The Pentagon Theorem in Miquelian Möbius planes

Dedicated to the memory of Prof. Dr. Krishan Lal Duggal.

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Abstract

We give an algebraic proof of the Pentagon Theorem. The proof works in all Miquelian Möbius planes obtained from a separable quadratic field extension. In particular, the theorem holds in every finite Miquelian plane. The arguments also reveal that the five concyclic points in the Pentagon Theorem are either pairwise distinct or identical to one single point. In addition we identify five additional quintuples of points in the pentagon configuration which are concyclic.

1 Introduction

The classical version of Miquel's Pentagon Theorem on the Riemann sphere can be formulated as follows:

Theorem 1. *Let $h_1, \dots, h_5$ be five different Möbius circles which intersect each other at a point I and such that any three of them only meet in I. Then, for $i \in \{1, \dots, 5\}$, h_{i-1} and h_{i+1} meet in I and a second point Q_i, and h_{i-2} and h_{i+2} meet in I and a second point S_i (indices read cyclically). Let k_i be the Möbius circle through S_i, Q_{i-1}, Q_{i+1}. Then, for $i \in \{1, \dots, 5\}$, k_{i-1} and k_{i+1} meet in Q_i and a second point P_i. Then the points $P_1, \dots, P_5$ all lie on one common Möbius circle c.*

The situation is shown in Figure 1. Miquel's original proof can be found in [5, Théorème III]. It is based on classical angle theorems. An algebraic proof was believed to be remarkably difficult. So far, only one computer assisted algebraic proof, based on null bracket algebra, has been published in [4]. In the present article we want to present a simple algebraic proof which is based on the cross ratio. This proof works not only for the classical Möbius plane, but for all Miquelian Möbius planes obtained from a separable quadratic field extension for which arguments with angles are not available.

The assumption that the circles h_i intersect (not touch) each other in I implies that the points Q_i and S_i are different from I. In addition, since we assume that any three of the circles h_i only meet in I, we have that the 10 points Q_i, S_i are pairwise distinct. The fact that $P_i \neq Q_i$ will follow below from Lemma 3. The assumptions can be relaxed if one is interested in degenerate cases of the configuration.

The article is organized as follows: In Section 2 we briefly present the theory of Miquelian Möbius planes. Section 3 contains the actual algebraic proof of the Pentagon Theorem in Miquelian Möbius planes obtained from a separable quadratic field extension. The reader who is only interested in the classical case can skip Section 2 and directly read Section 3 by simply ignoring the general framework. The proof will also reveal that the points Q_i, $P_{i-2}, P_{i+2}, S_{i-2}, S_{i+2}$ lie on a Möbius circle c_i for all $i \in \{1, \ldots, 5\}$. In Section 4 we show how the approach can be used to compute the points S_i and P_i in terms of the points Q_i, which culminates in a second algebraic proof of the Pentagon Theorem. In addition, we will see that the points P_i are either pairwise distinct or all identical.

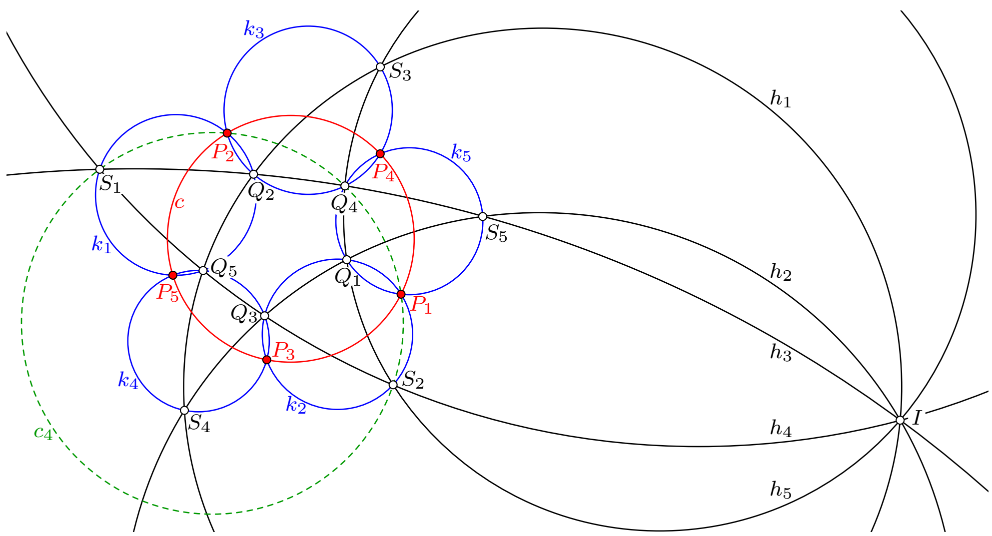


Figure 1: The classical Pentagon Theorem.

