widetilde{X} = (\widetilde{X}(t) : t \geq 0)$, where $\widetilde{X}(t) = -ct + \eta\widetilde{Z}_\alpha(t)$ and $\widetilde{Z}_\alpha$ denotes α–stable Lévy motion with skewness parameter $\widetilde{\beta} = -1$. Then $\widetilde{X}$ is a spectrally negative Lévy process starting from 0 with characteristic exponent $\widetilde{\xi}(s) = -cs + \kappa s^\alpha$, $\kappa = \eta^\alpha/\cos(\pi K(\alpha)/2)$. It follows that

$$P\left[\inf_{t\geq 0}(x+ct+\eta Z_\alpha(t)) < 0\right] = P\left[f\widetilde{X}(x) < \infty\right]\ ,$$

where the first passage functional $f\widetilde{X}$ is defined in (2.6). From (2.9) we have that

$$E\left[e^{-\sigma(f\widetilde{X})(x)}\right] = e^{-\tilde{\eta}(\sigma)x}\ .$$

Taking the limit as $\sigma \downarrow 0$, we get

$$P\left[f\widetilde{X}(x) < \infty\right] = e^{-\tilde{\eta}(0)x}\ ,$$

where $\widetilde{\eta}(0)$ is the positive root of the equation $\widetilde{\xi}(s) = -cs + \kappa s^\alpha = 0$, yielding $\widetilde{\eta}(0) = (c/\kappa)^{1/\tilde{\alpha}}$. □

3.2 Asymptotic Behaviour of the Ruin Probability

The purpose of this section is the investigation of the asymptotic behaviour of the ruin probability $\Psi_Z(x)$ defined in (3.5) as x becomes large. Notice that we can write $\phi(x) := 1 - \Psi_Z(x) = K * U(x)$, where K is the distribution function of the random geometric sum $X_1 + \cdots + X_N$, all X_i having distribution $F_I * U$. Roughly speaking the tail behaviour of ϕ is then related to that of F_I and/or U. Intuitively we can think of a balance, putting on each scale the tails of F_I and U, respectively. Both weights then contribute to the tail behaviour of ϕ when holding in equilibrium; see Theorem 3.9. If the mass of one tail exceeds the one of the other, the equilibrium is disturbed and it is solely the dominant distribution that affects the asymptotic behaviour of ϕ. See Theorem 3.11 when the tail of F_I dominates and Theorem 3.12 when the perturbation U is the relevant quantity.
In the following we shall carry out the above heuristic reasoning in more mathematical detail. We start with the case when the tail decays of F_I and U are of the same order, that is to say when $\overline{F_I} \in RV_{-\tilde{\alpha}}$, where $\tilde{\alpha} = \alpha - 1$ and α equals the index of stability of the stable Lévy motion.

3.2.1 $\overline{F} \in RV_{-\alpha}$

First observe that by Karamata's Theorem 2.17 the conditions $\overline{F} \in RV_{-\alpha}$ and $\overline{F_I} \in RV_{-\tilde{\alpha}}$ are equivalent.

Theorem 3.9 *Let $\Psi_Z(x)$ denote the ruin probability defined in* (3.5). *Suppose that $1 < \alpha < 2$ and $\overline{F} \in RV_{-\alpha}$, i.e. $\overline{F}(x) = x^{-\alpha} L(x)$ for some slowly varying function L. Then*

$$\begin{aligned}\Psi_Z(x) &\sim \frac{1}{\theta}\,\overline{F_I}(x) + \frac{\kappa}{\gamma\,\Gamma(1-\tilde{\alpha})}\,x^{-\tilde{\alpha}} \\ &\sim \left(\frac{L(x)}{\theta\mu\tilde{\alpha}} + \frac{\kappa}{\gamma\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}}\,, \quad x \to \infty\,,\end{aligned} \tag{3.23}$$

where $\gamma = E[R_Z(1)] = c - \lambda\mu > 0$, $\tilde{\alpha} = \alpha - 1$ and $\kappa = \eta^{\alpha} / \cos\left(\frac{\pi K(\alpha)}{2}\right)$.

REMARKS.

1. When $\alpha \nearrow 2$, the second summand in (3.23) tends to 0 and hence $\Psi_Z(x) \sim \overline{F}_I(x)/\theta$, a result which can be found in Veraverbeke [81] for distribution functions $F_I \in \mathcal{S}$.

2. Whereas monotonicity in α for $\Psi_Z(x)$ is still an open problem, the desired property holds for the above asymptotic form for $\Psi_Z(x)$. Define $W(\alpha, x) = (L(x)(\theta\mu\tilde{\alpha})^{-1} + \kappa(\Gamma(1-\tilde{\alpha})\gamma)^{-1})\, x^{-\tilde{\alpha}}$. Then for $1 < \alpha_1 \le \alpha_2 < 2$ and $x \ge 1$ one has $W(\alpha_1, x) \ge W(\alpha_2, x)$. □

PROOF OF THEOREM 3.9. With $\phi = 1 - \Psi_Z$ we can write

$$\begin{aligned}\phi &= (1-\rho)\sum_{n=0}^{\infty} \rho^n \left(F_I^{*n} * U^{*(n+1)}\right) \\ &= (1-\rho)\sum_{n=0}^{\infty} \rho^n H^{*n} * U = K * U \ ,\end{aligned}$$

where $H = F_I * U$ and $K = (1-\rho)\sum_{n=0}^{\infty} \rho^n H^{*n}$. Since $\overline{F} \in RV_{-\alpha}$, we conclude from Karamata's Theorem 2.17 that the tail of F_I behaves as

$$\overline{F}_I(x) \sim \frac{L(x)}{\tilde{\alpha}\mu}\, x^{-\tilde{\alpha}} \ , \quad x \to \infty \, . \tag{3.24}$$

From Proposition 2.13 and Proposition 3.3 (c) we obtain

$$\overline{H}(x) = 1 - F_I * U(x) \sim \left(\frac{L(x)}{\tilde{\alpha}\mu} + \frac{1}{a\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}} \ , \quad x \to \infty \, ,$$

and therefore $H \in \mathcal{S}$. Consequently, by virtue of Proposition 2.15, $\lim_{x\to\infty} \overline{K}(x)/\overline{H}(x) = \rho/(1-\rho)$ or

$$\begin{aligned}\overline{K}(x) &\sim \left(\frac{\rho}{1-\rho}\right)\left(\frac{L(x)}{\tilde{\alpha}\mu} + \frac{1}{a\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}} \\ &= A(x)\, x^{-\tilde{\alpha}} \ , \quad x \to \infty \, ,\end{aligned}$$

say. Finally,

$$\Psi_Z(x) = 1 - K * U(x)$$

$$
\begin{aligned}
&\sim \left(A(x) + \frac{1}{a\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}} \\
&= \left(\left(\frac{\rho}{1-\rho}\right)\frac{L(x)}{\tilde{\alpha}\mu} + \frac{\kappa}{\gamma\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}}
\end{aligned}
$$

or equivalently, because of (3.24),

$$
\Psi_Z(x) \sim \frac{1}{\theta}\,\overline{F}_I(x) + \frac{\kappa}{\gamma\,\Gamma(1-\tilde{\alpha})}\,x^{-\tilde{\alpha}}\ , \quad x \to \infty\,.
$$

□

The following corollary is an immediate consequence of Theorem 3.9. We consider the case where $\overline{F} \in RV_{-\alpha}$ and assume that the slowly varying function L tends to a finite positive constant as $x \to \infty$. Typical examples where this condition is fulfilled are Pareto distributed claims.

Corollary 3.10 *Suppose that* $\overline{F}(x) = x^{-\alpha}L(x)$, $1 < \alpha < 2$ *and* $\lim_{x\to\infty} L(x) = \ell_0$ *with* $0 < \ell_0 < \infty$. *Then*

$$
\begin{aligned}
\Psi_Z(x) &\sim \left(\frac{\ell_0}{\theta\tilde{\alpha}\mu} + \frac{\kappa}{\gamma\,\Gamma(1-\tilde{\alpha})}\right) x^{-\tilde{\alpha}} \\
&= \frac{\kappa(1+\rho b)}{\gamma\,\Gamma(1-\tilde{\alpha})}\,x^{-\tilde{\alpha}}\ , \quad x \to \infty\,,
\end{aligned}
$$

where $\rho = \lambda\mu/c$, $b = \lim_{x\to\infty} \overline{F}_I(x)/\overline{U}(x) = \ell_0 a\Gamma(1-\tilde{\alpha})/(\tilde{\alpha}\mu)$. □

In the proof of Theorem 3.9 we essentially make use of the explicitness of the functions ϕ and U. However, the same result can be obtained in a more general way. The key step in the following approach is that the Lévy measure of a stable process is regularly varying with index α. We claim that the Laplace exponent $\xi(s)$ allows the following representation (making the spectral negativity of the process $(R_Z(t) - x : t \geq 0)$ more transparent):

$$
\begin{aligned}
\xi(s) &= cs + \int_{-\infty}^{0} (e^{su} - 1)\,\Pi_1(du) + \int_{-\infty}^{0} (e^{su} - 1 - su)\,\Pi_2(du) \\
&= cs + I_1(s) + I_2(s)\ ,
\end{aligned}
$$

say. Indeed, setting $\Pi_1(u,0) = \lambda(1 - F(-u))$, $u \le 0$, one gets $I_1(s) = -\lambda\mu s \hat{f}_I(s)$. Recall from (3.7) the definition of κ and define Π_2 as

$$\Pi_2(du) = \frac{q}{|u|^{1+\alpha}}\,\mathbb{I}_{(-\infty,0)}(u)\,du\ , \quad \text{where} \quad q = \frac{\kappa\alpha\tilde{\alpha}}{\Gamma(1-\tilde{\alpha})}\ .$$

We then obtain by integration by parts $I_2(s) = \kappa s^\alpha$ and indeed the above representation of $\xi(s)$ coincides with (3.7). Note that $-\lambda\mu = \int_{-\infty}^0 u\,\Pi_1(du)$. Set $\Pi = \Pi_1 + \Pi_2$ and keep in mind that under the net profit condition $\gamma = c - \lambda\mu > 0$. Introduce the notation $\hat{h}(s) = \int_0^\infty e^{-sx}h(x)\,dx$, where $h(x) = \int_{-\infty}^{-x}\Pi(y)\,dy$. Consequently,

$$\begin{aligned}\xi(s) &= \gamma s + \int_{-\infty}^0 (e^{su} - 1 - su)\,\Pi(du)\\ &= \gamma s + s^2\hat{h}(s)\ ,\end{aligned}$$

where the last equality follows by two–fold integration by parts. With this we obtain

$$1 - \frac{\gamma s}{\xi(s)} = \frac{s\hat{h}(s)}{\gamma + s\hat{h}(s)}\ .$$

Multiply both sides by $\gamma/(s\hat{h}(s))$ and then letting $s \downarrow 0$ yields

$$\frac{\gamma}{s\hat{h}(s)}\left(1 - \frac{\gamma s}{\xi(s)}\right) = \frac{1}{1 + s\hat{h}(s)/\gamma} \to 1\ , \quad s \downarrow 0\ ,$$

which shows that

$$1 - \frac{\gamma s}{\xi(s)} \sim \frac{1}{\gamma}\,s\hat{h}(s)\ , \quad s \downarrow 0\ .$$

Therefore

$$\begin{aligned} s\int_0^\infty e^{-sx}\Psi_Z(x)\,dx &= 1 - \frac{\gamma s}{\xi(s)}\\ &\sim \frac{1}{\gamma}\,s\hat{h}(s)\,, \quad s \downarrow 0\\ &= \frac{1}{\gamma}\,s\int_0^\infty e^{-sx}\left(\int_{-\infty}^{-x}\Pi(y)\,dy\right)\,dx\ .\end{aligned}$$

Because of the definition of Π_2 and the assumption on F it follows that the function Π belongs to the class $RV_{-\tilde{\alpha}}$. Applying Theorem 2.18 then yields

$$\begin{aligned}\Psi_Z(x) &\sim \frac{1}{\gamma}\int_{-\infty}^{-x}\Pi(y)\,dy\ , \quad x\to\infty\\ &= \frac{1}{\gamma}\int_x^{\infty}\lambda\overline{F}(y)\,dy + \frac{1}{\gamma}\int_x^{\infty}\frac{q}{\alpha y^{\alpha}}\,dy\\ &= \frac{1}{\theta}\,\overline{F_I}(x) + \frac{\kappa}{\gamma\,\Gamma(1-\tilde{\alpha})}\,x^{-\tilde{\alpha}}\\ &\sim \left(\frac{1}{\theta}\,\frac{L(x)}{\tilde{\alpha}\mu} + \frac{\kappa}{\gamma\,\Gamma(1-\tilde{\alpha})}\right)x^{-\tilde{\alpha}}\ , \quad x\to\infty\,.\end{aligned}$$

3.2.2 $F_I \in \mathcal{S}$, $\overline{U}(x) = o(\overline{F_I}(x))$

We next consider the case where $F_I \in \mathcal{S}$ and $\overline{U}(x) = o(\overline{F_I}(x))$. The above conditions are satisfied for instance when $\overline{F} \in RV_{-\delta}$ with $\delta < \alpha$.

Theorem 3.11 *Suppose that $F_I \in \mathcal{S}$ and that $\overline{U}(x) = o(\overline{F_I}(x))$. Then*

$$\Psi_Z(x) \sim \frac{1}{\theta}\,\overline{F_I}(x)\ , \quad x\to\infty\,.$$

Remark. The above asymptotic version of $\Psi_Z(x)$ is independent of the perturbation Z_α. □

Proof. Again we can write $\phi = K * U$ with $K = (1-\rho)\sum_{n=0}^{\infty}\rho^n H^{*n}$ and $H = F_I * U$. From the assumptions and Propositions 2.14 and 2.15 we conclude that $H \in \mathcal{S}$, $\overline{H} \sim \overline{F_I}$ and $K \in \mathcal{S}$. Moreover,

$$\overline{K}(x) \sim \frac{1}{\theta}\,\overline{F_I}(x)\ , \quad x\to\infty\,.$$

Consequently, since $\overline{U}(x) = o(\overline{F_I}(x))$, we have $\overline{U}(x) = o(\overline{K}(x))$. Together with $K \in \mathcal{S}$ we conclude that $\phi = K * U \in \mathcal{S}$ and $\Psi_Z(x) = \overline{\phi}(x) \sim \overline{K}(x)$ as $x\to\infty$ (Proposition 2.14) which ends the proof. □

One may also consider the case where $\overline{F_I}(x) = o(\overline{U}(x))$, which means that the perturbation law has heavier tails than the claim size law. This condition is fulfilled for instance when the claim size distribution F is exponential or when $\overline{F} \in RV_\delta$ with $\delta > \alpha$. However, from a modelling point of view, this assumption may not be very relevant. The following theorem is added for the sake of completeness. The proof is based on the same arguments as the proof of Theorem 3.11.

