Sets and Multisets of Range Uniqueness for Polynomials

Lorenz Halbeisen, Norbert Hungerbühler, Salome Schumacher*

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Abstract

A set $S \subseteq \mathbb{R}$ is called a set of range uniqueness (SRU) for the set $\mathcal{P}_n$ of real polynomials of degree at most n, if for all $f, g \in \mathcal{P}_n$, $f[S] = g[S] \implies f = g$. We show that for every natural number n, there are SRUs for $\mathcal{P}_n$ of cardinality $2n+1$, but there are no such SRUs of size $2n$. We also construct SRUs for the set $\mathcal{P}$ of all real polynomials.

Key words: sets of range uniqueness, polynomials, multisets of range uniqueness, magic sets, unique range, Vandermonde
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1 Introduction

Let F be a set of functions from $\mathbb{R}$ to $\mathbb{R}$. Then a set $S \subseteq \mathbb{R}$ is called a *set of range uniqueness (SRU) for F* if the following implication holds: For all $f, g \in F$,

$$f[S] = g[S] \implies f = g$$

where $f[S] := \{y \in \mathbb{R} \mid \exists x \in S \ (f(x) = y)\}$. A set $S \subseteq \mathbb{R}$ is called a *multiset of range uniqueness (MSRU) for F* if the above implication holds when $f[S]$ and $g[S]$ are interpreted as multisets, where multisets are collections in which the elements can appear more than once. The concepts SRU and MSRU carry over in the obvious way to functions on $\mathbb{C}$ instead of $\mathbb{R}$.

Clearly, if S is an (M)SRU for a set F, then S is also an (M)SRU for any subset $G \subseteq F$. On the other hand, we will say that S is a *disassociating (M)SRU for $G \subseteq F$* if S is an (M)SRU for G, but not for F.

The question of the existence of SRUs has been studied in the past quite intensively. For example, SRUs always exist (*i.e.*, provable in ZFC) for the set of all Lebesgue-measurable functions on $\mathbb{R}$, as has been shown by Burke and Ciesielski in [2]. In [4] Diamond, Pomerance, and Rubel construct SRUs for the set $C^\omega(\mathbb{C})$ of entire functions: In particular, for $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$, $\{\frac{1}{n} \mid n \in \mathbb{N}^*\}$,

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$\left\{\frac{1}{n!} \mid n \in \mathbb{N}^*\right\}$ and $\left\{\frac{1}{\ln(n+1)} \mid n \in \mathbb{N}^*\right\}$ are SRUs for $C^\omega(\mathbb{C})$. Notice that, for example, $S := \left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\}$ is not an SRU for the set of functions $C^\infty(\mathbb{C})$, since

$$f(x+iy) = \begin{cases} \exp\left(-\frac{1}{x^2}\right)\sin\left(\frac{\pi}{x}\right) & \text{for } x+iy \neq 0, \\ 0 & \text{for } x+iy = 0, \end{cases}$$

and the zero-function $g(x) = 0$ agree on S. Hence, $S = \left\{\frac{1}{n} \mid n \in \mathbb{N}^*\right\}$ is a dis-associating SRU for $C^\omega(\mathbb{C}) \subseteq C^\infty(\mathbb{C})$. The continuum hypothesis implies the existence of an SRU for the class $C^n(\mathbb{R})$ of continuous nowhere constant functions from $\mathbb{R}$ to $\mathbb{R}$ (see the work [1] of Berarducci and Dikranjan). Halbeisen, Lischka and Schumacher have replaced the continuum hypothesis by a weaker condition (see [5]), but the existence of such a set is not provable in ZFC. In [3], Burke and Ciesielski have shown that a meager SRU for the family of continuous functions satisfying the *Luzin N-condition* always exists for the class of differentiable functions and the class of absolutely continuous functions.

If we consider the full regularity spectrum of function spaces, we see that the question of SRUs for polynomials has not yet been touched. It is the aim of this article to close this gap. We start in Section 2 by constructing SRUs for the set $\mathbb{R}[x]$ of real polynomials in one variable. Surprisingly, the question of an SRU for the finite dimensional vector spaces of polynomials of bounded degree is then much harder to answer (see Sections 3 and 4).

2 An MSRU and an SRU for the set of polynomials

The aim of this section is to construct an SRU for the set $\mathbb{R}[x]$ of real polynomials in one variable which is not an SRU for the set of entire functions.

Theorem 1. *The set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers is an MSRU for the set $\mathbb{R}[x]$ of real polynomials in one variable.*

Proof. Let $p \in \mathbb{R}[x]$ be a polynomial. We want to show that p can be reconstructed from the multiset $p[\mathbb{N}]$. To do this, we arrange the multiset $p[\mathbb{N}]$ in ascending order $\xi_0 \leq \xi_1 \leq \xi_2 \leq \ldots$ if $p[\mathbb{N}]$ is bounded from below, and in descending order if $p[\mathbb{N}]$ is bounded from above. In what follows it suffices to consider the first case, the second case is analogous.

There exists $\alpha \geq 0$ such that p is monotone increasing on $[\alpha, \infty)$. Let

$$M := \max\left\{p(x) \mid x \in [0, \alpha]\right\}.$$

Then there is a number $\beta \geq \alpha$, $\beta \in \mathbb{N}$, such that $p(\beta) \geq M$. It follows that

$$\xi_n = p(n) \text{ for all } n \in \mathbb{N}, n \geq \beta. \tag{1}$$

Hence, any two polynomials which have the same image of $\mathbb{N}$ must agree on an end-segment of $\mathbb{N}$ and are therefore equal. $\square$

In order to actually identify the polynomial p from its multiset $p(\mathbb{N})$, one can proceed as follows: Consider the difference operator Δ acting on the set of sequences:

$$\Delta(a_0, a_1, a_2, \ldots) := (a_1 - a_0, a_2 - a_1, \ldots).$$

We apply Δ repeatedly to the sequence $\xi := (\xi_0, \xi_1, \xi_2, \ldots)$: $\Delta^{n+1}(\xi) = \Delta(\Delta^n(\xi))$. Then it follows from (1) that there is an iteration $\Delta^g(\xi)$ which has a constant tail $\Delta^g(\xi) = (*, *, \ldots, *, c, c, c, \ldots)$, $c \neq 0$. We conclude that g is the degree of the polynomial p, and that p is the unique interpolation polynomial of this degree through the points $(\beta, \xi_\beta), (\beta+1, \xi_{\beta+1}), \ldots, (\beta+g, \xi_{\beta+g})$.

Remarks.

1. Observe that $\mathbb{N}$ is not an SRU for the set of polynomials $\mathbb{R}[x]$: For example, for the polynomials $p(x) = x(x-1)$ and $q(x) = x(x+1)$, we have that the sets $p[\mathbb{N}]$ and $q[\mathbb{N}]$ agree.

2. We also remark, that there is no algorithm which would allow to compute p from the multiset $p[\mathbb{N}]$, since one cannot verify in finitely many steps if a certain iteration $\Delta^n(\xi)$ has a constant tail.

3. It is easy to see that every cofinite subset of $\mathbb{N}$ is also an MSRU for the set of polynomials $\mathbb{R}[x]$. On the other hand, a finite set cannot be an MSRU for the set of polynomials $\mathbb{R}[x]$.