2 Miquelian Möbius planes

We first briefly summarize the necessary general concepts and terminology. A Möbius plane is an incidence structure consisting of a set of points $\mathbb{P}$ and a set of blocks $\mathbb{B}$ which satisfies the following axioms (see, *e.g.*, [3, Chapter 6] or [1]):

(M1) For any three points P, Q, R, $P \neq Q$, $P \neq R$ and $Q \neq R$, there exists a unique block C which is incident with P, Q and R.

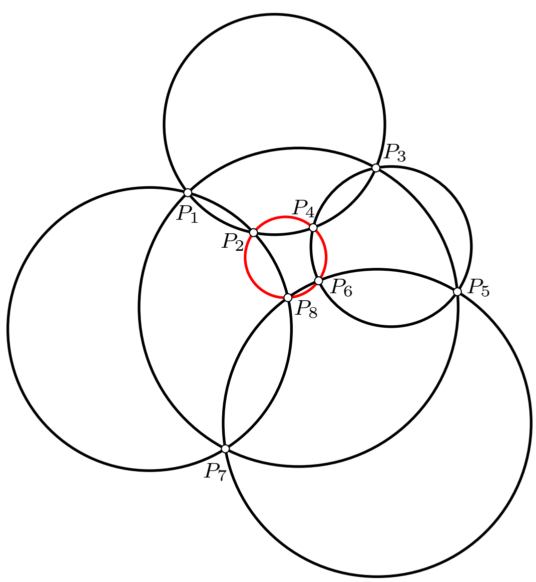
(M2) For any block C, and points P, Q with P incident with C and Q not incident with C, there exists a unique block D which is incident with P and Q but such that P is the only point incident with both, C and D.

(M3) There are four points P_1, P_2, P_3, P_4 which are not all incident with one block C. Moreover, every block C is incident with at least one point.

The "blocks" generalize the lines and circles of the classical Möbius plane. Blocks which have only one point in common are called parallel. In this case we also say that the blocks touch each other.

A Möbius plane is called Miquelian if in addition the Six Circles Theorem of Miquel [5, Théorème I] holds:

Theorem 2 (Miquel)**.** *If one can assign 8 points $P_1, \ldots, P_8$ to the corners of a cube in such a way that the points assigned to five of its faces each lie on a circle, then this is also the case for the points assigned to the 6th face (see Figure 2).*



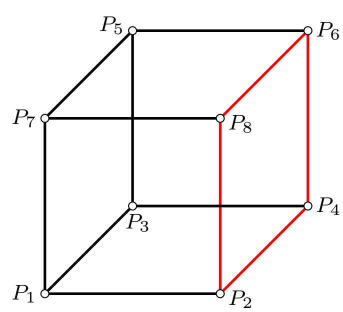


Figure 2: The Six Circles Theorem of Miquel.

It is a famous result by Chen [2] that a Miquelian Möbius plane is isomorphic to a Möbius plane $\mathfrak{M}(K, q)$ over a field K where $q(z) = z^2 + a_0 z + b_0$ is an irreducible polynomial with $a_0, b_0 \in K$. Here, the set of points in $\mathfrak{M}(K, q)$ is

$$\mathbb{P} := K^2 \cup \{\infty\},$$

where $\infty \notin K$, and the set of blocks $\mathbb{B}$ consists of

- *lines*, *i.e.*, the sets of solutions (x_1, x_2) of the equations $ux_1 + vx_2 + w = 0$ for $u, v, w \in K, (u, v) \neq (0, 0)$, and the element ∞, and
- *circles*, *i.e.*, the sets of solutions (x_1, x_2) of the equations $x_1^2 + a_0 x_1 x_2 + b_0 x_2^2 + ux_1 + vx_2 + w = 0$ for $u, v, w \in K$, if this set of solutions consists of more than one point.