Theorem 3.12 *Assume that* $\overline{F}_I(x) = o(\overline{U}(x))$. *Then one has*

$$\Psi_Z(x) \sim \frac{\kappa}{\gamma\,\Gamma(1-\tilde{\alpha})}\, x^{-\tilde{\alpha}}\,, \quad x \to \infty\,.$$

□

Chapter 4

Weak Convergence of Risk Processes

In this chapter we will consider weak approximations of a risk process by an α–stable Lévy motion with linear drift. The concept of diffusion approximations in risk theory originates from Hadwiger [46] who compared a discrete–time risk process with diffusion. This can be viewed, though theoretically not comparable with the modern approach, as the first treatment of diffusion approximations in risk theory. A more modern version, based on weak convergence, is due to Iglehart [50]. The idea here is to let the number of claims grow in a unit time interval and to make the claim sizes smaller in such a way that the risk process converges weakly to a diffusion. While the classical theory of diffusion approximation (as treated for instance in Schmidli [72]) requires short–tailed claims, our approximations are especially relevant whenever we have heavy–tailed claims. We will define a sequence of risk processes and show that under certain conditions they converge weakly to an α–stable Lévy motion with drift. We illustrate the convergence when the claim arrival process is a renewal process or a Pólya process. In Section 4.1 we investigate the convergence of functionals of the risk process. It will be shown that the finite–time passage probabilities converge. We derive upper bounds for the hitting time probabilities of the limiting process. Section 4.1 is closed with a numerical example where we compare the 'exact' finite–time ruin probabilities of a classical risk process with the hitting time probabilities of the corresponding stable Lévy motion. The values in Tables 4.1 and 4.2 indicate that stable processes work reasonably well as approximations to risk processes in the context of ruin type problems. In Section 4.2 we concern ourselves with the convergence of the infinite–time ruin probabilities. For this purpose we make an excursion to queuing theory.

The Donsker invariance principle (see for instance Billingsley [10, Theorem 10.1]) for a sequence of iid random variables with finite variance is fundamental to many applications, most often in conjunction with the continuous mapping theorem. It explains why Brownian motion can be taken as a reasonable approximation to many real processes which are in some sense related to sums of independent random variables. In the regime of heavy–tailedness, the analogous powerful result is a *stable* functional central limit theorem: Suppose that $(Y_k : k \in \mathbb{N})$ is a sequence of iid random variables with common distribution $F \in \mathcal{D}(\alpha, \beta)$, $1 < \alpha < 2$, i.e.

$$\frac{1}{\varphi(n)} \sum_{k=1}^{n} (Y_k - \mu) \Rightarrow Y \ , \qquad n \to \infty \, , \tag{4.1}$$

where $Y \sim S_\alpha(1, \beta, 0)$ and $\varphi(n) = n^{1/\alpha} L(n)$ for an appropriate slowly varying function L. Then, for $0 \le t \le 1$,

$$\frac{1}{\varphi(n)} \sum_{k=1}^{[nt]} (Y_k - \mu) \Rightarrow Z_\alpha(t) \ , \qquad n \to \infty \, ,$$

where Z_α denotes α–stable Lévy motion with index α and skewness parameter β. Moreover, $Z_\alpha(1) \stackrel{d}{=} Y$. For the notion of weak convergence in the space $\mathbb{D}$, see Definition 4.1 below.

Definition 4.1 *A sequence* $(X^{(n)} : n \in \mathbb{N})$ *of stochastic processes in* $\mathbb{D} = \mathbb{D}[0, \infty)$ *is said to converge weakly in the Skorokhod* J_1*–topology to a stochastic process* X *if for every bounded continuous functional* f *on* $\mathbb{D}$ *it follows that*

$$\lim_{n \to \infty} E\big[f(X^{(n)})\big] = E[f(X)] \ .$$

In this case one writes $X^{(n)} \Rightarrow X$, $n \to \infty$.

The J_1–metric for càdlàg functions on $[0,1]$ was introduced by Skorokhod [74]. It extends the uniform metric to applications where the jumps of the limiting process are not constrained to lie in a fixed countable subset of $\mathbb{R}$. The main difference between convergence in the uniform sense and convergence in the J_1–sense appears at the discontinuity points of a limit function. If $(x_n : n \in \mathbb{N})$ converges uniformly on compacta to x and if x has a jump at t_0, then each x_n, for large n, must have a jump of almost the same magnitude *precisely* at t_0. Skorokhod's metric still forces x_n to have a jump of almost the same magnitude, but not precisely at t_0. The extension of the Skorokhod J_1–topology to $\mathbb{D} = \mathbb{D}[0,\infty)$ has been given by Stone [77], see also Lindvall [56].
We also use the following mode of convergence:

Definition 4.2 *A sequence $(X^{(n)} : n \in \mathbb{N})$ of stochastic processes in $\mathbb{D}$ is said to converge in probability in the Skorokhod J_1–topology to a stochastic process X if for all $\varepsilon > 0$*

$$\lim_{n\to\infty} P\left[d\big(X^{(n)}, X\big) \geq \varepsilon \right] = 0 \ ,$$

where d denotes the Skorokhod J_1–metric. We write $X^{(n)} \xrightarrow{P} X$.

If a is a constant–valued random element then $X^{(n)} \xrightarrow{P} a$ if and only if $X^{(n)} \Rightarrow a$. Notice that if $X^{(n)} \Rightarrow X$, $Y^{(n)} \xrightarrow{P} 0$ then $X^{(n)} + Y^{(n)} \Rightarrow X$; see Billingsley [10, p. 28].

We define a sequence $(R^{(n)} : n \in \mathbb{N})$ of risk processes as follows. For every $n \in \mathbb{N}$ let $x^{(n)} > 0$ denote the initial risk reserve, $c^{(n)} > 0$ the premium rate and $N^{(n)}$ the corresponding point process of claim arrivals. The claim sizes are denoted by $(Y_k^{(n)} : k \in \mathbb{N})$. So

$$R^{(n)}(t) = x^{(n)} + c^{(n)} t - \sum_{k=1}^{N^{(n)}(t)} Y_k^{(n)} \ . \tag{4.2}$$

In what follows we assume that the claims are of the form $Y_k^{(n)} = Y_k/\varphi(n)$, where $(Y_k : k \in \mathbb{N})$ is a sequence of iid random variables

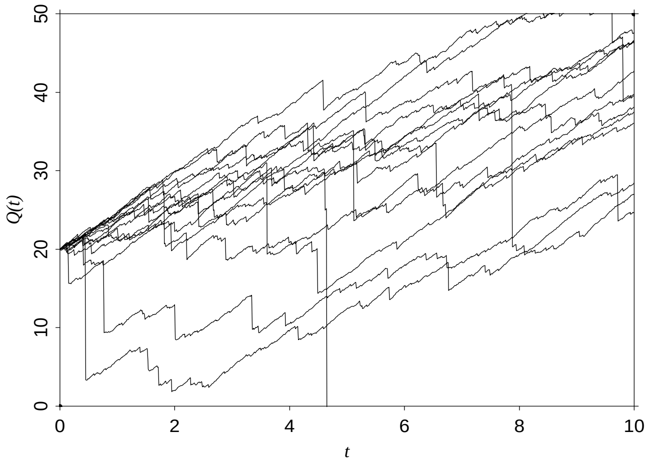

Figure 4.1: *Simulations of the process* $Q = (Q(t) : t \geq 0)$, *where* $Q(t) = x + ct - \lambda^{1/\alpha} Z_\alpha(t)$ *with* $x = 20$, $c = 5$, $\lambda = 1$, $\alpha = 1.2$, $\beta = 1$.

with common distribution function $F \in \mathcal{D}(\alpha, \beta)$ and mean μ such that (4.1) holds. Because $\alpha = 2$ leads to the well–known Brownian diffusion approximation, we shall restrict ourselves to the case $1 < \alpha < 2$. This means that the random variables Y_k do not have finite second moments and their distribution exhibits long–tail behaviour. The condition $\alpha > 1$ guarantees a finite mean of Y, see Corollary 2.23.

Consider a process $Q = (Q(t) : t \geq 0)$ given by

$$Q(t) = x + ct - \lambda^{1/\alpha} Z_\alpha(t) \ , \tag{4.3}$$

where x and c are positive numbers and $(Z_\alpha(t) : t \geq 0)$ is α–stable Lévy motion with $Z_\alpha(1) \stackrel{d}{=} Y \sim S_\alpha(1, \beta, 0)$. Here λ is some positive constant which will be specified in the following theorem. Observe that the process in (4.3) contrary to the Brownian case ($\alpha = 2$) exhibits jumps. For some simulated sample paths of Q, see Figure 4.1. We now show that the sequence $(R^{(n)} : n \in \mathbb{N})$ under certain conditions converges weakly to Q.

Theorem 4.3 *Let* $(Y_k : k \in \mathbb{N})$ *be a sequence of iid random variables with common distribution function* $F \in \mathcal{D}(\alpha, \beta)$ *and let* $(N^{(n)} : n \in \mathbb{N})$ *be a sequence of point processes such that*

$$\frac{N^{(n)}(t) - \lambda n t}{\varphi(n)} \xrightarrow{P} 0\,, \quad n \to \infty\,, \tag{4.4}$$

for some positive constant λ. *Assume also that*

$$\lim_{n\to\infty} \left(c^{(n)} - \lambda n \frac{\mu}{\varphi(n)} \right) = c\,, \qquad \lim_{n\to\infty} x^{(n)} = x\,.$$

Then, as $n \to \infty$,

$$\begin{aligned} R^{(n)}(t) &= x^{(n)} + c^{(n)} t - \frac{1}{\varphi(n)} \sum_{k=1}^{N^{(n)}(t)} Y_k \\ &\Rightarrow Q(t) = x + ct - \lambda^{1/\alpha} Z_\alpha(t) \end{aligned} \tag{4.5}$$

in the Skorokhod J_1*–topology.*

In order to prove Theorem 4.3 we will need the following theorem on random time transformation, see Whitt [84], where it is shown that the composition mapping is continuous. See also Durrett and Resnick [25] for some background.

Proposition 4.4 *Let* $(Z_n : n \in \mathbb{N})$, Z *be processes in* $\mathbb{D}$ *and suppose that* $Z_n \Rightarrow Z$. *Let* $(N_n : n \in \mathbb{N})$ *be a sequence of processes with non-decreasing sample paths starting from 0 such that* $N_n/n \Rightarrow \Lambda I$, *where* I *denotes the identity map on* $[0, \infty)$ *and* Λ *is a random variable with* $P[0 < \Lambda < \infty] = 1$. *For each* $n \in \mathbb{N}$, Z_n *and* N_n *are assumed to be defined on the same probability space. Then* $\psi(Z_n, N_n/n) \Rightarrow \psi(Z, \Lambda)$, *where* $\psi(f, g) = f \circ g$ *denotes the composition mapping.* $\square$

We are now able to prove Theorem 4.3.

Proof of Theorem 4.3. Let us write $R^{(n)}(t)$ in the following form

$$R^{(n)}(t) = x^{(n)} + c^{(n)} t - \frac{1}{\varphi(n)} \sum_{k=1}^{N^{(n)}(t)} Y_k$$

$$
\begin{aligned}
&= x^{(n)} + t\left(c^{(n)} - \lambda n \frac{\mu}{\varphi(n)}\right) - \mu\left(\frac{N^{(n)}(t) - \lambda n t}{\varphi(n)}\right) \\
&\quad - \frac{1}{\varphi(n)} \sum_{k=1}^{N^{(n)}(t)} (Y_k - \mu) \ .
\end{aligned}
$$

From assumption (4.4) we obtain

$$
\mu\left(\frac{N^{(n)}(t) - \lambda n t}{\varphi(n)}\right) \xrightarrow{P} 0\,, \quad n \to \infty\,.
$$

From (4.1) we deduce that

$$
Z_n(t) = \frac{1}{\varphi(n)} \sum_{k=1}^{[nt]} (Y_k - \mu) \Rightarrow Z_\alpha(t)\,, \quad n \to \infty\,, \tag{4.6}
$$

see Skorokhod [75]. Now $1/\varphi(n) \sum_{k=1}^{N^{(n)}(t)} (Y_k - \mu)$ is a random time transformation of Z_n, so we conclude from (4.4), (4.6) and Proposition 4.4 with the constant–valued random variable $\lambda > 0$ that

$$
\begin{aligned}
\frac{1}{\varphi(n)} \sum_{k=1}^{N^{(n)}(t)} (Y_k - \mu) = Z_n\left(\frac{N^{(n)}(t)}{n}\right) &\Rightarrow Z_\alpha(\lambda t)\,, \qquad n \to \infty\,, \\
&\stackrel{d}{=} \lambda^{1/\alpha} Z_\alpha(t)\,.
\end{aligned}
$$

In the last equality we used the self–similarity property (3.3) of α–stable Lévy motions. Because

$$
x^{(n)} + t\left(c^{(n)} - \lambda n \frac{\mu}{\varphi(n)}\right) - \mu\left(\frac{N^{(n)}(t) - \lambda n t}{\varphi(n)}\right) \xrightarrow{P} x + ct
$$

the proof is complete. □

The previous theorem shows that the distribution of a risk process in the long–tailed case can be approximated by the distribution of a process of the type (4.3). In the next sections we investigate the question of the convergence of the associated ruin probabilities.

The limit theorem $R^{(n)} \Rightarrow Q$ holds for more general point processes than a Poisson process.