4. As a last remark we would like to mention that for any transcendental number τ, $\{\tau\}$ is an SRU for the set $\mathbb{Q}[x]$ of rational polynomials. The reason is that the reals in the field $\mathbb{Q}(\tau)$ form an infinite dimensional vector space over $\mathbb{Q}$ with basis $\{\tau^n \mid n \in \mathbb{N}\}$. With a similar argument one can show, for example, that for each prime p and for every $n \in \mathbb{N}^*$, $\{p^{\frac{1}{n+1}}\}$ is a disassociating SRU for $\mathcal{Q}_n \subseteq \mathcal{Q}_{n+1}$, where $\mathcal{Q}_n$ and $\mathcal{Q}_{n+1}$ denote the rational polynomials of degree at most n and $n+1$, respectively.

Theorem 2. *The set $S := \mathbb{N} \cup \left\{n + \frac{1}{n} \mid n \in \mathbb{N}, n > 0\right\}$ is an SRU for the set $\mathbb{R}[x]$ of real polynomials in one real variable. S is not an SRU for the set $C^\omega(\mathbb{C})$ of entire functions.*

Proof. Let $p \in \mathbb{R}[x]$ be a polynomial. We will show, that p can be reconstructed from the set $p[S]$. To do so, we first sort the set $p[S]$ in ascending order $\xi_0 < \xi_1 < \xi_2 < \ldots$ if $p[S]$ is bounded from below, and in descending order if $p[S]$ is bounded from above. We consider only the first case, the second is analogous.

Let, as in the proof of Theorem 1, $\beta \in \mathbb{N}$ be such that p is monotone increasing on $[\beta, \infty)$ and $p(x) \leq p(\beta)$ for all $x \in [0, \beta]$. In particular, the values $p(n)$ and $p\left(n + \frac{1}{n}\right)$ are distinct for all $n \geq \beta$. Hence, for some $k \in \mathbb{N}$ we have:

$$\xi_k = p(\beta) < \xi_{k+1} = p\Big(\beta + \frac{1}{\beta}\Big) < \xi_{k+2} =$$
$$= p(\beta+1) < \xi_{k+3} = p\Big(\beta + 1 + \frac{1}{\beta+1}\Big) < \ldots$$

If we apply repeatedly the difference operator Δ to the two sequences $\xi_{\text{even}} := (\xi_{2n})_{n\in\mathbb{N}}$ and $\xi_{\text{odd}} := (\xi_{2n+1})_{n\in\mathbb{N}}$ we will find that, depending on the parity of k, exactly one of the sequences $\Delta^g(\xi_{\text{even}})$ or $\Delta^g(\xi_{\text{odd}})$ has a constant tail $(*, *, \ldots, *, c, c, c, \ldots)$, $c \neq 0$, for some $g \in \mathbb{N}$. In fact, if k is even, then $\Delta^g(\xi_{\text{even}})$ has a constant tail and g is the degree of p, if k is odd, then $\Delta^g(\xi_{\text{odd}})$ has a constant tail and g is the degree of p. Observe that for the the sequence $\eta = (p(n + \frac{1}{n}))_{n\in\mathbb{N}^*}$ the m-th difference sequence $\Delta^m(\eta)$ can never have a zero tail for some $m \in \mathbb{N}$ (and hence, no constant tail for $m-1$). This is because the n-th term in the sequence $\Delta^m(\eta)$ is given by a rational function with a pole in 0 evaluated in n. Such a function cannot have infinitely many zeros.

Now, we consider the unique interpolation polynomial q of degree g through the points $(0, \xi_k), (1, \xi_{k+2}), (2, \xi_{k+4}), \ldots$. Then p must be one of the polynomials $q_j(x) := q(x - j)$, $j \in \mathbb{N}$, namely the only q_j for which $q_j(j + \frac{1}{j}) = \xi_{k+1}$.

It remains to show that S is not an SRU for the set of entire functions. Indeed, according to the Weierstrass product theorem, there is an entire function with zeros exactly in S and which therefore agrees with the zero function on S. $\square$

By applying Theorem 2 separately to the real and imaginary part of complex polynomials, one obtains the following:

Corollary 3. *The set* $S := \mathbb{N} \cup \left\{n + \frac{1}{n} \mid n \in \mathbb{N}^*\right\}$ *is a disassociating SRU for the set* $\mathbb{C}[z]$ *of complex polynomials in one complex variable considered as a subset of the entire functions* $C^\omega(\mathbb{C})$.

3 An SRU for $\mathcal{P}_n$ of size $2n + 1$

In this section we will show that for every $s \geq 2n + 1$ there is an SRU of size s for the set $\mathcal{P}_n$ of all real polynomials of degree at most n. For this, we first introduce a special type of directed graphs.

Definition 4. A *directed graph* G is a pair (V, E), where V is a set (the *vertices* of G) and $E \subseteq V \times V$ (the *edges* of G). The elements of E are denoted (v_i, v_j), where $v_i, v_j \in V$. For $v \in V$, we define

$$\begin{aligned} \text{indegree}_G(v) &:= \left|\left\{v' \in V : (v', v) \in E\right\}\right|, \\ \text{outdegree}_G(v) &:= \left|\left\{v' \in V : (v, v') \in E\right\}\right|. \end{aligned}$$

Before we consider special directed graphs, let us give a few general definitions:

Definition 5. Let $G = (V, E)$ be a directed graph.

- A *cycle* is a subgraph $C = (V_C, E_C)$ of G with $V_C = \{c_0, c_1, \ldots, c_{m-1}\}$ and $E_C = \{(c_i, c_{(i+1) \bmod m}) \mid i \in \mathbb{N}\}$ for an $m \geq 2$.
- A *loop* is a subgraph $L = (V_L, E_L)$ of G with $V_L = \{w\}$ and $E_L = \{(w, w)\}$.
- A *path* is a subgraph $P = (V_P, E_P)$ of G with $V_P = \{p_0, p_1, \ldots, p_{m-1}\}$ and $E_C = \{(p_i, p_{i+1}) \mid 0 \leq i \leq m - 2\}$ for an $m \geq 2$.

Let $k, n \in \mathbb{N}^*$ with $k \geq 2n$ and let $\{x_0, x_1, \ldots, x_k\} \subseteq \mathbb{R}$. For all $0 \leq i \leq k$ let $v_i := (x_i, x_i^2, \ldots, x_i^n)$. The following family $\mathcal{G}$ of directed graphs will play a crucial role in the construction of SRUs of size $2n + 1$ for the set $\mathcal{P}_n$:

$\mathcal{G}$ is the family of all directed graphs $G = (V, E)$ with vertex set $V = \{v_0, v_1, \ldots, v_k\}$ and a set E of directed edges (v_i, v_j), such that for each $v \in V$ we have

$$\text{indegree}_G(v) \geq 1 \quad \text{and} \quad \text{outdegree}_G(v) \geq 1\,.$$

Definition 6. Let $l \in \mathbb{N}$. Cycles and loops $C_0 = (V_{C_0}, E_{C_0}), \ldots, C_l = (V_{C_l}, E_{C_l})$ are called *obviously different* if for every $0 \leq i \leq l$ there is a $y_i \in V_{C_i}$ with

$$y_i \notin \left(\bigcup_{j=0}^{l} V_{C_j}\right) \setminus V_{C_i}.$$

We partition the family $\mathcal{G}$ of directed graphs $G = (V, E)$ into two parts, namely the graphs of type 1_n and the graphs of type 2_n.

Definition 7. A graph $G = (V, E) \in \mathcal{G}$ is of *type* 1_n *iff* there are at most n obviously different cycles and loops in G. Otherwise G is of *type* 2_n.