A point is called *incident* with a block, if it is an element of the block. Let E be the splitting field of q. Hence, there are $\alpha_1, \alpha_2 \in E$ such that $q(z) = (z + \alpha_1)(z + \alpha_2)$, and E is a two dimensional vector space over K with basis $\{1, \alpha_1\}$ or $\{1, \alpha_2\}$. Since every point $(x_1, x_2) \in K^2$ can be represented by $z = x_1 + \alpha_1 x_2$ or $z = x_1 + \alpha_2 x_2$, we can identify K^2 with E. If q is separable, *i.e.*, $\alpha_1 \neq \alpha_2$, then the mapping

$$\bar{} : E \to E, \quad z = x_1 + \alpha_1 x_2 \;\mapsto\; \bar{z} = x_1 + \alpha_2 x_2 = x_1 + a_0 x_2 - \alpha_1 x_2$$

is an involutorial automorphism of E (observe that $\alpha_1 + \alpha_2 = a_0$). Hence we have

$$x_1 = \frac{\alpha_1 \bar{z} - \alpha_2 z}{\alpha_1 - \alpha_2}, \quad x_2 = \frac{z - \bar{z}}{\alpha_1 - \alpha_2},$$

and the equation of a line $ux_1 + vx_2 + w = 0$ can be written in the form

$$\bar{c}z + c\bar{z} = r \text{ with } c \in E \setminus \{0\} \text{ and } r \in K. \tag{1}$$

Similarly, the equation of a circle $x_1^2 + a_0 x_1 x_2 + b_0 x_2^2 + ux_1 + vx_2 + w = 0$ can be written as a quadratic equation of the form

$$(z - c)(\bar{z} - \bar{c}) = r \text{ for } r \in K \setminus \{0\} \text{ and } c \in E \tag{2}$$

(use $x_1^2 + a_0 x_1 x_2 + b_0 x_2^2 = z\bar{z}$ for $z = x_1 + \alpha_1 x_2$). For $K = \mathbb{R}$ and $q(z) = z^2 + 1$ we have $E = \mathbb{C}$ and we are in the situation of the classical model of the Möbius plane. Another example is the Galois field $K = GF(t)$ for an odd prime power $t = p^n$, and $q(z) = z^2 - \alpha$ for a non-square $\alpha \in GF(t)$. Then, $GF(t)(\alpha) \cong GF(t^2)$ and the conjugation is given by the Frobenius automorphism $z \mapsto \bar{z} = z^t$. Notice also, that every finite extension of a finite field is separable. Hence, our proof shows that Theorem 1 is valid in each Miquelian Möbius plane $\mathfrak{M}(K, q)$ if q is separable, and in particular in every finite Miquelian Möbius plane.

3 A simple algebraic proof of Miquel's Pentagon Theorem

The reader who skipped Section 2 should consider the points $P, Q, \ldots$ in this section als elements of the complex plane, and z is a complex variable. In this case, $\bar{z}$ means complex conjugation, and take $E = \mathbb{C}$ and $K = \mathbb{R}$.

The equation of a line through two different points P, Q is given by

$$(P - z)(\bar{Q} - \bar{z}) = (\bar{P} - \bar{z})(Q - z).$$

Indeed, $z = P$ and $z = Q$ are solutions of this equation and expanded it has the required form (1) of a line. Hence, three different points P, Q, z lie on a line if and only if

$$\frac{P-z}{Q-z} \in K \setminus \{0,1\}.$$

Similarly, a circle through three different points P, Q, R (which do not lie on a line) is given by

$$(P-Q)(R-z)(\bar{P}-\bar{z})(\bar{R}-\bar{Q}) = (\bar{P}-\bar{Q})(\bar{R}-\bar{z})(P-z)(R-Q)$$

since $z = P, z = Q, z = R$ are solutions of this equation and expanded it has the form (2) of a circle. Thus, the fact that four different points P, Q, R, z lie on a circle can be expressed by the cross ratio

$$\frac{P-Q}{R-Q} \cdot \frac{R-z}{P-z} \in K \setminus \{0,1\},$$

and this is still true, if P, Q, R, z lie on a line.

Observe that the group of Möbius transformations

$$z \mapsto \frac{az+b}{cz+d}, \quad \det\begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0, \quad a,b,c,d \in E,$$

(with the usual convention $1/0 = \infty, 1/\infty = 0$) is sharply 3-transitive on the set of points and maps blocks (*i.e.*, the set of circles and lines) to blocks.