(I) Renewal case

Let $N=(N(t):t\geq 0)$ be an arbitrary renewal process:

$$N(t)=\max\left\{n:\sum_{k=1}^{n}T_k\leq t\right\}\ ,$$

where the inter–occurrence times $(T_k:k\in\mathbb{N})$ are assumed to be iid positive random variables with mean $1/\lambda$ and variance σ^2. We define $N^{(n)}(t)=N(nt)$. Let $B=(B(t):t\geq 0)$ denote standard one–dimensional Brownian motion. Then

$$\frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{[nt]}\left(T_k-\frac{1}{\lambda}\right)\Rightarrow B(t)\ ,\quad n\to\infty\,,$$

which implies

$$\frac{N(nt)-\lambda nt}{\sqrt{n}}\Rightarrow\sigma\lambda^{3/2}B(t)\ ,\quad n\to\infty\,, \tag{4.7}$$

see Billingsley [10, p. 148]. Since $\alpha<2$ we conclude from (4.7) that

$$\frac{N(nt)-\lambda nt}{n^{1/\alpha}}\xrightarrow{P}0\ ,\quad n\to\infty\,.$$

(II) Mixed Poisson case

Apart from the homogeneous Poisson process the negative binomial process is often used for the claim arrival process in insurance applications. A negative binomial process or Pólya process belongs to the class of mixed Poisson processes. A recent textbook treatment of mixed Poisson processes is Grandell [44].

Definition 4.5 *Let $\widetilde{N}$ be a homogeneous Poisson process with intensity 1 and Λ a random variable with $P[\Lambda>0]=1$, independent of $\widetilde{N}$. Then the process*

$$N=\widetilde{N}\circ\Lambda=(\widetilde{N}(\Lambda t):t\geq 0)$$

is called a mixed Poisson process. The random variable Λ is called structure variable.

A mixed Poisson process has stationary increments, however the independence condition (independent increments) is violated. The stochastic variation of the claim number intensity can be interpreted as random changes of the Poisson parameter from its expected value λ. The most common choice for the distribution of the structure variable Λ is certainly the gamma distribution whose density function is given by

$$f_\Lambda(x) = \frac{\delta^\gamma}{\Gamma(\gamma)}\, x^{\gamma-1}\, e^{-\delta x} \;, \quad x \geq 0 \,. \tag{4.8}$$

We use the notation $\Lambda \sim \Gamma(\gamma, \delta)$ to indicate that the random variable Λ has a gamma distribution with density function given in (4.8).

Definition 4.6 *A mixed Poisson process N is called a negative binomial process or Pólya process if $\Lambda \sim \Gamma(\gamma, \delta)$.*

We then have for a Pólya process N

$$\begin{aligned} P[N(t) = n] &= P\big[\widetilde{N}(\Lambda t) = n\big] \\ &= \int_0^\infty P\big[\widetilde{N}(\Lambda t) = n \mid \Lambda = \lambda\big]\, f_\Lambda(\lambda)\, d\lambda \\ &= \int_0^\infty e^{-\lambda t}\, \frac{(\lambda t)^n}{n!}\, \frac{\delta^\gamma}{\Gamma(\gamma)}\, \lambda^{\gamma-1}\, e^{-\delta\lambda}\, d\lambda \\ &= \binom{\gamma+n-1}{n} \left(\frac{\delta}{\delta+t}\right)^\gamma \left(\frac{t}{\delta+t}\right)^n \,, \end{aligned}$$

i.e. $N(t)$ has a negative binomial distribution. Let N be a Pólya process with structure variable Λ. We define a sequence $N^{(n)}$ of point processes by $N^{(n)}(t) = N(nt) = \widetilde{N}(\Lambda nt)$. Since $(\widetilde{N}(nt) - nt)/\sqrt{n} \Rightarrow B(t)$, $n \to \infty$, it follows from Proposition 4.4 that

$$\frac{N^{(n)}(t) - E[\Lambda]nt}{\sqrt{n}} = \frac{\widetilde{N}(\Lambda nt) - E\big[\widetilde{N}(\Lambda nt)\big]}{\sqrt{n}} \Rightarrow B(\Lambda t) \;, \qquad n \to \infty$$

and consequently, since $\alpha < 2$,

$$\frac{N^{(n)}(t) - E[\Lambda]nt}{\varphi(n)} \xrightarrow{P} 0 \;, \qquad n \to \infty \,,$$

where $\varphi(n) = n^{1/\alpha}L(n)$ for some slowly varying function L. Hence, Theorem 4.3 holds in the mixed Poisson case with $\lambda = E[\Lambda]$. The limiting process $Q_\Lambda = (Q_\Lambda(t) : t \geq 0)$ is then given by

$$Q_\Lambda(t) = x + ct - Z_\alpha(\Lambda t) \ .$$

If $\Lambda \sim \Gamma(\gamma, \delta)$ and $\alpha = 2$ one can explicitly compute $P[\inf_{s\geq 0}(x + cs + B(\Lambda s)) < 0]$. It follows that

$$\begin{aligned}
P&\left[\inf_{s\geq 0}(x + cs + B(\Lambda s)) < 0\right] \\
&= \int_0^\infty P\left[\inf_{s\geq 0}(x + cs + B(\Lambda s)) < 0 \,\middle|\, \Lambda = \lambda\right] f_\Lambda(\lambda)\, d\lambda \\
&= \int_0^\infty e^{-2cx/\lambda}\, \frac{\delta^\gamma}{\Gamma(\gamma)}\, \lambda^{\gamma-1}\, e^{-\delta\lambda}\, d\lambda \\
&= \frac{2\,\delta^\gamma}{\Gamma(\gamma)}\, \frac{K_\gamma(\sqrt{8c\delta x})}{(\delta/(2cx))^{\gamma/2}} \ ,
\end{aligned}$$

where $K_\gamma(\omega) = 1/2 \int_0^\infty y^{\gamma-1} \exp\{-\omega(1/y + y)/2\}\, dy$ denotes the modified Bessel function of the third kind. For the second–last equality we refer to Asmussen [1, Corollary 3.5, p. 265].

4.1 Convergence of the Finite-Time Ruin Probabilities

Risk theory has traditionally been concerned with evaluating the probability that the risk reserve becomes negative. We have proposed an approximation for a risk process with infinite claim size variance and now we want to investigate the convergence of the associated ruin probabilities. Let X be a stochastic process starting from $x > 0$ and define

$$\tau = \tau(X) = \inf\{t \geq 0 \,|\, X(t) < 0\} \tag{4.9}$$

if the set $\{t \geq 0 \,|\, X(t) < 0\}$ is not empty and $+\infty$ otherwise. In order to derive the result in Theorem 4.9 which permits us to compare the finite–time ruin probability with the 'ruin probability' of the corresponding weak approximation, we need the following technical lemma. Let $X = (X(t) : t \geq 0)$ be a càdlàg process. We define the jump process $\Delta X = (\Delta X(t) : t \geq 0)$ by $\Delta X(t) = X(t) - X(t-)$. Then

$$\{\Delta X \neq 0\} = \bigcup_{n=1}^{\infty} \{(\omega, t) \,|\, t \geq 0\,,\, t = T_n(\omega)\}\ ,$$

where $(T_n : n \in \mathbb{N})$ are stopping times with respect to the natural filtration of X; see Jacod and Shiryaev [51, Proposition 1.32, p. 8].

Lemma 4.7 *Let $Q = (Q(t) : t \geq 0)$ be the process given in* (4.3) *and let $(\tau_n : n \in \mathbb{N})$ be the sequence of all jump times of Q. Then*

$$P\left[\bigcup_{n=1}^{\infty} \{Q(\tau_n -) = 0\}\right] = 0\ . \tag{4.10}$$

Proof. It suffices to prove the property on a finite time interval. So we restrict the domain of the process Q to the interval $[0,1]$ and use the series representation of α–stable Lévy motion to get insight into the structure of Q. If Z_α is standard α–stable Lévy motion on $[0,1]$ with index $1 < \alpha < 2$ we can write

$$Z_\alpha(t) = C_\alpha^{1/\alpha} \sum_{i=1}^{\infty} \left(\gamma_i \Gamma_i^{-1/\alpha} \mathbb{I}_{\{\tau_i \leq t\}} - \beta t b_i^{(\alpha)}\right)\ ,$$

see Samorodnitsky and Taqqu [71, p. 151]. Here $(\gamma_i : i \in \mathbb{N})$ is a sequence of iid random variables satisfying

$$P[\gamma_i = 1] = 1 - P[\gamma_i = -1] = \frac{1+\beta}{2}\ .$$

The sequence $(\Gamma_i : i \in \mathbb{N})$ consists of arrival times of a Poisson process with intensity 1 and $(\tau_i : i \in \mathbb{N})$ is a sequence of iid random variables uniformly distributed on $[0,1]$. These three sequences are independent. Finally, C_α and $b_i^{(\alpha)}$ are some constants. The direction of a jump is

thus governed by $(\gamma_i : i \in \mathbb{N})$ and the height of the jumps, viewed in decreasing order, is distributed as the $-1/\alpha$ power of the arrival times of a Poisson process with unit intensity. In particular, $\Gamma_1^{-1/\alpha}$ is the height of the highest jump, $\Gamma_2^{-1/\alpha}$ is the height of the second highest jump and so on. Non–symmetric stable Lévy motions can be regarded as pure jump processes with a deterministic linear trend.
Hence, the process Q allows the following representation:

$$Q(t) = u + ct - (\lambda C_\alpha)^{1/\alpha} \sum_{i=1}^{\infty} \left(\gamma_i \Gamma_i^{-1/\alpha} \mathbb{I}_{\{\tau_i \le t\}} - \beta t b_i^{(\alpha)} \right) .$$

We show that $P[Q(\tau_k-) = 0] = 0$. We have

$$\begin{aligned} P[Q(\tau_k-) = 0] &= P[\Delta Q(\tau_k) = Q(\tau_k)] \\ &= P\Big[u + c\tau_k + (\lambda C_\alpha)^{1/\alpha} \beta \tau_k b_k^{(\alpha)} \\ &\quad - (\lambda C_\alpha)^{1/\alpha} \sum_{i=1, i\neq k}^{\infty} \left(\gamma_i \Gamma_i^{-1/\alpha} \mathbb{I}_{\{\tau_i \le \tau_k\}} - \beta \tau_k b_i^{(\alpha)} \right) = 0 \Big] . \end{aligned}$$

Using conditional probability and independence we obtain

$$\begin{aligned} &P[Q(\tau_k-) = 0] \\ &= \int_0^1 P\Big[u + c\tau_k + (\lambda C_\alpha)^{1/\alpha} \beta \tau_k b_k^{(\alpha)} \\ &\quad - (\lambda C_\alpha)^{1/\alpha} \sum_{i=1, i\neq k}^{\infty} \left(\gamma_i \Gamma_i^{-1/\alpha} \mathbb{I}_{\{\tau_i \le \tau_k\}} - \beta \tau_k b_i^{(\alpha)} \right) = 0 \;\Big|\; \tau_k = t \Big] \, dt \\ &= \int_0^1 P\Big[u + ct + (\lambda C_\alpha)^{1/\alpha} \beta t b_k^{(\alpha)} \\ &\qquad - (\lambda C_\alpha)^{1/\alpha} \sum_{i=1, i\neq k}^{\infty} \left(\gamma_i \Gamma_i^{-1/\alpha} \mathbb{I}_{\{\tau_i \le t\}} - \beta t b_i^{(\alpha)} \right) = 0 \Big] \, dt = 0 . \end{aligned}$$

In the last equality we used

$$\begin{aligned} P\Big[&u + ct + (\lambda C_\alpha)^{1/\alpha} \beta \, t \, b_k^{(\alpha)} \\ &- (\lambda C_\alpha)^{1/\alpha} \sum_{i=1, i\neq k}^{\infty} \left(\gamma_i \Gamma_i^{-1/\alpha} \mathbb{I}_{\{\tau_i \le t\}} - \beta t b_i^{(\alpha)} \right) = 0 \Big] \end{aligned}$$

$$= P\big[u + ct - \lambda^{1/\alpha} Z_\alpha(t) = 0 \;\big|\; \tau_k > t\big] = 0$$

for all $0 < t < 1$. This concludes the proof. □

Now we are able to state and prove the basic limit result, Theorem 4.9. We will apply the continuous mapping theorem stated below (Proposition 4.8) to translate the question of stochastic convergence to the question of deterministic convergence in the underlying sample space. Suppose that $(X_n : n \in \mathbb{N})$ and X are random elements with values in a separable metric space and $(f_n : n \in \mathbb{N})$, f are Borel measurable functions mapping this separable metric space into another. Then the following proposition holds, see Billingsley [10, Theorem 5.1]

Proposition 4.8 (Continuous mapping theorem)

(*i*) *If $X_n \Rightarrow X$ and f is continuous almost surely with respect to the distribution of X, then $f(X_n) \Rightarrow f(X)$.*

(*ii*) *If $X_n \Rightarrow X$ and $f_n(x_n) \to f(x)$ for all $x \in A$ and $(x_n : n \in \mathbb{N})$ with $x_n \to x$ for some A with $P[X \in A] = 1$ then $f_n(X_n) \Rightarrow f(X)$.*

□

Theorem 4.9 *Let τ be the ruin time defined in* (4.9)*. If $R^{(n)} \Rightarrow Q$, $n \to \infty$, and Q is defined in* (4.3)*, then*

$$\tau(R^{(n)}) \Rightarrow \tau(Q)\ , \qquad n \to \infty\,. \tag{4.11}$$

Proof. Let x_n and x be càdlàg functions such that $x_n \to x$ in the Skorokhod topology. Assume that $\tau(x_n)$ does not converge to $\tau(x)$. Then we can find a subsequence $\tau(x_n)$ which tends to $t_0 \neq \tau(x)$. We first assume that $t_0 > \tau(x)$ and $\tau(x) < \infty$, we also allow for $t_0 = \infty$. We have $x(\tau(x)) < 0$. We can find $\delta > 0$ such that $\tau(x) + \delta < t_0$ and $x(\tau(x) + \delta) = x([\tau(x) + \delta]-)$ and $\varepsilon < -x(\tau(x) + \delta)$ for some $\varepsilon > 0$. Then, for large n, we obtain $|x_n(\tau(x) + \delta) - x(\tau(x) + \delta)| < \varepsilon$. Hence, for $n \geq n_0$, $x_n(\tau(x) + \delta) < 0$ which is a contradiction.