In Sections 3.1 and 3.2, we consider graphs of type 1_n and we will show in Proposition 20, that for every graph $G = (V, E)$ of type 1_n and all sets $U \in \mathbb{R}^{k+1}$ which are open in the box topology, there is a $(2n+1) \times (2n+1)$-matrix

$$M_G(x_0, x_1, \ldots, x_k) = \begin{pmatrix} 1 & v_{i_0} & -v_{j_0} \\ 1 & v_{i_1} & -v_{j_1} \\ \vdots & \vdots & \vdots \\ 1 & v_{i_{2n}} & -v_{j_{2n}} \end{pmatrix}$$

with $i_l, j_l \in \{0, 1, \ldots, k\}$ (for $0 \leq l \leq 2n$) and $(v_{i_l}, v_{j_l}) \in E$ (for $0 \leq l \leq 2n$), and an open set $U_G \subseteq U$ in the box topology, such that for all $(x_0, x_1, \ldots, x_k) \in U_G$ we have

$$\det\big(M_G(x_0, x_1, \ldots, x_k)\big) \neq 0\,. \tag{2}$$

Concerning graphs $H = (V, E)$ of type 2_n, let $C_0 = (V_{C_0}, E_{C_0}), \ldots, C_n = (V_{C_n}, E_{C_n})$ be $n+1$ obviously different loops and cycles. Let $x_{i_0}, x_{i_1}, \ldots, x_{i_n}$ be $n+1$ vertices of H such that for each $0 \leq l \leq n$, $x_{i_l} \in V_{C_l}$ and

$$x_{i_l} \notin \left(\bigcup_{m=0}^{n} V_{C_m}\right) \setminus V_{C_l}.$$

We will show in Section 3.3 that for every open set $U \subseteq \mathbb{R}^{k+1}$ in the box topology there is an open set $U_H \subseteq U$ in the box topology such that for all $(x_0, x_1, \ldots, x_k) \in U_H$ we have

$$\det\big(M_H(x_0, x_1, \ldots, x_k)\big) \neq 0\,, \tag{3}$$

where

$$M_H(x_0, x_1, \ldots x_k) = \begin{pmatrix} |V_{C_0}| & \sum_{x \in V_{C_0}} x & \sum_{x \in V_{C_0}} x^2 & \cdots & \sum_{x \in V_{C_0}} x^n \\ |V_{C_1}| & \sum_{x \in V_{C_1}} x & \sum_{x \in V_{C_1}} x^2 & \cdots & \sum_{x \in V_{C_1}} x^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & \sum_{x \in V_{C_n}} x & \sum_{x \in V_{C_n}} x^2 & \cdots & \sum_{x \in V_{C_n}} x^n \end{pmatrix}.$$

As a consequence of (2) and (3), and since $|\mathcal{G}| < \infty$, we can find a point $(m_0, m_1, \dots, m_k) \in \mathbb{R}^{k+1}$ such that for all $G \in \mathcal{G}$ of type 1_n

$$\det\left(M_G(m_0, m_1, \dots, m_k)\right) \neq 0$$

and for all $H \in \mathcal{G}$ of type 2_n

$$\det\left(M_H(m_0, m_1, \dots, m_k)\right) \neq 0.$$

This leads to the following

Theorem 8. *The set $S := \{m_0, m_1, \dots, m_k\}$ is an SRU for $\mathcal{P}_n$.*

Proof. Assume towards a contradiction that S is not an SRU for $\mathcal{P}_n$. So, there are two polynomials

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

and

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

such that $f \neq g$ but $f[S] = g[S]$. Let $G = (V, E)$ with

$$V := S \text{ and } E := \{(m_i, m_j) \mid f(m_i) = g(m_j)\}.$$

Note that $G \in \mathcal{G}$. There are two cases:

Case 1: G is of type 1_n.
In this case

$$M_G(m_0, m_1 \dots, m_k) = \begin{pmatrix} 1 & v_{i_0} & -v_{j_0} \\ 1 & v_{i_1} & -v_{j_1} \\ \vdots & \vdots & \vdots \\ 1 & v_{i_{2n}} & -v_{j_{2n}} \end{pmatrix}$$

has non-zero determinant. Note that for all $0 \leq l \leq n$ we have that

$$f(m_{i_l}) = g(m_{j_l}) \iff$$
$$(a_0 - b_0) + (a_1 m_{i_l} + \cdots + a_n m_{i_l}^n) - (b_1 m_{j_l} + \cdots + b_n m_{j_l}^n) = 0.$$

So, f and g satisfy the following system of linear equations:

$$M_G(m_0, \dots, m_k) \cdot \begin{pmatrix} a_0 - b_0 \\ a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det(M_G(m_0,\dots,m_k)) \neq 0$, this equation has a unique solution, namely

$$\begin{pmatrix} a_0 - b_0 \\ a_1 \\ \vdots \\ a_n \\ b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore, $f = g$, which is a contradiction to our assumption that S is not an SRU.

Case 2: G is of type 2_n.
In this case

$$M_H(m_0,\dots,m_k) = \begin{pmatrix} |V_{C_0}| & \sum_{x\in V_{C_0}} x & \sum_{x\in V_{C_0}} x^2 & \cdots & \sum_{x\in V_{C_0}} x^n \\ |V_{C_1}| & \sum_{x\in V_{C_1}} x & \sum_{x\in V_{C_1}} x^2 & \cdots & \sum_{x\in V_{C_1}} x^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & \sum_{x\in V_{C_n}} x & \sum_{x\in V_{C_n}} x^2 & \cdots & \sum_{x\in V_{C_n}} x^n \end{pmatrix}$$

with $n+1$ obviously disjoint cycles $C_0,\dots,C_n$. For all $0 \le i \le n$ we have that

$$\sum_{m\in V_{C_i}} (f-g)(m) = 0.$$

In other words, we have to solve the following system of linear equations:

$$M_H(m_0,\dots,m_k) \cdot \begin{pmatrix} a_0 - b_0 \\ a_1 - b_1 \\ \vdots \\ a_n - b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since $\det(M_H(m_0,\dots,m_k)) \neq 0$ this equation has a unique solution, namely

$$\begin{pmatrix} a_0 - b_0 \\ a_1 - b_1 \\ \vdots \\ a_n - b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Therefore, $f = g$, which is again a contradiction.

□

It remains to prove the equations (2) and (3), respectively.

3.1 Graphs of type 1_n

Definition 9. Let $G = (V, E) \in \mathcal{G}$ be a directed graph and let $G' = (V', E') \subseteq G$. For each vertex $v \in V'$ we define

$$\deg_{G'}(v) := \text{indegree}_{G'}(v) + \text{outdegree}_{G'}(v).$$

Moreover, for all $v \in V \setminus V'$ we define $\deg_{G'}(v) := 0$.

Definition 10. Let $n \in \mathbb{N}^*$ and let $G = (V, E)$ be a graph of type 1_n with $|V| \geq 2n + 1$. A *nice sequence* of length $m \in \mathbb{N}$ of G is a sequence of graphs

$$G_0 = (V_0, E_0) \subseteq G_1 = (V_1, E_1) \subseteq \cdots \subseteq G_m = (V_m, E_m) \subseteq G = (V, E)$$

with the following properties: For all $0 \leq i \leq m$

1. we have that $|E_i| \in \{2i, 2i + 1\}$;
2. there are at most i obviously different loops and cycles in G_i;
3. we have that $E_{i+1} \setminus E_i$ has one of the following forms:
   - $E_{i+1} \setminus E_i = \{(v_j, v_j), (v_k, v_l)\}$ with $\deg_{G_i}(v_j) = 0$, and $\deg_{G_i}(v_k) = 0$ or $\deg_{G_i}(v_l) = 0$;
   - $E_{i+1} \setminus E_i = \{(v_j, v_k), (v_l, v_j)\}$ with $\deg_{G_i}(v_j) = 0$.