The following result will be useful below (see Figure 3).

Lemma 3. *Let c_1, c_2 be circles which touch each other in a point P. Let h_1, h_2 be two lines through P. Then the line through the second intersections of c_1 with h_1 and h_2 is parallel to the line through the second intersections of c_2 with h_1 and h_2.*

Proof. By a suitable linear Möbius transformation we may assume that c_1 is given by the equation $(z-1)(\bar{z}-1) = 1$ and $P = 0$. Then the equation of c_2 has the form $(z-u)(\bar{z}-\bar{u}) = u\bar{u}$ for some $u = \bar{u} \in K$, and h_i is given by $\bar{a}_i z + a_i \bar{z} = 0$ for some $a_i \in E$. The second intersection of c_1 with h_i is $P_i = 1 - \frac{a_i}{\bar{a}_i}$, and the second intersection of c_2 with h_i is $Q_i = u(1 - \frac{a_i}{\bar{a}_i})$. So indeed, the line through P_1, P_2 and the line through Q_1, Q_2 are parallel. □

We are now ready to give the new, simple algebraic proof of Miquel's Pentagon Theorem. As a side result, we identify five additional quintuples of points in the pentagon configuration which are concyclic (see also Miquel's original proof in [5, Théorème III]).

The Möbius transformation $z \mapsto 1/(z-I)$ maps the point I to the point ∞. Hence we may assume without loss of generality that I is the point ∞. We use the cross ratio in the following way, where we assume for the moment that all numerators and denominators

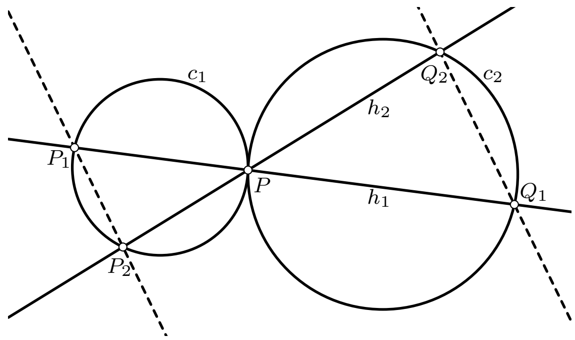


Figure 3: Illustration of Lemma 3. The dashed lines are parallel.

are different from 0:

$$
\begin{aligned}
P_3, Q_1, Q_3, S_2 \in k_2 &\implies \mu_1 := \frac{S_2 - Q_1}{Q_3 - Q_1} \cdot \frac{Q_3 - P_3}{S_2 - P_3} \in K \\
P_3, Q_3, Q_5, S_4 \in k_4 &\implies \mu_2 := \frac{Q_3 - S_4}{Q_5 - S_4} \cdot \frac{Q_5 - P_3}{Q_3 - P_3} \in K \\
Q_5, S_3, S_4 \in h_1 &\implies \mu_3 := \frac{S_4 - Q_5}{S_3 - Q_5} \in K \\
Q_1, Q_3, S_4 \in h_2 &\implies \mu_4 := \frac{Q_1 - Q_3}{S_4 - Q_3} \in K \\
Q_1, S_2, S_3 \in h_5 &\implies \mu_5 := \frac{S_3 - S_2}{Q_1 - S_2} \in K
\end{aligned}
$$

The product of the values μ_i is

$$\alpha := \mu_1\mu_2\mu_3\mu_4\mu_5 = \frac{Q_5 - P_3}{S_2 - P_3} \cdot \frac{S_2 - S_3}{Q_5 - S_3}.$$

Similarly, by mirroring the indices $1 \leftrightarrow 4, 2 \leftrightarrow 3$ we have

$$
\begin{aligned}
P_2, Q_4, Q_2, S_3 \in k_3 &\implies \nu_1 := \frac{S_3 - Q_4}{Q_2 - Q_4} \cdot \frac{Q_2 - P_2}{S_3 - P_2} \in K \\
P_2, Q_2, Q_5, S_1 \in k_1 &\implies \nu_2 := \frac{Q_2 - S_1}{Q_5 - S_1} \cdot \frac{Q_5 - P_2}{Q_2 - P_2} \in K \\
Q_5, S_2, S_1 \in h_4 &\implies \nu_3 := \frac{S_1 - Q_5}{S_2 - Q_5} \in K \\
Q_4, Q_2, S_1 \in h_3 &\implies \nu_4 := \frac{Q_4 - Q_2}{S_1 - Q_2} \in K \\
Q_4, S_3, S_2 \in h_5 &\implies \nu_5 := \frac{S_2 - S_3}{Q_4 - S_3} \in K
\end{aligned}
$$