Now let $t_0 < \tau(x)$, we also allow for $\tau(x) = \infty$, then $x_n(\tau(x_n)) < 0$ and $x_n(\tau(x_n))$ tends to $x(t_0)$ or $x(t_0-)$ (more precisely, a subsequence of

$x_n(\tau(x_n))$ has that property). Thus $x(t_0) \leq 0$ or $x(t_0-) \leq 0$. Because $t_0 < \tau(x)$ and $\tau(x)$ is the first instant where the process x becomes negative, we can exclude the case $x(t_0) < 0$. Hence, $x(t_0) = 0$ or in case of a jump at t_0, $x(t_0-) \leq 0$. Now, if $x(t_0) = 0$ we use the strong Markov property and the fact that the process $(ct - Z_\alpha(t) : t \geq 0)$ crosses 0 infinitely many times in every right neighbourhood of $t = 0$. This follows from the law of the iterated logarithm for stable processes (see Fristedt [34, p. 361] and Mijnheer [58, p. 45]). In particular, we can find $\delta > 0$ such that $t_0 + \delta < \tau(x)$ and $x(t_0 + \delta) < 0$. Now suppose that $x(t_0-) \leq 0$. Applying Lemma 4.7, we conclude that $x(t_0-) < 0$ for almost all trajectories. Again we can find $\delta > 0$ such that $x(t_0 + \delta) < 0$ which is a contradiction.
We have shown that the functional τ is almost surely continuous. The assertion now follows from Proposition 4.8 (ii). □

Remark. If ruin at time t is caused by a jump, we have $\Delta Q(t) \neq 0$. But $P[\Delta Q(t) \neq 0] = 0$ because there are no fixed discontinuities of the process Q. When $\Delta Q(t) = 0$, we have $P[\tau(Q) = t] = P[Q(t) = 0] = 0$. Combining the above, we conclude that $P[\tau(Q) = t] = 0$. Consequently, Theorem 4.9 shows that the finite–time passage probabilities converge:

$$\lim_{n\to\infty} P\big[\tau(R^{(n)}) \leq t\big] = P[\tau(Q) \leq t] \ ,$$

or equivalently

$$\lim_{n\to\infty} P\left[\inf_{0\leq s\leq t} R^{(n)}(s) < 0\right] = P\left[\inf_{0\leq s\leq t} Q(s) < 0\right] \ .$$

Hence we can approximate the finite–time ruin probabilities by the probability of the first 0-downcrossing of the process (4.3). In general, we do however not know when $R^{(n)} \Rightarrow Q$ implies

$$\lim_{n\to\infty} P\left[\inf_{t\geq 0} R^{(n)}(t) < 0\right] \stackrel{?}{=} P\left[\inf_{t\geq 0} Q(t) < 0\right] \ .$$

An answer to that question is given in Section 4.2. □

In order to get some appreciation on how Theorem 4.9 can be used for practical purposes we consider a classical risk process and its corresponding weak approximation. We establish the theoretical result which permits us to compare the two processes. To this end let $(N(t) : t \geq 0)$ be a Poisson process with intensity λ. Define two processes $R^{(n)}$ and Q in the following way:

$$R^{(n)}(t) = x_0 + c^{(n)}t - \frac{1}{\varphi(n)} \sum_{k=1}^{N(nt)} Y_k \ ,$$
$$Q(t) = x_0 + \gamma t - \lambda^{1/\alpha} Z_\alpha(t) \ .$$

Recall that condition (4.4) is fulfilled in the Poisson case. So, by Theorem 4.3, $R^{(n)} \Rightarrow Q$ if and only if $\lim_{n\to\infty}(c^{(n)} - \lambda n\mu/\varphi(n)) = \gamma$. Now define

$$\Psi^{(n)}(x_0, t_0) = P\big[\tau(R^{(n)}) \leq t_0\big] = P\left[\inf_{0\leq s\leq t_0} R^{(n)}(s) < 0\right] \ ,$$
$$\Psi_W(x_0, t_0) = \ P\big[\tau(Q) \leq t_0\big] \ \ = P\left[\inf_{0\leq s\leq t_0} Q(s) < 0\right] \ .$$

Consequently, by Theorem 4.9, $\Psi^{(n)}(x_0, t_0) \to \Psi_W(x_0, t_0)$ as $n \to \infty$. Consider now a classical risk process R as defined in (2.20), i.e. $R(t) = x + ct - \sum_{k=1}^{N(t)} Y_k$ with claim size distribution $F \in \mathcal{D}(\alpha, \beta)$ and relative safety loading $\theta = c/(\lambda\mu) - 1 > 0$. Figures 4.2 and 4.3 depict some simulated sample paths of the processes R and Q, respectively. Denote the finite–time ruin probability by $\Psi(x, t)$. We then have, for each $n \in \mathbb{N}$,

$$\begin{aligned}
\Psi(x,t) &= P\left[\inf_{0\leq s\leq t}\left(x + cs - \sum_{k=1}^{N(s)} Y_k\right) < 0\right] \\
&= P\left[\inf_{0\leq s\leq t}\left(\frac{x}{\varphi(n)} + \frac{cs}{\varphi(n)} - \frac{1}{\varphi(n)}\sum_{k=1}^{N(s)} Y_k\right) < 0\right] \\
&= P\left[\inf_{0\leq s\leq t/n}\left(\frac{x}{\varphi(n)} + \frac{cns}{\varphi(n)} - \frac{1}{\varphi(n)}\sum_{k=1}^{N(ns)} Y_k\right) < 0\right] \ .
\end{aligned}$$

Put $\gamma = \theta\lambda n\mu/\varphi(n)$, $x_0 = x/\varphi(n)$ and $t_0 = t/n$. This leads to the approximation

$$\Psi(x, t) \cong \Psi_W(x_0, t_0) \ .$$

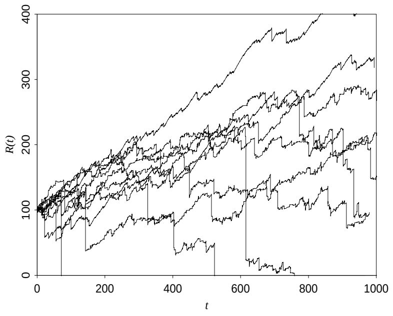

Figure 4.2: *Simulations of a classical risk process R (Pareto distributed claims with $\alpha = 1.5$).*

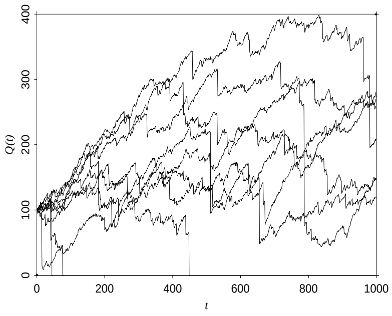

Figure 4.3: *Simulations of the corresponding weak approximation Q.*

If $F \in \mathcal{D}_{\mathcal{N}}(\alpha, \beta)$ it follows that

$$\Psi_W(x_0, t_0) = P\left[\inf_{0 \le s \le t} \bigl(x + \lambda\mu\theta s - \lambda^{1/\alpha} Z_\alpha(s)\bigr) < 0\right] . \tag{4.12}$$

In the Brownian case ($\alpha = 2$) one can explicitly calculate the right–hand side of (4.12); see Asmussen [1, Theorem 3.3, p. 263]. The case $1 < \alpha < 2$, however, turns out to be more delicate. Whenever the claim size distribution F has support $[0, \infty)$ the skewness parameter of Z_α equals $+1$: see Theorem 2.29. Consequently, the limiting process Q belongs to the class of spectrally negative Lévy processes. Proposition 2.7 in principle provides the necessary tools to evaluate (4.12) (either in explicit form or in terms of double Laplace transforms). However, in our situation the formulas in the aforementioned proposition do not allow to calculate the distribution of the infimum in closed form. Therefore numerical methods based on formulas (2.16) or (2.18) are required to compute (4.12). As a partial solution to our problem we will use the simple Monte Carlo method to estimate (4.12). In Tables 4.1 and 4.2 we have presented some numerical values for illustrative purposes.
Because of the absence of an explicit expression for (4.12) we first establish the asymptotic behaviour of (4.12) as $x \to \infty$ and thereafter we

shall derive upper bounds for $\Psi_W(x,t)$.

Proposition 4.10 *Let $Q = (Q(t) : t \geq 0)$ be given by $Q(t) = x + ct - \lambda^{1/\alpha} Z_\alpha(t)$. Assume that the skewness parameter β of the stable component Z_α satisfies $-1 < \beta \leq 1$. For $\tau(Q) = \inf\{t \geq 0 \,|\, Q(t) < 0\}$ and $C_{\alpha,1}$ defined in* (2.35) *we have*

$$P[\tau(Q) \leq t] \sim C_{\alpha,1} \left(\tfrac{1+\beta}{2}\right) \lambda \; t \; (x+ct)^{-\alpha} \;, \quad x \to \infty \,. \tag{4.13}$$

Proof. Since the distribution of $Z_\alpha(t)$ has a regularly varying right tail, it follows from Willekens [85] that

$$P[\tau(Q) \leq t] \sim P\big[ct - \lambda^{1/\alpha} Z_\alpha(t) < -x\big] \;, \qquad x \to \infty \,.$$

The assertion now follows from Proposition 2.22. □

We shall next derive upper bounds for $P[\tau(Q) \leq t]$, where the stable component is assumed to be symmetric. To this end we will use the Bochner theorem which says that symmetric α–stable Lévy motion can be obtained from Brownian motion by a random time change. More precisely, let $(X(t) : t \geq 0)$ be a totally skewed $\alpha/2$–stable Lévy motion, i.e. $X(t) \sim S_{\alpha/2}(t^{2/\alpha}, 1, 0)$ and let B denote standard Brownian motion, $B(1) \sim N(0,1) = S_2(1/\sqrt{2}, 0, 0)$. Assume that both processes are independent and defined on the same probability space. This entails the following equality in law; see for instance Janicki and Weron [52, p. 33],

$$B(X(t)) \stackrel{d}{=} Z_\alpha(t) \,, \tag{4.14}$$

where Z_α is symmetric α–stable Lévy motion such that

$$Z_\alpha(t) \sim S_\alpha \left(t^{1/\alpha} / \big(\sqrt{2}(\cos \tfrac{\pi\alpha}{4})^{1/\alpha}\big), 0, 0 \right) \,.$$

Now let $gB = (gB(t) : t \geq 0) = (\sup_{0 \leq s \leq t} B(s) : t \geq 0)$ denote the supremum process of B, see (2.7). Then gB is an adapted process with non–decreasing paths. Recall the reflection principle for Brownian motion, see for instance Karatzas and Shreve [53, p. 95].

Proposition 4.11 *Let $B = (B(t) : t \geq 0)$ denote standard Brownian motion ($B(0) = 0$ a.s.) and gB its maximum process. For $y \geq 0$, $z > 0$*

$$P[B(t) < z - y, gB(t) \geq z] = P[B(t) > y + z] \; . \tag{4.15}$$

□

Proposition 4.11 and equation (4.14) enable us to state the following 'reflection principle' for symmetric α–stable Lévy motion.

Theorem 4.12 *Let Z_α be symmetric α–stable Lévy motion and $y \geq 0$, $z > 0$, then*

$$P\left[Z_\alpha(t) < z - y, \sup_{0 \leq s \leq t} Z_\alpha(s) \geq z\right] \leq P\big[Z_\alpha(t) > y + z\big] \; . \tag{4.16}$$

If we take $y = 0$ in (4.16) and add to both sides $P[Z_\alpha(t) > z]$ we obtain the following corollary.

Corollary 4.13 *Let Z_α denote symmetric α–stable Lévy motion and let z be a positive number. Then*

$$P\left[\sup_{0 \leq s \leq t} Z_\alpha(s) \geq z\right] \leq 2\, P\big[Z_\alpha(t) > z\big] \; . \tag{4.17}$$

□

Proof of Theorem 4.12. Consider

$$\begin{aligned}
P_t(z,y) &= P\left[Z_\alpha(t) < z - y, \sup_{0 \leq s \leq t} Z_\alpha(s) \geq z\right] \\
&= P\left[B(X(t)) < z - y, \sup_{0 \leq s \leq t} B(X(s)) \geq z\right] \\
&= \int_{\mathbb{D} \times \mathbb{C}} \mathbb{I}\left\{b(x(t)) < z - y, \sup_{0 \leq s \leq t} b(x(s)) \geq z\right\} dM(x)\, dW(b) \\
&= \int_{\mathbb{D}} dM(x) \int_{\mathbb{C}} \mathbb{I}\left\{b(x(t)) < z - y, \sup_{0 \leq s \leq t} b(x(s)) \geq z\right\} dW(b) \; ,
\end{aligned}$$

where the last equality follows from the independence of the processes X and B. Here W denotes the Wiener measure on the space $\mathbb{C} = \mathbb{C}[0,\infty)$ of continuous functions and M is a measure on the space $\mathbb{D}$ induced by the process X. So

$$\begin{aligned} P_t(z,y) &= \int_{\mathbb{D}} P\left[B(x(t)) < z - y,\ \sup_{0\le s\le t} B(x(s)) \ge z\right] dM(x) \\ &\le \int_{\mathbb{D}} P\left[B(x(t)) < z - y,\ \sup_{0\le s\le x(t)} B(s) \ge z\right] dM(x)\ . \end{aligned}$$

By Proposition 4.11 we obtain

$$\begin{aligned} P_t(z,y) &\le \int_{\mathbb{D}} P[B(x(t)) > y + z]\, dM(x) \\ &= \int_{\mathbb{D}\times\mathbb{C}} \mathbb{I}\{b(x(t)) > y + z\}\, dM(x)\, dW(b) \\ &= P[B(X(t)) > y + z] = P[Z_\alpha(t) > y + z] \end{aligned}$$

which ends the proof. □

Now we can provide some upper bounds for the ruin probability of the process Q where the stable component Z_α is symmetric α–stable Lévy motion. Recall that for a stable random variable $X \sim S_\alpha(1,\beta,0)$ we denote by $G(x;\alpha,\beta)$ its cumulative distribution function.