Definition 11. Two directed graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called *undirected edge disjoint* if and only if the corresponding undirected graphs do not share any edges.

Lemma 12. *Let $n \in \mathbb{N}^*$. Every graph $G = (V, E)$ of type 1_n with $|V| \geq 2n+1$ has a nice sequence*

$$G_0 = (V_0, E_0) \ \subseteq \ G_1 = (V_1, E_1) \ \subseteq \ \ldots \ \subseteq \ G_m = (V_m, E_m) \ \subseteq \ G$$

of length m with $|E_m| \geq 2n + 1$.

Proof. Let $G = (V, E)$ be a graph of type 1_n. Let $\mathcal{L}$ be the set of all isolated loops of G. To be more precise

$$\mathcal{L} := \{(\{v\}, \{(v, v)\}) \subseteq G \mid \deg_G(v) = 2\}.$$

Notice that since G is of type 1_n, $|\mathcal{L}| \leq n$, and since $G \in \mathcal{G}$, at least $n + 1$ edges belong to cycles or paths.

<u>How to construct G_m.</u> (See also Example 13.)

We start with the empty graph $H_0 := (\emptyset, \emptyset)$.

Step 1: ***Adding cycles***

Let $C_0, C_1, \ldots, C_l$ be a maximal family of pairwise disjoint cycles. First, add $C_0 = (V_{C_0}, E_{C_0})$ to H_0, and then add a maximal subset $\mathcal{M} \subseteq \mathcal{L}$ to $H_0 + C_0$ with

$$|\mathcal{M}| \leq |E_{C_0}| - 2.$$

The resulting graph is called H_1^0. Furthermore, let $\mathcal{L}_1^0 := \mathcal{L} \setminus \mathcal{M}$. Repeat the same construction with respect to C_1, a maximal subset $\mathcal{M} \subseteq \mathcal{L}_1^0$, and the graph H_1^0, in order to obtain H_1^1, and so on. We define $H_1 = (V_{H_1}, E_{H_1}) := H_1^l$ and $\mathcal{L}_1 := \mathcal{L}_1^l$. Note that in this graph $|V_{H_1}| = |E_{H_1}|$.

Step 2: ***Adding paths***

Let $P_0 = (V_{P_0}, E_{P_0})$ be a maximal path in G which is undirected edge disjoint from H_1. In addition, we require that all vertices of P_1 (except possibly the first or the last one) are disjoint from the vertices in H_1. We allow P_0 to start and end in the same vertex if this vertex is in H_1. In this case, P_0 is a cycle which shares a vertex with one of the cycles $C_0, \dots, C_l$. Since $G \in \mathcal{G}$, we have that if P_0 starts (or ends) in a vertex which is not in H_1, it starts (or ends) in a loop, and in this case, we add these loops to P_0. Let $l_0 \in \{0, 1, 2\}$ be the number of loops in P_0. There are two cases:

- $|\mathcal{L}_1| \leq |E_{P_0}| - l_0 - 1$
  If $|E_{P_0}| + |\mathcal{L}_1|$ is odd, then remove the first edge (which might be a loop) from the path P_0. Otherwise, do not modify P_0. Then add P_0 and $\mathcal{L}_1$ to H_1. This new graph is called H_2^0 and we define $\mathcal{L}_2^0 := \emptyset$. Note that there is a surjection from the set of all edges of H_2^0 to the set of all vertices of H_2^0.
- $|\mathcal{L}_1| > |E_{P_0}| - l_0 - 1$, *i.e.*, $|\mathcal{L}_1| \geq |E_{P_0}| - l_0$
  Let $\mathcal{M} \subseteq \mathcal{L}_1$ be a $(|E_{P_0}| - l_0 - 1)$-element subset. Now, remove the first edge (which might be a loop) from P_0, add this new path to H_1, and add $\mathcal{M}$ to H_1. The resulting graph is called H_2^0. Moreover we define $\mathcal{L}_2^0 := \mathcal{L}_1 \setminus \mathcal{M}$. Note that there is a surjection from the set of all edges of H_2^0 to the set of all vertices of H_2^0.

Repeat the same construction with respect to H_2^0 and $\mathcal{L}_2^0$, in order to obtain H_2^1, and so on. Finally, let $m := \left\lfloor \frac{|E_{G_m}|}{2} \right\rfloor$, denote the resulting graph $G_m = (V_{G_m}, E_{G_m})$ and the resulting set of loops $\mathcal{L}_m$. Note that by construction, $|V_{G_m}| \leq |E_{G_m}|$, and since $|\mathcal{L}| \leq n$, $\mathcal{L}_m = \emptyset$.

How to construct G_i for $1 \leq i \leq n$. (See also Example 14.)

We start with the graph G_m and first construct G_{m-1}. For this let $C_0, \dots, C_l$ be the pairwise disjoint cycles from Step 1 and let $P_0, \dots, P_s$ be the paths from Step 2 in the order we added them to the graph. For each $0 \leq i \leq l$ let $\mathcal{M}_i \subseteq \mathcal{L}$ be the set of all loops we added to the graph together with the cycle C_i. And for each $0 \leq j \leq s$ let $\mathcal{N}_j \subseteq \mathcal{L}$ be the set of all loops we added to the graph together with the path P_j. First of all we will completely remove E_{P_s} from G_m. This is possible because $|E_{P_S}| + |\mathcal{N}_s|$ is even.

Case 1: There is a loop (v, v) in $P_s = (V_{P_s}, E_{P_s})$.

We define

$$G_{m-1} := (V_{G_m}, E_{G_m} \setminus \{(a, b) \in E_{P_s} \mid a = v \text{ or } b = v\}).$$

Remove the vertex v and the corresponding edges from P_s.

Case 2: We are not in Case 1 and there is a vertex v in P_s with $\deg_{G_m}(v) = 1$.
If $\mathcal{N}_s \neq \emptyset$ let e_0 be a loop from $\mathcal{N}_s$. Moreover let $e_1 \in E_{P_s}$ be the edge that contains v. Define

$$G_{m-1} := (V_{G_m}, E_{G_m} \setminus \{e_0, e_1\}).$$

Remove v and e_1 from P_s.
If $\mathcal{N}_s = \emptyset$ there is a vertex $w \in V_{P_s}$ with $\deg_{G_m}(w) = 2$. We define

$$G_{m-1} := (V_{G_m}, E_{G_m} \setminus \{(a,b) \in E_{P_s} \mid a = w \text{ or } b = w\}).$$

Remove w and the corresponding edges from P_s.

Case 3: We are not in one of the previous cases.
There is a vertex $v \in V_{P_s}$ with $\deg_{G_m}(v) = 2$. We define

$$G_{m-1} := (V_{G_m}, E_{G_m} \setminus \{(a,b) \in E_{P_s} \mid a = v \text{ or } b = v\}).$$

Remove v and the corresponding edges from P_s.

