On The Complexity Of Hamel Bases Of Infinite Dimensional Banach Spaces

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Abstract

We call a subset S of a topological vector space V *linearly Borel*, if for every finite number n, the set of all linear combinations of S of length n is a Borel subset of V. It will be shown that a Hamel base of an infinite dimensional Banach space can never be linearly Borel. This answers a question of Anatolij Plichko.

In the sequel, let X be any infinite dimensional Banach space. A subset S of X is called **linearly Borel (w.r.t. X)**, if for every positive integer n, the set of all linear combinations with n vectors of S is a Borel subset of X. Since X is a complete metric space, X is a **Baire space**, *i.e.*, a space in which non-empty open sets are not meager (*cf.* [1, Section 3.9]). Moreover, all Borel subsets of X have the **Baire property**, *i.e.*, for each Borel set S, there is an open set $\mathcal{O}$ such that $\mathcal{O}\Delta S$ is meager, where $\mathcal{O}\Delta S = (\mathcal{O} \setminus S) \cup (S \setminus \mathcal{O})$.

This is already enough to prove the following.

Theorem. If X is an infinite dimension Banach space and H is a Hamel base of X, then H is not linearly Borel (w.r.t. X).

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Proof: Let X be any infinite dimensional Banach space over the field $\mathbb{F}$ and let H be any Hamel base of X. For a positive integer n, let $[H]^n$ be the set of all n-element subsets of H and let

$$H_n := \left\{ \sum_{i=1}^{n} \alpha_i h_i : \ \alpha_1, \dots, \alpha_n \in \mathbb{F} \setminus \{0\} \textit{ and } \{h_1, \dots, h_n\} \in [H]^n \right\}.$$

Assume towards a contradiction that H is linearly Borel. Then, by definition, for each positive integer n, H_n is Borel, and hence, by the facts mentioned above, H_n has the Baire property. Since H is a Hamel base of X, we get

$$X = \bigcup_{n=1}^{\infty} H_n \,,$$

and because X is a Baire space, there must be a least positive integer m such that H_m is not meager. Because H_m has the Baire property and is not meager, there is a non-empty open set $\mathcal{O}$ such that $\mathcal{O} \Delta H_m$ is meager. Since H is a Hamel base, $\mathcal{O} \setminus H_m$ cannot be empty, and therefore, $\mathcal{O} \setminus H_m$ is a non-empty meager set. Let $B_{v,r}$ denote the open ball with center $v \in X$ and radius r. Let $x \in H_m \cap \mathcal{O}$ and let ε be such that $B_{x,2\varepsilon} \subseteq \mathcal{O}$. Take any $y \in H_{3m+1}$ with $\|y\| < \varepsilon$, then $B_{x+y,\varepsilon} \subseteq \mathcal{O}$. The following map is a homeomorphism from $B_{x,2\varepsilon}$ into $B_{x+y,\varepsilon}$:

$$\begin{array}{rrcl} \Phi : & B_{x,2\varepsilon} & \longrightarrow & B_{x+y,\varepsilon} \\ & z & \longmapsto & x + y + \frac{1}{2}(z - x) \end{array}$$

Since $\mathcal{O} \setminus H_m$ is meager, both sets, $B_{x,2\varepsilon} \setminus H_m$ as well as $B_{x+y,\varepsilon} \setminus H_m$, are meager, and further, by the definition of Φ, also $B_{x+y,\varepsilon} \setminus \Phi[H_m]$ is meager, where $\Phi[H_m] := \{\Phi(z) : z \in H_m \cap B_{x,2\varepsilon}\}$. Now, because we have chosen $y \in H_{3m+1}$, $\Phi[H_m] \cap H_m = \emptyset$, and hence,

$$B_{x+y,\varepsilon} = \big(B_{x+y,\varepsilon} \setminus H_m\big) \cup \big(B_{x+y,\varepsilon} \setminus \Phi[H_m]\big) \,,$$

which implies that the open set $B_{x+y,\varepsilon}$, as the union of two meager sets, is meager. But this is a contradiction to the fact that X is a Baire space. $\dashv$

References

[1] Charalambos D. Aliprantis and Kim C. Border: "Infinite Dimensional Analysis: A Hitchhiker's Guide." *2nd Ed.*, Springer-Verlag, Berlin · Heidelberg (1999).