Theorem 4.14 *Let the process $Q = (Q(t) : t \ge 0)$ be given by $Q(t) = x + ct - \lambda^{1/\alpha} Z_\alpha(t)$, where the stable component Z_α is assumed to be symmetric ($\beta = 0$). For $\tau(Q) = \inf\{t \ge 0 \,|\, Q(t) < 0\}$ one has*

$$P[\tau(Q) \le t] \le 2\,P\big[Z_\alpha(t) > x\lambda^{-1/\alpha}\big] = 2\,\overline{G}\big(x/(\lambda t)^{1/\alpha};\alpha,0\big)\ .$$

Proof. Using Corollary 4.13 we obtain

$$\begin{aligned} P[\tau(Q) > t] &= P\left[\inf_{0\le s\le t} (x + cs - \lambda^{1/\alpha} Z_\alpha(s)) \ge 0\right] \\ &= P\left[\sup_{0\le s\le t} (-x - cs + \lambda^{1/\alpha} Z_\alpha(s)) \le 0\right] \end{aligned}$$

$$= 1 - P\left[\sup_{0\le s\le t}(-x - cs + \lambda^{1/\alpha} Z_\alpha(s)) > 0\right]$$
$$\ge 1 - P\left[\sup_{0\le s\le t} Z_\alpha(s) \ge x\lambda^{-1/\alpha}\right]$$
$$\ge 1 - 2\,P\big[Z_\alpha(t) > x\lambda^{-1/\alpha}\big] \;.$$

□

In order to prove an analogous result for the non–symmetric case, we need the following lemma.

Lemma 4.15 *Let Z_α be α–stable Lévy motion with $\alpha \neq 1$ and skewness parameter $|\beta| \le 1$. Then one has*

$$(i)\quad P[Z_\alpha(t) > 0] = \frac{1}{2} + \frac{1}{\pi\alpha}\arctan\left(\beta\tan\frac{\pi\alpha}{2}\right) =: \varrho\;,$$

$$(ii)\quad P\left[\sup_{0\le s\le t} Z_\alpha(s) \ge z\right] \le \frac{1}{\varrho}\,P[Z_\alpha(t) > z]\;, \qquad z > 0\,.$$

Proof. (i) Because of the scaling property (3.3) we have $Z_\alpha(t) \stackrel{d}{=} t^{1/\alpha} Z_\alpha(1)$ and hence $P[Z_\alpha(t) > 0] = P[Z_\alpha(1) > 0] = \varrho$, independent of t. From Zolotarev [88, p. 79] we have

$$G(0;\alpha,\beta) = \frac{1}{2}\left(1 - \beta_B\,\frac{K(\alpha)}{\alpha}\right)\;,$$

where $K(\alpha)$ and β_B are defined in (2.30) and (2.31), respectively, yielding $G(0;\alpha,\beta) = 1/2 - \arctan(\beta\tan\frac{\pi\alpha}{2})/(\pi\alpha)$. Finally,

$$P[Z_\alpha(t) > 0] = P[Z_\alpha(1) > 0] = \overline{G}(0;\alpha,\beta)$$
$$= \frac{1}{2} + \frac{1}{\pi\alpha}\arctan\left(\beta\tan\frac{\pi\alpha}{2}\right)\;.$$

Note that in the case $\beta = 1,\ 1 < \alpha \le 2$ we have $G(0;\alpha,1) = 1/\alpha$.
(ii) Recall from (2.6) the definition of the first passage functional. Then, for fixed $z > 0$, $T = (fZ_\alpha)(z) = \inf\{u \ge 0 \,|\, Z_\alpha(u) > z\}$ is a stopping

time and $\{T \le t\} = \{\sup_{0\le s\le t} Z_\alpha(s) \ge z\}$. Write

$$\begin{aligned} P\left[\sup_{0\le s\le t} Z_\alpha(s) \ge z\right] &= P\left[\sup_{0\le s\le t} Z_\alpha(s) \ge z, Z_\alpha(t) > z\right] \\ &\quad + P\left[\sup_{0\le s\le t} Z_\alpha(s) \ge z, Z_\alpha(t) \le z\right] . \end{aligned} \tag{4.18}$$

Because $\{Z_\alpha(t) > z\} \subset \{\sup_{0\le s\le t} Z_\alpha(s) \ge z\}$, the first summand on the right hand side of (4.18) reduces to $P[Z_\alpha(t) > z]$. Define a process $Z_\alpha^* = (Z_\alpha^*(t) : t \ge 0)$ as $Z_\alpha^*(t) = Z_\alpha(T+t) - Z_\alpha(T)$. It is obvious that

$$\left\{\sup_{0\le s\le t} Z_\alpha(s) \ge z, Z_\alpha(t) \le z\right\} \subset \left\{\sup_{0\le s\le t} Z_\alpha(s) \ge z, Z_\alpha^*(t-T) \le 0\right\}$$

and hence, using the strong Markov property, we obtain

$$P\left[\sup_{0\le s\le t} Z_\alpha(s) \ge z, Z_\alpha(t) \le z\right] \le P\left[\sup_{0\le s\le t} Z_\alpha(s) \ge z\right](1-\varrho) .$$

So

$$P\left[\sup_{0\le s\le t} Z_\alpha(s) \ge z\right] \le P\big[Z_\alpha(t) > z\big] + P\left[\sup_{0\le s\le t} Z_\alpha(s) \ge z\right](1-\varrho)$$

or equivalently

$$P\left[\sup_{0\le s\le t} Z_\alpha(s) \ge z\right] \le \frac{1}{\varrho} P[Z_\alpha(t) > z] .$$

□

Using the same arguments as in the proof of Theorem 4.14 we obtain an upper bound for the hitting time probability in the non–symmetric case.

Theorem 4.16 *Let $Q = (Q(t) : t \ge 0)$ be given by $Q(t) = x + ct - \lambda^{1/\alpha} Z_\alpha(t)$, where $\alpha \ne 1$ and $|\beta| \le 1$. For $\tau(Q) = \inf\{t \ge 0 \,|\, Q(t) < 0\}$ we have*

$$P[\tau(Q) \le t] \le \frac{1}{\varrho} P\big[Z_\alpha(t) > x\lambda^{-1/\alpha}\big] = \overline{G}\big(x/(\lambda t)^{1/\alpha}; \alpha, \beta\big)/\varrho ,$$

where ϱ is given in Lemma 4.15. □

Similar arguments as those used in the proof of Lemma 4.15 yield the following result.

Theorem 4.17 *Let Q and $\tau(Q)$ be given as in Theorem 4.16. Then*

$$P[\tau(Q) \le t] \le \frac{\overline{G}((x+ct)/(\lambda t)^{1/\alpha};\alpha,\beta)}{\overline{G}(ct/(\lambda t)^{1/\alpha};\alpha,\beta)} \ . \tag{4.19}$$

□

We close this section with a numerical example where we compare the finite–time ruin probability $\Psi(x,t)$ of a classical risk process R with $\Psi_W(x,t) = P[\tau(Q) \le t]$, the probability of the first 0–downcrossing of its weak approximation, see the derivation yielding to (4.12). An explicit formula for $\Psi(x,t)$ can only be found in terms of Laplace transforms. Thorin and Wikstad [80] proceed in this way and evaluate $\Psi(x,t)$ numerically by inverting a double Laplace transform. In our approach we use a simple Monte Carlo method to estimate both $\Psi(x,t)$ as well as $\Psi_W(x,t)$. For simulations of stable variables and processes we refer to Weron [83]. Where comparisons are possible, our simulation results for $\Psi(x,t)$ agree with the 'exact' values of Thorin and Wikstad [80]. As an example, we consider Pareto distributed claims with shape parameter $1 < \alpha < 2$ and $\nu > 0$, i.e.

$$F(x) = 1 - (\nu/x)^{\alpha} \ , \quad x \ge \nu \ .$$

The mean μ is given by $\nu\alpha/\tilde{\alpha}$. Because $x^{\alpha}\overline{F}(x) = \nu^{\alpha}$, we conclude that $F \in \mathcal{D}_{\mathcal{N}}(\alpha,1)$, see Proposition 2.31. In our simulations ν equals 0.6 and $\alpha \in \{1.1, 1.2, 1.3, 1.4, 1.5, 1.8\}$, the latter values are typical for heavy–tailed insurance portfolios, see Embrechts, Klüppelberg and Mikosch [31]. The relative safety loading θ is assumed to be 5%. We ran $N =$ 100000 simulations in order to estimate $\Psi(x,t)$ and $\Psi_W(x,t)$ for $t =$ 1000. The relative error ε is given by $\varepsilon = (\hat{\Psi}_W - \hat{\Psi})/\hat{\Psi}$.
The values in Tables 4.1 and 4.2 indicate that $\Psi_W(x,t)$ works reasonably well as an approximation for the finite–time ruin probabilities. Small values of α provide better approximations than values α near 2. We explain this by the fact that the convergence of convolutions to the limiting stable law is slow when $1.5 < \alpha < 2$, see the Remark at the

end of Section 2.3. Further numerical work has shown that the upper bounds derived in Theorems 4.16 and 4.17 are very crude.

x	α	$\hat{\Psi}(x,t) \pm 1.96\hat{\sigma}/\sqrt{N}$	$\hat{\Psi}_W(x,t) \pm 1.96\hat{\sigma}/\sqrt{N}$	ε
5	1.1	$(44.7 \pm 0.31)10^{-2}$	$(42.6 \pm 0.31)10^{-2}$	-4.84%
10	1.1	$(41.3 \pm 0.26)10^{-2}$	$(39.8 \pm 0.30)10^{-2}$	-3.65%
20	1.1	$(37.2 \pm 0.30)10^{-2}$	$(36.6 \pm 0.30)10^{-2}$	-1.52%
50	1.1	$(32.0 \pm 0.29)10^{-2}$	$(31.7 \pm 0.29)10^{-2}$	-1.04%
100	1.1	$(27.3 \pm 0.28)10^{-2}$	$(27.6 \pm 0.28)10^{-2}$	1.31%
200	1.1	$(22.7 \pm 0.26)10^{-2}$	$(22.9 \pm 0.26)10^{-2}$	1.12%
1000	1.1	$(11.9 \pm 0.20)10^{-2}$	$(12.3 \pm 0.20)10^{-2}$	3.17%
5	1.2	$(60.8 \pm 0.30)10^{-2}$	$(59.1 \pm 0.30)10^{-2}$	-2.87%
10	1.2	$(55.7 \pm 0.31)10^{-2}$	$(54.7 \pm 0.31)10^{-2}$	-1.79%
20	1.2	$(49.9 \pm 0.31)10^{-2}$	$(49.6 \pm 0.31)10^{-2}$	-0.65%
50	1.2	$(41.4 \pm 0.31)10^{-2}$	$(41.3 \pm 0.31)10^{-2}$	-0.13%
100	1.2	$(33.7 \pm 0.29)10^{-2}$	$(33.9 \pm 0.29)10^{-2}$	0.43%
200	1.2	$(26.2 \pm 0.27)10^{-2}$	$(26.4 \pm 0.27)10^{-2}$	0.79%
1000	1.2	$(9.61 \pm 0.18)10^{-2}$	$(9.93 \pm 0.19)10^{-2}$	3.34%
5	1.3	$(68.6 \pm 0.29)10^{-2}$	$(67.1 \pm 0.29)10^{-2}$	-2.28%
10	1.3	$(62.3 \pm 0.30)10^{-2}$	$(61.4 \pm 0.30)10^{-2}$	-1.38%
20	1.3	$(54.8 \pm 0.31)10^{-2}$	$(54.5 \pm 0.31)10^{-2}$	-0.66%
50	1.3	$(43.6 \pm 0.31)10^{-2}$	$(43.2 \pm 0.31)10^{-2}$	-1.00%
100	1.3	$(33.6 \pm 0.29)10^{-2}$	$(33.7 \pm 0.29)10^{-2}$	0.36%
200	1.3	$(23.6 \pm 0.26)10^{-2}$	$(23.5 \pm 0.26)10^{-2}$	-0.67%
1000	1.3	$(5.78 \pm 0.15)10^{-2}$	$(5.83 \pm 0.15)10^{-2}$	0.85%

Table 4.1: *Comparison of the finite–time ruin probabilities of a classical risk process (intensity $\lambda = 1$, safety loading $\theta = 5\%$, Pareto distributed claims with $\nu = 0.6$ and shape parameters $\alpha \in \{1.1, 1.2, 1.3\}$) with the ruin probabilities of the corresponding weak approximation. The time horizon t equals* 1000.

x	α	$\hat{\Psi}(x,t) \pm 1.96\hat{\sigma}/\sqrt{N}$	$\hat{\Psi}_W(x,t) \pm 1.96\hat{\sigma}/\sqrt{N}$	ε
5	1.4	$(72.2 \pm 0.28)10^{-2}$	$(71.0 \pm 0.28)10^{-2}$	-1.61%
10	1.4	$(65.0 \pm 0.30)10^{-2}$	$(64.3 \pm 0.30)10^{-2}$	-1.10%
20	1.4	$(56.4 \pm 0.31)10^{-2}$	$(55.4 \pm 0.31)10^{-2}$	-1.71%
50	1.4	$(42.4 \pm 0.31)10^{-2}$	$(41.4 \pm 0.31)10^{-2}$	-2.18%
100	1.4	$(30.6 \pm 0.29)10^{-2}$	$(29.5 \pm 0.28)10^{-2}$	-3.73%
200	1.4	$(19.1 \pm 0.24)10^{-2}$	$(18.1 \pm 0.24)10^{-2}$	-5.08%
1000	1.4	$(3.14 \pm 0.11)10^{-2}$	$(2.88 \pm 0.10)10^{-2}$	-8.19%
5	1.5	$(74.2 \pm 0.27)10^{-2}$	$(72.8 \pm 0.28)10^{-2}$	-1.88%
10	1.5	$(66.2 \pm 0.29)10^{-2}$	$(64.6 \pm 0.30)10^{-2}$	-2.28%
20	1.5	$(55.9 \pm 0.31)10^{-2}$	$(53.9 \pm 0.31)10^{-2}$	-3.63%
50	1.5	$(39.4 \pm 0.30)10^{-2}$	$(37.0 \pm 0.30)10^{-2}$	-6.16%
100	1.5	$(25.7 \pm 0.27)10^{-2}$	$(23.5 \pm 0.26)10^{-2}$	-8.66%
200	1.5	$(13.7 \pm 0.21)10^{-2}$	$(12.0 \pm 0.20)10^{-2}$	-12.12%
1000	1.5	$(1.50 \pm 0.08)10^{-2}$	$(1.29 \pm 0.07)10^{-2}$	-13.96%
5	1.8	$(74.4 \pm 0.27)10^{-2}$	$(72.4 \pm 0.27)10^{-2}$	-2.77%
10	1.8	$(63.1 \pm 0.30)10^{-2}$	$(58.7 \pm 0.30)10^{-2}$	-6.88%
20	1.8	$(48.2 \pm 0.31)10^{-2}$	$(40.8 \pm 0.30)10^{-2}$	-15.40%
50	1.8	$(25.5 \pm 0.27)10^{-2}$	$(16.7 \pm 0.23)10^{-2}$	-34.54%
100	1.8	$(11.0 \pm 0.19)10^{-2}$	$(5.36 \pm 0.14)10^{-2}$	-51.19%
200	1.8	$(3.30 \pm 0.11)10^{-2}$	$(1.39 \pm 0.07)10^{-2}$	-57.74%
1000	1.8	$(0.14 \pm 0.02)10^{-2}$	$(0.07 \pm 0.02)10^{-2}$	-52.21%

Table 4.2: *Comparison of the finite–time ruin probabilities of a classical risk process (intensity $\lambda = 1$, safety loading $\theta = 5\%$, Pareto distributed claims with $\nu = 0.6$ and shape parameters $\alpha \in \{1.4, 1.5, 1.8\}$) with the ruin probabilities of the corresponding weak approximation. The time horizon t equals* 1000.