After doing this process $k_s := \frac{|E_{P_s}|+|\mathcal{N}_s|}{2}$ many times, we found a sequence

$$G_m \supseteq G_{m-1} \supseteq \cdots \supseteq G_{m-k_s}$$

of graphs. Do the same with all other paths $P_{s-1}, \dots, P_1, P_0$.

Without loss of generality assume that G_{m-k_s} contains only cycles from Step 1 and the loops $\bigcup_{i=0}^{l} \mathcal{M}_i$. We will now remove all but at most one edge of C_l from G_{m-k_s}.

Case 1: Each vertex in V_{C_l} has degree 2 or 0.
Let $v \in V_{C_l}$ with degree $\deg_{G_{m-k_s}}(v) = 2$. We define

$$G_{m-k_s-1} := (V_{G_{m-k_s}}, E_{G_{m-k_s}} \setminus \{(a,b) \in E_{C_l} \mid a = v \text{ or } b = v\}).$$

Remove v and the corresponding edges from C_l.

Case 2: There is a vertex v in V_{C_l} with degree 1.
If $\mathcal{M}_l \neq \emptyset$ let e_0 be a loop from $\mathcal{M}_l$. Moreover, let $e_1 \in E_{C_l}$ be the edge that contains v. Define

$$G_{m-k_s-1} := (V_{G_{m-k_s}}, E_{G_{m-k_s}} \setminus \{e_0, e_1\}).$$

Remove e_1 and v from C_l. If $\mathcal{M}_l = \emptyset$ there is a vertex $w \in V_{C_l}$ with $\deg_{G_{m-k_s}}(w) = 2$. We define

$$G_{m-k_s-1} := (V_{G_{m-k_s}}, E_{G_{m-k_s}} \setminus \{(a,b) \in E_{P_s} \mid a = w \text{ or } b = w\}).$$

Remove w and the corresponding edges from C_l.

Repeat this process until $|E_{C_l}| \leq 1$, and then, repeat this procedure again with all other cycles. So, we found a sequence of graphs

$$G_{m-k_s} \supseteq G_{m-k_s-1} \supseteq \cdots \supseteq G_t$$

for some $t \in \mathbb{N}$. If $|E_{G_t}| \geq 2$, then any two distinct edges $e_0, e_1 \in E_{G_t}$ are from two different disjoint cycles. So, we can remove them. The resulting graph is called G_{t-1}. Redo this process until we found a graph with at most one edge. $\square$

Example 13. In this example we will construct the graph G_9 for the following graph G of type 1_n:

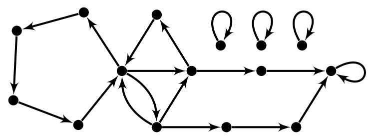


Figure 1: Graph $G = (V, E)$.

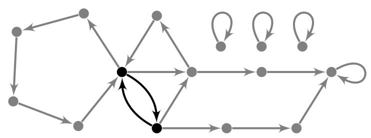


Figure 2: Cycle C_0.

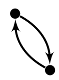


Figure 3: Graph H_1.

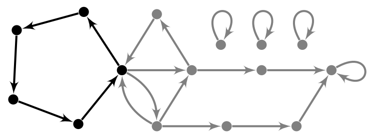


Figure 4: Path P_0.

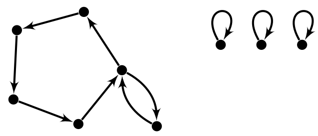


Figure 5: Graph H_2^0.

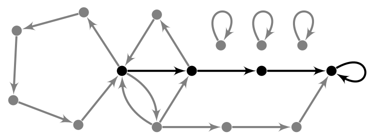


Figure 6: Path P_1.

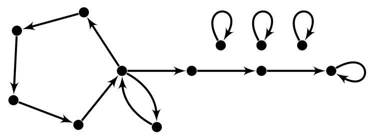


Figure 7: Graph H_2^1.

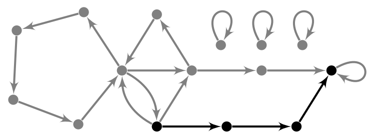


Figure 8: Path P_2.

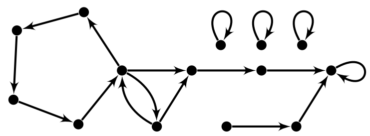


Figure 9: Graph H_2^2.

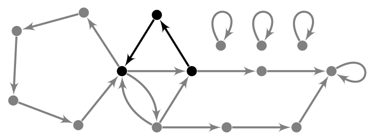


Figure 10: Path P_3.

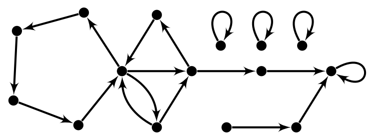


Figure 11: Graph $H_2^3 = G_9$.

Example 14. In this example we will construct a nice sequence for the graph G of Example 13. We start with the graph $G_m = G_9$ we found in Example 13:

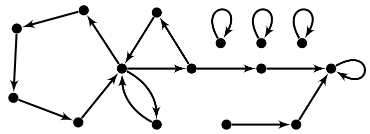


Figure 12: Graph G_9.

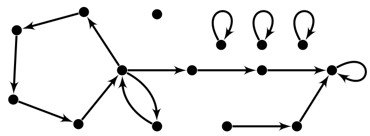


Figure 13: Graph G_8.

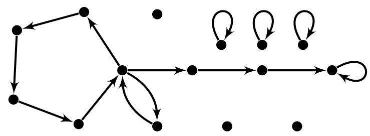


Figure 14: Graph G_7.

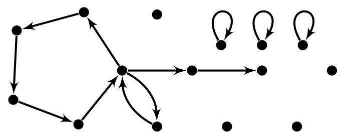


Figure 15: Graph G_6.

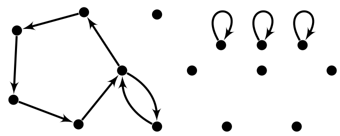


Figure 16: Graph G_5.

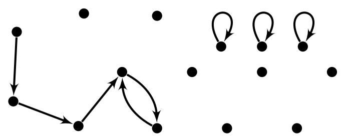


Figure 17: Graph G_4.

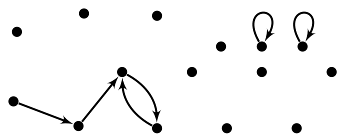


Figure 18: Graph G_3.

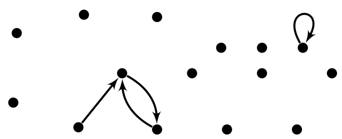


Figure 19: Graph G_2.

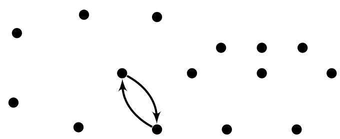


Figure 20: Graph G_1.

Corollary 15. *Let $n \in \mathbb{N}^*$, every graph $G = (V, E)$ of type 1_n with $|V| \geq 2n+1$ has a nice sequence*

$$G_0 = (V_0, E_0) \subseteq G_1 = (V_1, E_1) \subseteq \ldots \subseteq G_n = (V_n, E_n) \subseteq G$$

with $|E_n| = 2n + 1$.