4.2 Convergence of the Infinite-Time Ruin Probabilities

The aim of this section is to prove the convergence of the infinite–time ruin probabilities when we approximate a classical risk process with an α–stable Lévy motion with drift. Our approach follows Asmussen [1, VIII.6], where a similar result is shown for the Brownian diffusion approximation. Recall that there is a one–to–one correspondence between the ruin probability of a risk process and the distribution of the virtual waiting time in a $GI/G/1$ queuing model. In turn, the waiting times are related to the maximum of random walks. So it is convenient to first state and prove the main result (Theorem 4.19) in random walk terms and thereafter reformulate in terms more natural for queues and risk theory.

In what follows we first describe the $GI/G/1$ queuing model. The symbols $GI/G/1$ stand for a queuing system with general input (arrivals occur according to a renewal process), general service times (service times of successive customers are iid) and one server. Let T_n be the interarrival time between the nth and the $(n+1)$st arriving customer, and we assume that $(T_n : n \geq 0)$ are iid with common distribution function $A(x) = P[T_n \leq x]$. To describe the service mechanism let U_n be the service time of the nth arriving customer and suppose that $(U_n : n \geq 0)$ is a sequence of iid random variables with common distribution function $B(x) = P[U_n \leq x]$. Define the traffic intensity υ by

$$\upsilon = \frac{E[U_n]}{E[T_n]} .$$

If $\upsilon < 1$ then on average the server is able to cope with the load and not to fall hopelessly behind. We assume that there is one server and that he serves customers on a first in, first out basis (FIFO). A relevant process is $(W_n : n \geq 0)$, the waiting time of customer n until his services commences. For $n \geq 0$, define $X_n = U_n - T_n$. Unless otherwise stated, we assume that customer 0 has just arrived at time 0 to an empty queue. With this notation, the following proposition holds; see Asmussen [1, p. 181].

Proposition 4.18 *The actual waiting time process* $(W_n : n \geq 0)$ *satisfies*

$$W_{n+1} = (W_n + X_n)^+ \ , \tag{4.20}$$

$$W_n \stackrel{d}{=} M_n \ , \tag{4.21}$$

where $M_n = \max_{0 \leq \ell \leq n} S_\ell$ *with* $S_n = X_0 + \cdots + X_{n-1}$, $S_0 = 0$. □

Relation (4.20) tells us that the process $(W_n : n \geq 0)$ has the same transition mechanism as the random walk $(S_n : n \geq 0)$ except when S_n crosses from positive to negative values, the process W_n then stays at zero. If $\upsilon < 1$ it follows from the strong law of large numbers that $S_n \to -\infty$. Thus $P[W_n \leq x] \to P[W_\infty \leq x] = P[\max_{\ell \geq 0} S_\ell \leq x]$. However, it is in general difficult to compute the distribution of the maximum of a random walk with negative drift. If an approximation to the distribution of W_∞ can be found as $\upsilon \uparrow 1$ we call it a heavy traffic approximation. A central rôle in this context plays the Mittag–Leffler distribution which occurs as a limit of the equilibrium waiting time distribution when X_i has infinite variance (but finite mean).

Recall that $\mathbb{D} = \mathbb{D}[0,\infty)$ is the space of all real–valued right continuous functions on $[0,\infty)$ with limits from the left. For any stochastic process X in $\mathbb{D}$, the associated supremum functional gX defined in (2.7) is continuous in any of Skorokhod's topologies. Hence, weak convergence $X_n \Rightarrow X$ in $\mathbb{D}$ implies weak convergence $gX_n \Rightarrow gX$ in $\mathbb{D}$ by virtue of the continuous mapping theorem, Theorem 4.8.

Let $(X_n^{(k)})_{n \in \mathbb{N}}$, $k = 1, 2, \ldots$, be a family of sequences of iid random variables with $\mu^{(k)} = E[\,X_i^{(k)}\,] < 0$, $k = 1, 2, \ldots$. By $S_n^{(k)} = X_1^{(k)} + \cdots + X_n^{(k)}$ $(S_0^{(k)} = 0)$ we denote the corresponding random walks. Set $M_n^{(k)} = \max_{0 \leq \ell \leq n} S_\ell^{(k)}$. We assume that $X_i^{(k)} \in \mathcal{D}_{\mathcal{N}}(\alpha, 1)$, i.e.

$$\frac{1}{\varphi(k)} \sum_{i=1}^{k} \big(X_i^{(k)} - \mu^{(k)} \big) \Rightarrow Z \ , \quad k \to \infty \ ,$$

where $\varphi(k) = k^{1/\alpha}$ and $Z \sim S_\alpha(1,1,0)$ with index $1 < \alpha < 2$. Recall from Proposition 2.28 that $E[\,|X_i^{(k)}|^\delta\,] < \infty$ for $\delta < \alpha$ and $E[\,|X_i^{(k)}|^\alpha\,] = \infty$.

Theorem 4.19 *Let* $(S_n^{(k)} : k \in \mathbb{N})$ *be a sequence of random walks with* $X_i^{(k)} \in \mathcal{D}_{\mathcal{N}}(\alpha, 1)$, $1 < \alpha < 2$, $\mu^{(k)} < 0$, $\lim_{k\to\infty} \mu^{(k)} = 0$ *and assume that* $\limsup_{k\to\infty} \left(|\mu^{(k)}|\right)^{-(\alpha-\varepsilon)} E[\,|X_i^{(k)} - \mu^{(k)}|^{\alpha-\varepsilon}\,] = L(\varepsilon) < \infty$ *for* $0 < \varepsilon < \tilde{\alpha}/\alpha$. *Then*

$$\lim_{k\to\infty} P\left[\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} M^{(k)} > y\right] = \sum_{n=0}^{\infty} \frac{(-a)^n}{\Gamma(1+\tilde{\alpha}n)}\, y^{\tilde{\alpha}n} \ ,$$

where $M^{(k)} = \max_{\ell \geq 0} S_\ell^{(k)}$, $\tilde{\alpha} = \alpha - 1$ *and* $a = \cos(\frac{\pi K(\alpha)}{2})$ *with* $K(\alpha) = \alpha - 2$.

The proof of Theorem 4.19 basically involves weak convergence in $\mathbb{D}$. So let $Z_\alpha = (Z_\alpha(t) : t \geq 0)$ denote α–stable Lévy motion with skewness parameter $\beta = 1$ (only upward jumps) and $1 < \alpha < 2$. Consider the process $Q = (Q(t) : t \geq 0)$ given by

$$Q(t) = -t + Z_\alpha(t).$$

The process Q is a spectrally positive Lévy process, that is, $-Q$ is spectrally negative. It is characterized by $E[e^{-sQ(t)}] = e^{t\xi(s)}$, $\xi(s) = s + \kappa s^\alpha$, where $1/\kappa = a = \cos(\frac{\pi K(\alpha)}{2})$, see (3.7).

Proposition 4.20 *Under the conditions of Theorem* 4.19, *it holds for any* $T < \infty$ *that*

$$\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} M^{(k)}_{[T/|\mu^{(k)}|^{\alpha/\tilde{\alpha}}]} \Rightarrow \sup_{0\leq t\leq T} Q(t) \ , \quad k \to \infty \, , \tag{4.22}$$

where $[\,\cdot\,]$ *denotes the integer part.*

Proof. Subject to the conditions of "asymptotic negligibility" of the double sequence $X_i^{(k)}$ we have for every sequence $(c_k : k \in \mathbb{N})$ with $\lim_{k\to\infty} c_k = \infty$,

$$\frac{S^{(k)}_{[c_k t]} - [c_k t]\mu^{(k)}}{\varphi(c_k)} \Rightarrow Z_\alpha(t) \ , \quad k \to \infty \, ;$$

see for instance Prohorov [66, Theorem 3.2]. Set $c_k = |\mu^{(k)}|^{-\alpha/\tilde{\alpha}}$. Then we can write

$$\begin{aligned}
&\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} S^{(k)}_{[t/\mu^{(k)}|^{\alpha/\tilde{\alpha}}]} \\
&\qquad = \frac{S^{(k)}_{[t/|\mu^{(k)}|^{\alpha/\tilde{\alpha}}]} - [t/|\mu^{(k)}|^{\alpha/\tilde{\alpha}}]\,\mu^{(k)}}{1/|\mu^{(k)}|^{1/\tilde{\alpha}}} + \frac{[t/|\mu^{(k)}|^{\alpha/\tilde{\alpha}}]\,\mu^{(k)}}{1/|\mu^{(k)}|^{1/\tilde{\alpha}}} \\
&\qquad \Rightarrow Z_\alpha(t) - t\,, \quad k \to \infty\,.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\sup_{0\le t\le T}\left(\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} S^{(k)}_{[t/|\mu^{(k)}|^{\alpha/\tilde{\alpha}}]}\right) &= \left|\mu^{(k)}\right|^{1/\tilde{\alpha}} M^{(k)}_{[T/|\mu^{(k)}|^{\alpha/\tilde{\alpha}}]} \\
&\Rightarrow \sup_{0\le t\le T}\left(-t + Z_\alpha(t)\right)\,, \quad k \to \infty\,.
\end{aligned}$$

□

The proof of Theorem 4.19 now rests on the following lemma due to von Bahr and Esseen [82].

Lemma 4.21 *Let $X_1, \ldots, X_n$ be a sequence of random variables satisfying $E[X_{m+1}|S_m] = 0$ a.s., $S_n = X_1 + \cdots + X_n$. If $E[\,|X_i|^r\,] < \infty$, $1 \le r \le 2$, then*

$$E[\,|S_n|^r\,] \le 2\sum_{i=1}^{n} E[\,|X_i|^r\,]\,.$$

□

Remark. The condition $E[\,X_{m+1}|S_m\,] = 0$ a.s. is fulfilled if the random variables $(X_i : i \in \mathbb{N})$ are independent and have zero mean. □

We are finally in a position to prove Theorem 4.19.

Proof of Theorem 4.19. Define $Y^{(k)} = |\mu^{(k)}|^{1/\tilde{\alpha}} M^{(k)}$ and set $N^{(k)} = [T/|\mu^{(k)}|^{\alpha/\tilde{\alpha}}]$. We can write

$$Y^{(k)} = \left|\mu^{(k)}\right|^{1/\tilde{\alpha}} M^{(k)}_{N^{(k)}} \bigvee \sup_{n > N^{(k)}}\left(|\mu^{(k)}|^{1/\tilde{\alpha}} S^{(k)}_n\right)\,, \tag{4.23}$$

where $a \vee b = \max(a, b)$. From Proposition 4.20 we obtain for the first term on the right hand side of (4.23)

$$\lim_{T\to\infty}\lim_{k\to\infty} P\left[\left|\mu^{(k)}\right|^{1/\tilde\alpha} M^{(k)}_{N^{(k)}} > y\right]$$
$$\begin{aligned} &= \lim_{T\to\infty} P\left[\sup_{0\le t\le T}(-t + Z_\alpha(t)) > y\right] \\ &= P\left[\sup_{t\ge 0}(-t + Z_\alpha(t)) > y\right] \qquad (4.24) \\ &= \sum_{n=0}^{\infty} \frac{(-a)^n}{\Gamma(1+\tilde\alpha n)}\, y^{\tilde\alpha n} \, . \end{aligned}$$

The last equality follows from (3.20). We now want to show that the second term on the right–hand side of (4.23) is asymptotically negligible for large values of T and k:

$$\limsup_{T\to\infty}\limsup_{k\to\infty} P\left[\sup_{n>N^{(k)}}\left(\left|\mu^{(k)}\right|^{1/\tilde\alpha} S^{(k)}_n\right) > 0\right] = 0\,. \qquad (4.25)$$

Choose $0 < \varepsilon < \tilde\alpha/\alpha$ which implies that $\alpha\varepsilon/\tilde\alpha < 1$, $\tilde\alpha - \varepsilon > 0$ and $\alpha - \varepsilon > 1$, then

$$P\left[\sup_{n>N^{(k)}}\left(\left|\mu^{(k)}\right|^{1/\tilde\alpha} S^{(k)}_n\right) > 0\right]$$
$$\begin{aligned} &= P\left[\sup_{n>N^{(k)}}\left(\frac{S^{(k)}_n}{n} - \mu^{(k)}\right) > -\mu^{(k)}\right] \\ &\le \left(\frac{1}{|\mu^{(k)}|}\right)^{\alpha-\varepsilon} E\left[\left|\frac{S^{(k)}_{N^{(k)}}}{N^{(k)}} - \mu^{(k)}\right|^{\alpha-\varepsilon}\right] \\ &= \left(\frac{1}{|\mu^{(k)}|}\right)^{\alpha-\varepsilon}\left(\frac{1}{N^{(k)}}\right)^{\alpha-\varepsilon} E\left[\left|S^{(k)}_{N^{(k)}} - \mu^{(k)}\, N^{(k)}\right|^{\alpha-\varepsilon}\right] \, , \end{aligned}$$

where we have used Kolmogorv's inequality for the backwards submartingale $(S^{(k)}_n/n - \mu^{(k)})_{n\ge N^{(k)}}$; see for instance Chung [17, Theorem 9.4.1]. Using Lemma 4.21 we further conclude that

$$
\begin{aligned}
P&\left[\sup_{n>N^{(k)}}\left(\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} S_n^{(k)}\right)>0\right]\\
&\le\left(\frac{1}{|\mu^{(k)}|}\right)^{\alpha-\varepsilon}\left(\frac{1}{N^{(k)}}\right)^{\alpha-\varepsilon} 2\,N^{(k)}\,E\left[\left|X_i^{(k)}-\mu^{(k)}\right|^{\alpha-\varepsilon}\right]\\
&=\left(\frac{1}{|\mu^{(k)}|}\right)^{\alpha-\varepsilon}\left(N^{(k)}\right)^{-\tilde{\alpha}+\varepsilon} 2\,E\left[\left|X_i^{(k)}-\mu^{(k)}\right|^{\alpha-\varepsilon}\right] .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\limsup_{k\to\infty} P&\left[\sup_{n>N^{(k)}}\left(\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} S_n^{(k)}\right)>0\right]\\
&\le\frac{2}{T^{\tilde{\alpha}-\varepsilon}}\limsup_{k\to\infty}\left(\frac{1}{|\mu^{(k)}|}\right)^{\alpha\varepsilon/\tilde{\alpha}} E\left[\left|X_i^{(k)}-\mu^{(k)}\right|^{\alpha-\varepsilon}\right]\\
&\le\frac{2}{T^{\tilde{\alpha}-\varepsilon}}\limsup_{k\to\infty}\left(\frac{1}{|\mu^{(k)}|}\right)^{\alpha-\varepsilon} E\left[\left|X_i^{(k)}-\mu^{(k)}\right|^{\alpha-\varepsilon}\right]\\
&=\frac{2L(\varepsilon)}{T^{\tilde{\alpha}-\varepsilon}} .
\end{aligned}
$$

Assertion (4.25) now follows as T tends to infinity. Recall that

$$
Y^{(k)}=\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} M^{(k)}=\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} M_{N^{(k)}}^{(k)}\bigvee\sup_{n>N^{(k)}}\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} S_n^{(k)} ,
$$

where again we have used the abbreviation $N^{(k)}=[T/|\mu^{(k)}|^{\alpha/\tilde{\alpha}}]$. We have that

$$
\begin{aligned}
P\left[Y^{(k)}>y\right]&\le P\left[\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} M_{N^{(k)}}^{(k)}>y\right]+P\left[\sup_{n>N^{(k)}}\left|\mu^{(k)}\right|^{1/\tilde{\alpha}} S_n^{(k)}>0\right]\\
&=P_1+P_2 ,
\end{aligned}
$$

say, which implies by (4.24) and (4.25) that

$$
\begin{aligned}
\limsup_{k\to\infty} P\left[Y^{(k)}>y\right]&\le\lim_{T\to\infty}\limsup_{k\to\infty} P_1+\lim_{T\to\infty}\limsup_{k\to\infty} P_2\\
&=P\left[\sup_{t\ge 0}(-t+Z_\alpha(t))\right]\\
&=\sum_{n=0}^{\infty}\frac{(-a)^n}{\Gamma(1+\tilde{\alpha}n)}\,y^{\tilde{\alpha}n} .
\end{aligned}
\tag{4.26}
$$

On the other hand, $\{Y^{(k)} > y\} \supset \{|\mu^{(k)}|^{1/\tilde{\alpha}} M^{(k)}_{N^{(k)}} > y\}$ and therefore

$$\begin{aligned}\liminf_{k\to\infty} P\big[\, Y^{(k)} > y \,\big] &\geq \lim_{T\to\infty} \liminf_{k\to\infty} P\left[\, \left|\mu^{(k)}\right|^{1/\tilde{\alpha}} M^{(k)}_{N^{(k)}} > y \,\right] \\ &= P\left[\, \sup_{t\geq 0}(-t + Z_\alpha(t)) > y \,\right] \\ &= \sum_{n=0}^{\infty} \frac{(-a)^n}{\Gamma(1+\tilde{\alpha}n)}\, y^{\tilde{\alpha}n} \ .\end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27) we obtain

$$\lim_{k\to\infty} P\big[\, Y^{(k)} > y \,\big] = \sum_{n=0}^{\infty} \frac{(-a)^n}{\Gamma(1+\tilde{\alpha}n)}\, y^{\tilde{\alpha}n} \ .$$

□

REMARK. In order to apply Kolmogorov's inequality a smaller exponent than α is required because $X_i \in \mathcal{D}_{\mathcal{N}}(\alpha, 1)$ and thus $E[\,|X_i|^\alpha\,] = \infty$, whereas $E[\,|X_i|^{\alpha-\varepsilon}\,] < \infty$ for $0 < \varepsilon \leq \alpha$; see Proposition 2.28. ■

We now reformulate Theorem 4.19 in terms of queuing systems. Recall that for a $GI/G/1$ queuing system we denote by W_n the actual waiting time of customer n, by U_n its service time and by T_n the interval between the arrival of customer n and $n+1$. Define $S_n = X_1 + \cdots + X_n$, $S_0 = 0$, where $X_n = U_n - T_n$. Furthermore, $M_n = \max_{0\leq \ell\leq n} S_\ell$. The following theorem is a consequence of Theorem 4.19 if we set $X^{(k)} = U^{(k)} - T^{(k)}$.

Theorem 4.22 *Consider $GI/G/1$ queuing systems indexed by k. For the kth model suppose that the interarrival time $T^{(k)}$ has distribution $A^{(k)}$ with mean $a^{(k)}$ and that the service time $U^{(k)}$ has distribution $B^{(k)}$ with mean $b^{(k)}$. Set $X^{(k)} = U^{(k)} - T^{(k)}$ and $\mu^{(k)} = b^{(k)} - a^{(k)}$. Assume that $X^{(k)} \in \mathcal{D}_{\mathcal{N}}(\alpha, 1)$, $1 < \alpha < 2$, $\upsilon^{(k)} = b^{(k)}/a^{(k)} < 1$ and $\lim_{k\to\infty} \upsilon^{(k)} = 1$. If $\limsup_{k\to\infty} \left(|\mu^{(k)}|\right)^{-(\alpha-\varepsilon)} E[\,|X^{(k)} - \mu^{(k)}|^{\alpha-\varepsilon}\,] < \infty$ for some $0 < \varepsilon < \tilde{\alpha}/\alpha$, then the equilibrium waiting time $Y^{(k)} = |\mu^{(k)}|^{1/\tilde{\alpha}} W^{(k)}$ satisfies*

$$\lim_{k\to\infty} P\left[\, \left|\mu^{(k)}\right|^{1/\tilde{\alpha}} W^{(k)} > y \,\right] = \sum_{n=0}^{\infty} \frac{(-a)^n}{\Gamma(1+\tilde{\alpha}n)}\, y^{\tilde{\alpha}n} \ ,$$

where $\tilde{\alpha} = \alpha - 1$ and $a = \cos(\frac{\pi K(\alpha)}{2})$ with $K(\alpha) = \alpha - 2$. □

As in Section 4.1 we may embed a classical risk process R with safety loading θ and claim size distribution $F \in \mathcal{D}(\alpha, 1)$, $1 < \alpha < 2$, in a sequence $R^{(n)}$ of risk processes converging to an α–stable Lévy motion. A classical risk process R corresponds to an initially empty $M/G/1$ queue with arrival intensity λ and service time distribution F. The saftey loading θ and the traffic intensity $\upsilon = E[U]/E[T]$ are related through $\upsilon = 1/(1+\theta)$. Moreover, $\Psi(x) = P[\,W > x\,]$, where W is the actual waiting time in the steady state. For more details on risk processes and the associated queues we refer to Asmussen [1, XIII.1]. In the terminology of Theorem 4.22 it is suggested to use the following approximation for the infinite–time ruin probability

$$\begin{aligned} \Psi(x) = P\left[\, |\tilde{\mu}|^{1/\tilde{\alpha}} W > |\tilde{\mu}|^{1/\tilde{\alpha}} x \,\right] &\cong \sum_{n=0}^{\infty} \frac{(-a\mu\theta)^n}{\Gamma(1+\tilde{\alpha}n)}\, x^{\tilde{\alpha}n} \\ &= E_{\tilde{\alpha}}(-a\mu\theta x^{\tilde{\alpha}}) = \Psi_W(x)\ , \end{aligned} \tag{4.28}$$

say, where $|\tilde{\mu}| = |E[U-T]| = \mu\theta$ is assumed to be small.

For illustrative purposes we consider a classical risk process R as given in (2.20). We assume that the claim size distribution is Pareto with shape parameter $1 < \alpha < 2$ and $\nu > 0$, i.e.

$$F(x) = 1 - (\nu/x)^{\alpha}\ , \quad x \geq \nu\ .$$

The mean μ is given by $\nu\alpha/\tilde{\alpha}$. From Proposition 2.31 it follows that $F \in \mathcal{D}_{\mathcal{N}}(\alpha, 1)$. In order to estimate $\Psi(x)$ we use the algorithms of Asmussen and Binswanger [3]. In our simulations $\alpha \in \{1.1, 1.2, 1.3, 1.4, 1.5, 1.8\}$. For the relative safety loading θ we assume $\theta = 5\%$. The estimate $\hat{\Psi}_{AB}$ for $\Psi(x)$ is based on $N = 10000$ simulations. The relative error ε is given by $\varepsilon = (\Psi_W - \hat{\Psi}_{AB})/\hat{\Psi}_{AB}$.

The most striking impression of Tables 4.3 and 4.4 is certainly the extremely good accuracy of the approximation for small values of α, say $\alpha < 1.5$, i.e. when the ruin probability is not too small. Contrary to the Brownian diffusion approximation ($\alpha = 2$) as treated for instance in Schmidli [72], the non–Gaussian approximation even works well for large values of x. In our case, the accuracy is still almost perfect. As α approaches 2 the 'exact' ruin probabilities are underestimated for large initial values of x. An explanation for this effect is that the tail decay of the Mittag–Leffler law tends to be of exponential order as $\alpha \nearrow 2$, whereas $\Psi(x)$ always decays like a power function, even for shape parameters $\alpha \geq 2$, see Theorem 2.16. According to Theorem 4.19 the approximation is rather sensitive with respect to $|\tilde{\mu}| = \mu\theta$, see Tables 4.5 and 4.6 in comparison with Tables 4.3 and 4.4, where $\nu = 0.6$ is replaced by $\nu = 1$, implying an increase of $|\tilde{\mu}|$. In Tables 4.5 and 4.6 we observe the same feature as in the Brownian diffusion approximation, namely an underestimation of the 'exact' ruin probabilities.

x	α	$\hat{\Psi}_{AB}(x) \pm 1.96\hat{\sigma}/\sqrt{N}$	$E_{\tilde{\alpha}}(-a\mu\theta x^{\tilde{\alpha}})$	ε
5	1.1	$(93.5 \pm 0.44)10^{-2}$	$94.0\ 10^{-2}$	0.50%
10	1.1	$(93.4 \pm 0.43)10^{-2}$	$93.6\ 10^{-2}$	0.21%
20	1.1	$(92.7 \pm 0.45)10^{-2}$	$93.2\ 10^{-2}$	0.52%
50	1.1	$(91.6 \pm 0.47)10^{-2}$	$92.5\ 10^{-2}$	1.04%
100	1.1	$(91.8 \pm 0.45)10^{-2}$	$92.1\ 10^{-2}$	0.28%
200	1.1	$(91.4 \pm 0.46)10^{-2}$	$91.5\ 10^{-2}$	0.21%
1000	1.1	$(89.5 \pm 0.50)10^{-2}$	$90.2\ 10^{-2}$	0.84%
5	1.2	$(92.1 \pm 0.45)10^{-2}$	$92.3\ 10^{-2}$	0.13%
10	1.2	$(90.9 \pm 0.48)10^{-2}$	$91.2\ 10^{-2}$	0.33%
20	1.2	$(89.7 \pm 0.50)10^{-2}$	$90.0\ 10^{-2}$	0.32%
50	1.2	$(87.9 \pm 0.53)10^{-2}$	$88.2\ 10^{-2}$	0.32%
100	1.2	$(86.4 \pm 0.55)10^{-2}$	$86.7\ 10^{-2}$	0.35%
200	1.2	$(84.5 \pm 0.58)10^{-2}$	$85.0\ 10^{-2}$	0.57%
1000	1.2	$(80.1 \pm 0.63)10^{-2}$	$80.4\ 10^{-2}$	0.41%
5	1.3	$(90.1 \pm 0.50)10^{-2}$	$90.3\ 10^{-2}$	0.18%
10	1.3	$(88.4 \pm 0.53)10^{-2}$	$88.3\ 10^{-2}$	−0.11%
20	1.3	$(85.8 \pm 0.57)10^{-2}$	$85.9\ 10^{-2}$	0.18%
50	1.3	$(81.6 \pm 0.63)10^{-2}$	$82.2\ 10^{-2}$	0.74%
100	1.3	$(79.2 \pm 0.65)10^{-2}$	$78.9\ 10^{-2}$	−0.36%
200	1.3	$(74.9 \pm 0.69)10^{-2}$	$75.1\ 10^{-2}$	0.32%
1000	1.3	$(64.4 \pm 0.75)10^{-2}$	$64.8\ 10^{-2}$	0.55%