Proof. Let $G = (V, E)$ be a graph of type 1_n. By Lemma 12, there is a nice sequence

$$H_0 = (V_0, E_0) \subseteq H_1 = (V_1, E_1) \subseteq \cdots \subseteq H_m = (V_m, E_m)$$

with $|E_i| \in \{2i, 2i+1\}$ (for all $0 \leq i \leq n$). If $|E_n| = 2n+1$, then we are done because

$$H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n$$

is a nice sequence with the right form. So, assume that $|E_n| = 2n$. In this case we have that $m \geq n+1$. Choose any $e_0 \in E_1$ (if possible, let e_0 be a loop). Then

$$\big(V_1, E_1 \setminus \{e_0\}\big) \ \subseteq \ \big(V_2, E_2 \setminus \{e_0\}\big) \ \subseteq \ \ldots \ \subseteq \ \big(V_{n+1}, E_{n+1} \setminus \{e_0\}\big)$$

is a nice sequence with the right form. □

3.2 Matrices of type 1_n

Let $k \geq n$, and for all $0 \leq i, j \leq k$ and all $0 \leq s \leq n$ define

$$1\_v_{i}\_ - v_j := (1, x_i, x_i^2, \ldots, x_i^s, -x_j, -x_j^2, \ldots, -x_j^s).$$

For every graph $G = (V, E)$ of type 1_n choose a nice sequence

$$G_0 = (V_0, E_0) \subseteq G_1 = (V_1, E_1) \subseteq \cdots \subseteq G_n = (V_n, E_n)$$

with $|E_n| = 2n+1$. For every graph G of type 1_n and all $0 \leq s \leq n$ let $M_{G_s}(x_0, \ldots, x_k)$ be a square matrix with pairwise different rows $1\_v_i\_ - v_j$ where $(v_i, v_j) \in E_{G_s}$. For all $0 \leq s \leq n$ we define

$$\mathcal{C}_s := \big\{ M_{G_s}(x_0, \ldots, x_k) \mid G \text{ is a graph of type } 1_n \big\}.$$

Furthermore, we define $M_G := M_{G_n}(x_0, \ldots, x_k)$.

Definition 16. Let $n \in \mathbb{N}^*$, let $k \geq 2n$, let $1 \leq s \leq n$, and let $C \in \mathcal{C}_s$. Assume that C has two rows of the form

$$1\_v_i\_ - v_j$$
$$1\_v_t\_ - v_l$$

with $0 \leq i, j, t, l \leq k$. Then we define $C^{1\_v_i\_ - v_j, 1\_v_t\_ - v_l}$ to be the matrix that we obtain from C by deleting the rows $1\_v_i\_ - v_j$ and $1\_v_t\_ - v_l$, as well as the $(s+1)$-th column and the $(2s+1)$-th column.

Lemma 17. *Let $n \in \mathbb{N}^*$, let $k \geq 2n$, let $1 \leq s \leq n$ and let $C \in \mathcal{C}_s$. Moreover, let $0 \leq i, j, t \leq k$ such that C has two rows of the form*

$$1\_v_i\_ - v_j$$
$$1\_v_j\_ - v_t$$

with $i \neq j$, $t \neq j$ and there are no other rows which contain v_j or $-v_j$. We assume that $\det(C^{1\_v_i\_ - v_j, 1\_v_j\_ - v_t}) \not\equiv 0$. Then we have that $\det(C) \not\equiv 0$.

Proof. First of all, we do a Laplace expansion of C along the row $1\_v_i\_ - v_j$. So, we have that

$$\det(C) = \epsilon_0 x_j^s \det(\overline{C}) + \gamma,$$

where $\overline{C}$ is the matrix that we obtain from C by deleting the row $1\_v_{i\_} - v_j$ and the $(2s+1)$-th column. Moreover, γ is a polynomial in which there is no term of the form x_j^{2s} and we have that $\epsilon_0 \in \{-1, 1\}$. Now we do a Laplace expansion along the remainders of the row $1\_v_{j\_} - v_t$. We get

$$\det(\overline{C}) = \epsilon_1 x_j^s \det(C^{1\_v_{i\_}-v_j, 1\_v_{j\_}-v_t}) + \delta,$$

where δ is a polynomial in which there is no term of the form x_j^s and $\epsilon_1 \in \{-1, 1\}$. So, we have that

$$\det(C) = \epsilon_0\epsilon_1 x_j^{2s} \det(C^{1\_v_{i\_}-v_j, 1\_v_{j\_}-v_t}) + \epsilon_0 x_j^s \delta + \gamma.$$

In the polynomial $\epsilon_0 x_j^s \delta + \gamma$ there is no term of the form x_j^{2s} and

$$\epsilon_0\epsilon_1 x_j^{2s} \det(C^{1\_v_{i\_}-v_j, 1\_v_{j\_}-v_t}) \not\equiv 0\,,$$

which concludes the proof of the lemma. □

Lemma 18. *Let $n \in \mathbb{N}^*$, let $k \geq 2n$, let $1 \leq s \leq n$ and let $C \in \mathcal{C}_s$. Moreover, let $0 \leq i, j, t \leq k$ such that C has two rows*

$$1\_v_{i\_} - v_i$$
$$1\_v_{j\_} - v_t$$

with $t \neq j$ and there are no other rows which contain $v_i, v_j, -v_i$ or $-v_j$. We assume that $\det(C^{1\_v_{i\_}-v_i, 1\_v_{j\_}-v_t}) \not\equiv 0$. Then we have that $\det(C) \not\equiv 0$.

Proof. First of all, we do a Laplace expansion of C along the row $1\_v_{i\_} - v_i$. So, we have that

$$\det(C) = \epsilon_0 x_i^s \det(\overline{C}) + \gamma,$$

where $\overline{C}$ is the matrix that we obtain from C by deleting the row $1\_v_{i\_} - v_i$ and the $(2s+1)$-th column. Moreover, γ is a polynomial in which there is no term of the form $x_i^s x_j^s$ and $\epsilon_0 \in \{-1, 1\}$. Now, we do a Laplace expansion along the remainders of the row $1\_v_{j\_} - v_t$. We get

$$\det(\overline{C}) = \epsilon_1 x_j^s \det(C^{1\_v_{i\_}-v_i, 1\_v_{j\_}-v_t}) + \delta,$$

where δ is a polynomial in which there is no term of the form x_j^s and $\epsilon_1 \in \{-1, 1\}$. So, we have that

$$\det(C) = \epsilon_0\epsilon_1 x_i^s x_j^s \det(C^{1\_v_{i\_}-v_i, 1\_v_{j\_}-v_t}) + \epsilon_0 x_i^s \delta + \gamma.$$

In the polynomial $\epsilon_0 x_i^s \delta + \gamma$ there is no term of the form $x_i^s x_j^s$ and

$$\epsilon_0\epsilon_1 x_i^s x_j^s \det(C^{1\_v_{i\_}-v_i, 1\_v_{j\_}-v_t}) \not\equiv 0\,,$$

which concludes the proof of the lemma. □

Lemma 19. *Let $n \in \mathbb{N}^*$, let $k \geq 2n$, let $1 \leq s \leq n$ and let $C \in \mathcal{C}_s$. Moreover, let $0 \leq i, j, t \leq k$ such that C has two rows*

$$1\_v_{i\_} - v_i$$
$$1\_v_{t\_} - v_j$$

with $t \neq j$ and there are no other rows which contain $v_i, v_j, -v_i$ or $-v_j$. We assume that $\det(C^{1\_v_{i\_}-v_i, 1\_v_{j\_}-v_t}) \not\equiv 0$. Then we have that $\det(C) \not\equiv 0$.

Proof. The proof is similar to the proof of Lemma 18. □

Proposition 20. *Let $n \in \mathbb{N}^*$, $k \geq 2n$ and $M_G = M_G(x_0, \ldots, x_k) \in \mathcal{C}_n$. Then for every open set $U \subseteq \mathbb{R}^{k+1}$ in the box topology there is an open set $U_G \subseteq U$ in the box topology such that*

$$\det(M_G) \neq 0$$

for all $(x_0, x_1, \ldots, x_k) \in U_G$.

Proof. It suffices to prove that $\det(M_G)$ is a non-zero polynomial in the $k+1$ variables $x_0, x_1, \ldots, x_k$. Let

$$G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n$$

be the nice sequence we used to construct M_G. Note that $M_{G_0} = \big(1\big)$, and therefore, $\det(M_{G_0}) = 1 \not\equiv 0$. Assume that for an i with $0 \leq i < n$, we have already shown that $\det(M_{G_i}) \not\equiv 0$. Now, we want to show that $\det(M_{G_{i+1}}) \not\equiv 0$. For this, let a and b be the two rows which are added to M_{G_i} in order to obtain $M_{G_{i+1}}$. Since the matrices M_{G_i} are constructed with a nice sequence, these two rows have one of the following three forms:

1. $a = 1\_v_{i\_} - v_j$ and $b = 1\_v_{j\_} - v_t$ with $0 \leq i, j, t \leq k$, $i \neq j, t \neq j$ and there are no other rows in $M_{G_{i+1}}$ which contain v_j or $-v_j$. In this case we apply Lemma 17.

2. $a = 1\_v_{i\_} - v_i$ and $b = 1\_v_{j\_} - v_t$ with $0 \leq i, j, t \leq k$, $t \neq j$ and there are no other rows in $M_{G_{i+1}}$ which contain $v_i, v_j, -v_i$ or $-v_j$. In this case we apply Lemma 18.

3. $a = 1\_v_{i\_} - v_i$ and $b = 1\_v_{t\_} - v_j$ with $0 \leq i, j, t \leq k$, $t \neq j$ and there are no other rows in $M_{G_{i+1}}$ which contain $v_i, v_j, -v_i$ or $-v_j$. In this case we apply Lemma 19.

So, we see that $\det(M_{G_{i+1}}) \not\equiv 0$, which concludes the proof of the proposition. □

3.3 Graphs and Matrices of type 2_n

Let

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

and

$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$$

be two polynomials and assume that the graph $G_{f,g}$ contains at least $n+1$ obviously different loops and cycles $C_0, C_1, \ldots, C_n$. For all $1 \leq i \leq n+1$ we have that

$$\sum_{x \in V_{C_i}} (f - g)(x) = 0.$$

The matrix belonging to this system of linear equations is given by

$$\underbrace{\begin{pmatrix} |V_{C_0}| & \sum_{x\in V_{C_0}} x & \sum_{x\in V_{C_0}} x^2 & \cdots & \sum_{x\in V_{C_0}} x^n \\ |V_{C_1}| & \sum_{x\in V_{C_1}} x & \sum_{x\in V_{C_1}} x^2 & \cdots & \sum_{x\in V_{C_1}} x^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & \sum_{x\in V_{C_n}} x & \sum_{x\in V_{C_n}} x^2 & \cdots & \sum_{x\in V_{C_n}} x^n \end{pmatrix}}_{=:M_{G_{f,g}}(x_0,\ldots x_k)} \begin{pmatrix} a_0-b_0 \\ a_1-b_1 \\ \vdots \\ a_n-b_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Our goal is to show that $\det\big(C(x_0,\ldots,x_k)\big) \not\equiv 0$ (*i.e.*, $\det(C(x_0,\ldots,x_k))$, depending on $x_0,\ldots,x_k$, is not the zero-function). Without loss of generality we can assume that for all $0 \le i \le n$ we have that $x_i \in V_{C_i}$ and

$$x_i \notin \left(\bigcup_{j=0}^{n} V_{C_j}\right) \setminus V_{C_i}.$$

Then we have that

$$\det(C(x_0,x_1,\ldots,x_n,0,\ldots,0)) = \det\begin{pmatrix} |V_{C_0}| & x_0 & x_0^2 & \cdots & x_0^n \\ |V_{C_1}| & x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ |V_{C_n}| & x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}$$

$$= \sum_{l=0}^{n}(-1)^{l+2}|V_{C_l}|\det\begin{pmatrix} x_0 & x_0^2 & \cdots & x_0^n \\ x_1 & x_1^2 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{l-1} & x_{l-1}^2 & \cdots & x_{l-1}^n \\ x_{l+1} & x_{l+1}^2 & \cdots & x_{l+1}^n \\ x_{l+2} & x_{l+2}^2 & \cdots & x_{l+2}^n \\ \vdots & \vdots & \ddots & \vdots \\ x_n & x_n^2 & \cdots & x_n^n \end{pmatrix}$$

$$= \sum_{l=0}^{n}(-1)^{l}|V_{C_l}| \prod_{\substack{0\le i<j\le n \\ i,j\neq l}} (x_j - x_i) \not\equiv 0.$$

Therefore, $\det(M_{G_{f,g}}(x_0,\ldots,x_k)) \not\equiv 0$. So, for every open set $U \subseteq \mathbb{R}^{k+1}$ in the box topology there is an open set $U_{G_{f,g}} \subseteq U$ in the box topology such that for all $(x_0,\ldots,x_k) \in U_{G_{f,g}}$

$$\det(M_{G_{f,g}}(x_0,\ldots,x_k)) \neq 0.$$

4 There are no SRUs of size $2n$ for $\mathcal{P}_n$

In this section we will show that for every $n \in \mathbb{N}$, whenever S is a set of cardinality $2n$, then there are two polynomials $f,g \in \mathcal{P}_n$ with $f \neq g$ and $f[S] = g[S]$. In other words, there are no SRUs for $\mathcal{P}_n$ of size $2n$.

Let $S = \{x_1, x_2, \ldots, x_{2n}\} \subseteq \mathbb{R}$ be a set with $2n$ pairwise different points. Without loss of generality we can assume that $0 < x_1 < x_2 < \cdots < x_{2n-1} < x_{2n}$. Our goal is to find two polynomials $f, g \in \mathcal{P}_n$ with $f \neq g$ and

$$f[S] = g[S].$$

In fact, these two polynomials will have the form

$$g(x) = \sum_{j=1}^{n} b_j x^j \quad \text{with } b_j \in \mathbb{R} \text{ for } j = 1, \ldots, n,$$

and

$$f(x) = 1 - g(x).$$

Moreover, they will even satisfy the equations

$$f(x_{2i}) = g(x_{2i-1}) \quad \text{and} \quad f(x_{2i-1}) = g(x_{2i}) \tag{4}$$

for all $1 \leq i < n$. In order to prove that such polynomials f and g exist, we have to show that the following linear equation is solvable:

$$\underbrace{\begin{pmatrix} x_1 + x_2 & x_1^2 + x_2^2 & \ldots & x_1^n + x_2^n \\ x_3 + x_4 & x_3^2 + x_4^2 & \ldots & x_3^n + x_4^n \\ \vdots & \vdots & \ddots & \vdots \\ x_{2n-1} + x_{2n} & x_{2n-1}^2 + x_{2n}^2 & \ldots & x_{2n-1}^n + x_{2n}^n \end{pmatrix}}_{=:A_n = A_n(x_1, x_2, \ldots, x_{2n-1}, x_{2n})} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

To see this, we will show that $\det(A_n) > 0$ for every $n \in \mathbb{N}^*$.