Table 4.3: *Comparison of the infinite–time ruin probabilities of a classical risk process ($\theta = 5\%$, Pareto distributed claims with $\nu = 0.6$ and shape parameter $\alpha \in \{1.1, 1.2, 1.3\}$) with the ruin probabilities of the corresponding weak approximation.*

x	α	$\hat{\Psi}_{AB}(x) \pm 1.96\hat{\sigma}/\sqrt{N}$	$E_{\tilde{\alpha}}(-a\mu\theta x^{\tilde{\alpha}})$	ε
5	1.4	$(88.2 \pm 0.54)10^{-2}$	$88.1\ 10^{-2}$	−0.13%
10	1.4	$(84.9 \pm 0.59)10^{-2}$	$84.8\ 10^{-2}$	−0.11%
20	1.4	$(81.4 \pm 0.64)10^{-2}$	$80.8\ 10^{-2}$	−0.72%
50	1.4	$(75.6 \pm 0.70)10^{-2}$	$74.2\ 10^{-2}$	−1.92%
100	1.4	$(70.4 \pm 0.74)10^{-2}$	$68.3\ 10^{-2}$	−3.03%
200	1.4	$(63.1 \pm 0.77)10^{-2}$	$61.7\ 10^{-2}$	−2.20%
1000	1.4	$(46.5 \pm 0.75)10^{-2}$	$44.8\ 10^{-2}$	−3.62%
5	1.5	$(85.7 \pm 0.59)10^{-2}$	$85.8\ 10^{-2}$	0.04%
10	1.5	$(82.0 \pm 0.64)10^{-2}$	$80.8\ 10^{-2}$	−1.52%
20	1.5	$(76.8 \pm 0.70)10^{-2}$	$74.5\ 10^{-2}$	−2.98%
50	1.5	$(66.2 \pm 0.77)10^{-2}$	$64.2\ 10^{-2}$	−3.03%
100	1.5	$(59.0 \pm 0.79)10^{-2}$	$55.2\ 10^{-2}$	−6.48%
200	1.5	$(49.9 \pm 0.78)10^{-2}$	$45.7\ 10^{-2}$	−8.45%
1000	1.5	$(28.5 \pm 0.65)10^{-2}$	$25.4\ 10^{-2}$	−10.80%
5	1.8	$(80.2 \pm 0.69)10^{-2}$	$78.4\ 10^{-2}$	−2.20%
10	1.8	$(71.2 \pm 0.78)10^{-2}$	$66.1\ 10^{-2}$	−7.23%
20	1.8	$(59.8 \pm 0.83)10^{-2}$	$49.9\ 10^{-2}$	−16.59%
50	1.8	$(40.5 \pm 0.80)10^{-2}$	$27.0\ 10^{-2}$	−33.33%
100	1.8	$(26.2 \pm 0.69)10^{-2}$	$13.9\ 10^{-2}$	−46.78%
200	1.8	$(14.9 \pm 0.50)10^{-2}$	$6.70\ 10^{-2}$	−54.95%
1000	1.8	$(3.13 \pm 0.15)10^{-2}$	$1.35\ 10^{-2}$	−56.87%

Table 4.4: *Comparison of the infinite–time ruin probabilities of a classical risk process ($\theta = 5\%$, Pareto distributed claims with $\nu = 0.6$ and shape parameter $\alpha \in \{1.4, 1.5, 1.8\}$) with the ruin probabilities of the corresponding weak approximation.*

x	α	$\hat{\Psi}_{AB}(x) \pm 1.96\hat{\sigma}/\sqrt{N}$	$E_{\tilde{\alpha}}(-a\mu\theta x^{\tilde{\alpha}})$	ε
5	1.1	$(94.1 \pm 0.42)10^{-2}$	$90.4\ 10^{-2}$	−3.91%
10	1.1	$(93.3 \pm 0.44)10^{-2}$	$89.8\ 10^{-2}$	−3.81%
20	1.1	$(93.1 \pm 0.44)10^{-2}$	$89.1\ 10^{-2}$	−4.25%
50	1.1	$(92.6 \pm 0.44)10^{-2}$	$88.2\ 10^{-2}$	−4.77%
100	1.1	$(92.5 \pm 0.44)10^{-2}$	$87.4\ 10^{-2}$	−5.50%
200	1.1	$(91.7 \pm 0.45)10^{-2}$	$86.7\ 10^{-2}$	−5.47%
1000	1.1	$(90.0 \pm 0.48)10^{-2}$	$84.7\ 10^{-2}$	−5.95%
5	1.2	$(92.8 \pm 0.44)10^{-2}$	$87.7\ 10^{-2}$	−5.49%
10	1.2	$(91.8 \pm 0.46)10^{-2}$	$86.1\ 10^{-2}$	−6.16%
20	1.2	$(90.9 \pm 0.48)10^{-2}$	$84.3\ 10^{-2}$	−7.20%
50	1.2	$(88.8 \pm 0.52)10^{-2}$	$81.8\ 10^{-2}$	−7.90%
100	1.2	$(87.4 \pm 0.54)10^{-2}$	$79.6\ 10^{-2}$	−8.92%
200	1.2	$(85.8 \pm 0.56)10^{-2}$	$77.2\ 10^{-2}$	−10.05%
1000	1.2	$(81.4 \pm 0.62)10^{-2}$	$71.0\ 10^{-2}$	−12.76%
5	1.3	$(91.1 \pm 0.48)10^{-2}$	$84.7\ 10^{-2}$	−7.02%
10	1.3	$(89.8 \pm 0.50)10^{-2}$	$81.8\ 10^{-2}$	−9.00%
20	1.3	$(88.0 \pm 0.53)10^{-2}$	$78.4\ 10^{-2}$	−10.92%
50	1.3	$(84.3 \pm 0.59)10^{-2}$	$73.2\ 10^{-2}$	−13.12%
100	1.3	$(81.1 \pm 0.63)10^{-2}$	$68.9\ 10^{-2}$	−15.11%
200	1.3	$(78.0 \pm 0.67)10^{-2}$	$64.1\ 10^{-2}$	−17.78%
1000	1.3	$(68.7 \pm 0.73)10^{-2}$	$52.0\ 10^{-2}$	−24.29%

Table 4.5: *Comparison of the infinite–time ruin probabilities of a classical risk process ($\theta = 5\%$, Pareto distributed claims with $\nu = 1$ and shape parameter $\alpha \in \{1.1, 1.2, 1.3\}$) with the ruin probabilities of the corresponding weak approximation.*

x	α	$\hat{\Psi}_{AB}(x) \pm 1.96\hat{\sigma}/\sqrt{N}$	$E_{\tilde{\alpha}}(-a\mu\theta x^{\tilde{\alpha}})$	ε
5	1.4	$(90.1 \pm 0.50)10^{-2}$	$81.5\ 10^{-2}$	−9.60%
10	1.4	$(87.9 \pm 0.54)10^{-2}$	$76.7\ 10^{-2}$	−12.68%
20	1.4	$(84.5 \pm 0.60)10^{-2}$	$71.2\ 10^{-2}$	−15.74%
50	1.4	$(79.1 \pm 0.66)10^{-2}$	$62.8\ 10^{-2}$	−20.65%
100	1.4	$(74.2 \pm 0.71)10^{-2}$	$55.7\ 10^{-2}$	−24.97%
200	1.4	$(67.8 \pm 0.75)10^{-2}$	$48.3\ 10^{-2}$	−28.73%
1000	1.4	$(51.9 \pm 0.77)10^{-2}$	$31.9\ 10^{-2}$	−38.56%
5	1.5	$(88.3 \pm 0.54)10^{-2}$	$78.0\ 10^{-2}$	−11.65%
10	1.5	$(85.2 \pm 0.60)10^{-2}$	$71.1\ 10^{-2}$	−16.59%
20	1.5	$(80.0 \pm 0.67)10^{-2}$	$62.9\ 10^{-2}$	−21.39%
50	1.5	$(72.3 \pm 0.74)10^{-2}$	$50.7\ 10^{-2}$	−29.90%
100	1.5	$(64.7 \pm 0.78)10^{-2}$	$41.2\ 10^{-2}$	−36.40%
200	1.5	$(56.7 \pm 0.80)10^{-2}$	$32.2\ 10^{-2}$	−43.36%
1000	1.5	$(34.9 \pm 0.71)10^{-2}$	$16.2\ 10^{-2}$	−53.67%
5	1.8	$(84.5 \pm 0.63)10^{-2}$	$67.2\ 10^{-2}$	−20.47%
10	1.8	$(78.3 \pm 0.71)10^{-2}$	$51.3\ 10^{-2}$	−34.52%
20	1.8	$(68.9 \pm 0.80)10^{-2}$	$33.6\ 10^{-2}$	−51.25%
50	1.8	$(52.3 \pm 0.84)10^{-2}$	$14.7\ 10^{-2}$	−71.78%
100	1.8	$(36.5 \pm 0.78)10^{-2}$	$7.09\ 10^{-2}$	−80.59%
200	1.8	$(22.7 \pm 0.64)10^{-2}$	$3.53\ 10^{-2}$	−84.44%
1000	1.8	$(5.30 \pm 0.23)10^{-2}$	$0.75\ 10^{-2}$	−85.83%

Table 4.6: *Comparison of the infinite–time ruin probabilities of a classical risk process ($\theta = 5\%$, Pareto distributed claims with $\nu = 1$ and shape parameter $\alpha \in \{1.4, 1.5, 1.8\}$) with the ruin probabilities of the corresponding weak approximation.*

Chapter 5

Conclusions

Stable distributions and stable processes are increasingly important in probability theory. In recent years there has been a considerable expansion of practical problems in which stable distributions appear in a natural way, for instance in the fields of engineering, physics, astronomy, economics and, more recently, in insurance and mathematical finance. The reasons why stable laws have become more and more popular in stochastic modelling are twofold. On the one hand, stable laws differ from the Gaussian one significantly. Stable distributions are leptokurtic; i.e. compared to the normal distribution they are typically heavy–tailed and more peaked around the center, a phenomenon which is commonly observed for asset return data. For the tail decay of a stable random variable $X \sim S_\alpha(\sigma, \beta, m)$ with $\alpha < 2$ we have $P[X > x] \sim kx^{-\alpha}$, $x \to \infty$. Consequently, no moments of order less than α exist. Also, the family of stable laws is reasonably flexible, since parameters regulating location, scale and especially skewness are available. On the other hand, stable laws and processes do still provide points of comparison with the Gaussian case, here we refer to the central limit theorem and the scaling–property.

Our aim in Chapter 3 was to consider a classical risk process endowed with an additional (heavy–tailed) source of randomness. For this reason

we add an α–stable Lévy motion to a classical risk process. We assume that the index of stability satisfies $1 < \alpha < 2$, because most applications of stable laws to empirical distributions have so far been concerned with this case. Contrary to Brownian motion, the sample paths of non–Gaussian stable motions have discontinuities. The direction of a jump is governed by the skewness parameter β. We choose $\beta = -1$ corresponding to the case where the Lévy measure Π attributes no mass to the positive half axis. This implies that the perturbation component only admits downward jumps which we interpret as certain extra random payments either involving the income side or the claim payment side. From an analytical point of view, both restrictions $1 < \alpha < 2$ and $\beta = -1$ ensure that a perturbed classical risk process belongs to the class of spectrally negative Lévy processes with finite mean function. Wiener–Hopf techniques are applicable to solve for the distribution of the infimum or equivalently, in the language of actuaries, for the ruin probability within infinite time. The resulting formula (3.17) is quite explicit and generalizes a result of Dufresne and Gerber [21]. As a by–product we obtain an explicit formula for the probability that an α–stable Lévy motion Z_α $(1 < \alpha < 2, \beta = -1)$ with positive linear drift ever attains a level $-x$, $x > 0$.
One can think of various generalizations of the perturbed classical risk model. The number of claims may be modelled by a more general point process than a Poisson process, e.g. a renewal process or a Cox process. Economic factors such as interest and inflation for example lead to a non–linear premium income. We may also drop the restriction $\beta = -1$ on the skewness parameter of the stable component. The random "noise" around the underlying risk process may then have jumps in both directions. All these extensions share the property that the perturbed risk process does no longer belong to the class of spectrally negative Lévy processes. As a consequence, Wiener–Hopf methods can *not* be applied any longer to evaluate the distribution of the infimum and so each of these extensions deserves further investigation. We do not believe that an explicit formula for the distribution of the infimum exists in these cases. As a partial solution one may try to establish inequalities or asymptotic estimates for the ruin probability, presumably via semi–martingale techniques.

The key step in Chapter 4 is the central limit type argument. The distribution of the sum of many small random effects which follow a heavy–tailed distribution can be approximated by a stable law. This allows us to construct a sequence $R^{(n)}$ of risk processes with heavy–tailed claim size distribution which converges to a limiting process Q. The process Q is α–stable Lévy motion with positive linear drift. If the risk process $R^{(n)}$ is approximated by a limiting process Q then, under some regularity conditions, the ruin probabilities for $R^{(n)}$ should also approximate the ruin probabilities for the limiting process Q. This "continuity" argument was developed in Section 4.1 for the finite time case and in Section 4.2 for an infinite time horizon. For Brownian motion with linear drift there is an explicit formula for the distribution of the infimum over a finite time interval, see for instance Asmussen [1, XII.3]. However, to our knowledge there is no analogous formula for the non–Gaussian stable case. Asmussen and Klüppelberg [4] consider a classical risk process with regularly varying claim size distribution $\bar{F} \in RV_{-\alpha}$, $\alpha > 1$, and obtain the following result:

$$\lim_{x\to\infty} \frac{\Psi(x,xt)}{\Psi(x)} = 1-(1+c(1-\rho)t)^{-\tilde{\alpha}} \ , \quad \tilde{\alpha} = \alpha - 1 \, ,$$

which suggests that the study of ruin probabilities within infinite time is also relevant for the finite time case when x is large. We conjecture that a similar result also holds for the hitting time probabilities of any spectrally negative Lévy process with regularly varying Lévy measure, e.g. an α–stable Lévy motion with linear drift. In studying the numerical computations in Sections 4.1 and 4.2 we see that stable processes work reasonably well in the context of ruin type approximations when α is small. When $1.5 < \alpha < 2$, the convergence of convolutions to their limiting stable distribution is rather slow, explaining the moderate accuracy of the ruin probability approximations in these cases.

Our interest in this work mainly focussed on probabilistic models in order to describe the dynamics of a stochastic process which incorporates heavy–tailed jumps. In doing so we neglected the statistical aspects. An insurer for instance who wants to adapt the model described in Chapter 3 for his insurance business is inevitably faced with statistical fitting problems. It would be interesting to see how well real data fit the model in Chapter 3. We hope that this work will stimulate further research in any of the possible directions mentioned above.

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Curriculum Vitae

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1975 – 1981	Primary school in Schlatt ZH
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1988 – 1993	Study of mathematics at ETH Zürich, specializing in probability theory. Diploma under the supervision of Prof. P. Embrechts: "Risk processes perturbed by diffusion". Walter Saxer–Versicherungs–Hochschulpreis for diploma dissertation
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