Definition 21. For every $n \in \mathbb{N}^*$ let π_n be the family of all permutations of $\{1, 2, \ldots, n\}$. For each $\sigma \in \pi_n$, let $\operatorname{sgn}(\sigma)$ be the *signum* of the permutation σ.

Definition 22. For every $n \in \mathbb{N}^*$ we define

$$Y^n := \big\{(y_1, y_2, \ldots, y_n) \in \mathbb{R}^n \mid y_i \in \{x_{2i-1}, x_{2i}\} \text{ for all } 1 \leq i \leq n\big\}.$$

Lemma 23. *For every $n \in \mathbb{N}^*$ we have that*

$$\det(A_n) = \sum_{(y_1, y_2, \ldots, y_n) \in Y^n} \sum_{\sigma \in \pi_n} (-1)^{\operatorname{sgn}(\sigma)} y_1^{\sigma(1)} y_2^{\sigma(2)} \ldots y_n^{\sigma(n)}.$$

Proof. The prove is by induction on n.

$n = 1$: We have that $\det(A_1) = x_1 + x_2$.

$n \mapsto n + 1$: We do a Laplace expansion of $A_{n+1} = A_{n+1}(x_1, x_2, \ldots, x_{2n+2})$ along the $(n+1)$-th column. So, we obtain

$$\det(A_{n+1}) = \sum_{i=1}^{n+1} (-1)^{n+1+i} (x_{2i-1}^{n+1} + x_{2i}^{n+1}) \det\big(A_n(x_1, \ldots, x_{2i-2}, x_{2i+1}, \ldots, x_{2n+2})\big).$$

Note that the number of inversions x_{2i-1} causes (or analogously x_{2i} causes) is equal to $n+1-i$ (e.g., if $n=3$ and $i=2$, then the number of inversions x_3 causes in the term $x_2^2 x_3^4 x_6^1 x_8^3$ is equal to 2). So, with the induction hypothesis we get that

$$\det(A_{n+1}) = \sum_{(y_1,y_2,\ldots,y_{n+1})\in Y^{n+1}} \sum_{\sigma\in\pi_{n+1}} (-1)^{\operatorname{sgn}(\sigma)} y_1^{\sigma(1)} \cdots y_{n+1}^{\sigma(n+1)}.$$

□

Lemma 24. *For every $n \in \mathbb{N}^*$ and all $y_1, y_2, \ldots, y_n \in \mathbb{R}$ let*

$$V_n(y_1, y_2, \ldots, y_n) := \begin{pmatrix} y_1 & y_1^2 & \cdots & y_1^n \\ y_2 & y_2^2 & \cdots & y_2^n \\ \vdots & \vdots & \ddots & \vdots \\ y_n & y_n^2 & \cdots & y_n^n \end{pmatrix}.$$

This is a Vandermonde matrix which satisfies

$$\det(V_n(y_1, \ldots, y_n)) = \sum_{\sigma\in\pi_n} (-1)^{\operatorname{sgn}(\sigma)} y_1^{\sigma(1)} y_2^{\sigma(2)} \cdots y_n^{\sigma(n)}. \tag{5}$$

Proof. It is well-known that

$$\det\big(V_n(y_1, \ldots, y_n)\big) = \left(\prod_{k=1}^n y_k\right)\left(\prod_{1\le i<j\le n} (y_j - y_i)\right)$$

and by expanding the right hand side we obtain (5). □

Corollary 25. *For all $n \in \mathbb{N}^*$ we have*

$$\det(A_n(x_1, x_2, \ldots, x_{2n})) > 0.$$

Proof. By combining Lemma 23 and Lemma 24 we get that

$$\det\left(A_n(x_1, x_2, \ldots, x_{2n})\right) = \sum_{(y_1,y_2,\ldots,y_n)\in Y^n} \det\left(V_n(y_1, \ldots, y_n)\right). \tag{6}$$

Finally, since

$$\det\big(V_n(y_1, \ldots, y_n)\big) = \left(\prod_{1\le i<j\le n} (y_j - y_i)\right)\left(\prod_{k=1}^n y_k\right) > 0$$

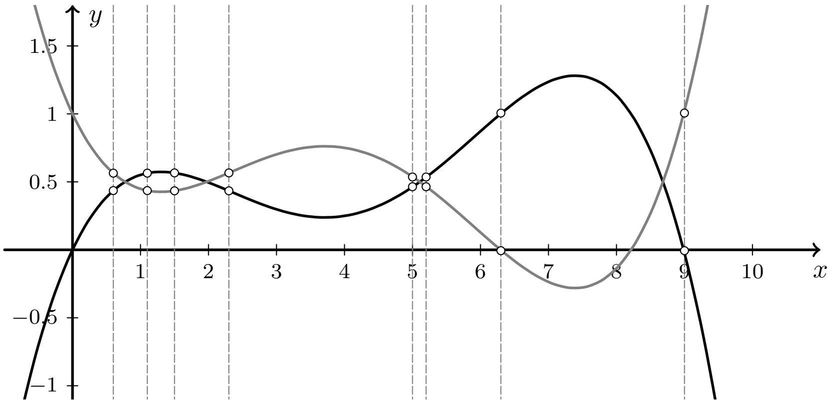
we obtain

$$\det(A_n(x_1, x_2, \ldots, x_{2n})) > 0$$

which completes the proof. □

Remark 26. *Note that (6) provides a formula for the determinant of the sum of two arbitrary Vandermonde matrices. Note also that the assumption $0 < x_0 < x_1 < \ldots, x_{2n-1} < x_{2n}$ is not necessary to derive this formula.*

Example 27. Let $S := \left\{\frac{3}{5}, \frac{11}{10}, \frac{3}{2}, \frac{23}{10}, 5, \frac{26}{5}, \frac{63}{10}, 9\right\}$. In the following picture we can see two polynomials f and g of degree 4 with $f[S] = g[S]$ but $f \neq g$. These polynomials indicate that S is not an SRU for $\mathcal{P}_4$.



Example 28. By definition each SRU for $\mathcal{P}_n$ is an MSRU for $\mathcal{P}_n$. In Section 4 equation (4) we saw that for every set $S = \{x_1, \dots, x_{2n}\}$ of size $2n$ there are polynomials $f, g \in \mathcal{P}_n$ with $f(x_{2i}) = g(x_{2i-1})$, which implies that the size of an MSRU for $\mathcal{P}_n$ is at least $2n + 1$. Observe that the set $S = \{0, 1, 4, 9, 16\}$ is an MSRU but not an SRU for quadratic polynomials: Indeed, for $f(x) = x^2 - 16x$ and $g(x) = -x^2 + 16x - 63$ we have $f[S] = g[S] = \{0, -15, -48, -63\}$ (f takes the value 0 twice, g takes the value -63 twice). Hence, in general, not every MSRU is an SRU for polynomials of bounded degree. Incidentally, the set $S = \{1, 4, 9, 16, 25\}$ *is* an an SRU for quadratic polynomials.

5 Open Questions

1. Is there a simple way to characterise SRUs and MSRUs for the set $\mathcal{P}_n$?

2. A set $M \subseteq \mathbb{R}$ is called a magic set for $\mathcal{P}_n$ if for all non-constant polynomials $f, g \in \mathcal{P}_n$, $f[M] \subseteq g[M] \implies f = g$. The question is now: Is there a magic set for $\mathcal{P}_n$ of size $2n + 1$? Note that since there is no SRU for $\mathcal{P}_n$ of size $2n$, there is no magic set for $\mathcal{P}_n$ of size $2n$.

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