The product of the values ν_i is

$$\beta := \nu_1\nu_2\nu_3\nu_4\nu_5 = \frac{Q_5 - P_2}{S_3 - P_2} \cdot \frac{S_3 - S_2}{Q_5 - S_2}.$$

Since $\alpha, \beta \in K$, it follows that also

$$\gamma := \frac{S_3 - P_2}{P_3 - P_2} \cdot \frac{P_3 - S_2}{S_3 - S_2} \in K.$$

Finally, we have

$$\begin{aligned}
Q_1, P_1, P_3, S_2 \in k_2 &\implies \xi_1 := \frac{P_3 - P_1}{Q_1 - P_1} \cdot \frac{Q_1 - S_2}{P_3 - S_2} \in K \\
Q_4, P_2, P_4, S_3 \in k_3 &\implies \xi_2 := \frac{S_3 - Q_4}{P_4 - Q_4} \cdot \frac{P_4 - P_2}{S_3 - P_2} \in K \\
Q_1, Q_4, P_1, P_4, \in k_5 &\implies \xi_3 := \frac{P_4 - Q_4}{Q_1 - Q_4} \cdot \frac{Q_1 - P_1}{P_4 - P_1} \in K \\
Q_1, Q_4, S_2, S_3 \in h_5 &\implies \xi_5 := \frac{Q_1 - Q_4}{S_3 - Q_4} \cdot \frac{S_3 - S_2}{Q_1 - S_2} \in K
\end{aligned}$$

Observe that we get the product

$$\xi_1 \xi_2 \xi_3 \xi_4 \gamma = \frac{P_4 - P_2}{P_3 - P_2} \cdot \frac{P_3 - P_1}{P_4 - P_1},$$

which is again an element of K. We conclude that P_1, P_2, P_3, P_4 lie on a common circle c_1. By shifting the index by one we also have that P_2, P_3, P_4, P_5 lie on a circle c_2. However, c_1 and c_2 have the three points P_2, P_3, P_4 in common and must therefore agree, which completes the proof of Theorem 1 if all numerators and denominators of the cross rations we used are different from 0. This is what we now check.

Recall first that the points Q_i, S_i, I are pairwise distinct. Notice also that the assertion of the theorem is trivially satisfied if three of the points P_i coincide or if two pairs of the points P_i coincide. In particular, we may assume that among the five pairs P_i, P_{i+2} at most one pair collapses, say $P_3 = P_5$. From Lemma 3 it follows that $P_i = Q_i$ is excluded since this would imply that to h_{i-2} and h_{i+2} are parallel. We can also exclude the case $S_2 = P_3$: Indeed, if we assume $S_2 = P_3$ we have that Q_3, Q_5, P_3 lie on the line h_4. But at the same time, these points define the block k_4 and hence $h_4 = k_4$. This would lead to $S_4 = I$ or $S_4 = Q_3$, which is not possible. Similarly, we have $S_3 \neq P_2$. Next, assume that $P_3 = Q_5$. Now, h_4 intersects k_2 in the points Q_3 and S_2. But if $Q_5 = P_3$, k_2 passes also through Q_5 which is a point of h_4. It follows that $S_2 = Q_5$ which is not possible. Similarly, we have $P_2 \neq Q_5$. Finally, suppose $P_2 = P_3$. Since k_1 and k_4 are both determined by $P_2 = P_3, P_5, Q_5$ it follows that $k_1 = k_4$ (unless $P_5 = P_2$, but in that case the assertion of the theorem is trivial). But this leads to $S_4 = Q_5$ or $S_4 = Q_2$ which is impossible. Therefore indeed, all numerators and denominators in the cross rations we used are different from 0.

Notice that $\alpha, \beta \in K$ implies that that the points Q_5, P_2, P_3, S_2, S_2 lie on a circle. By shifting the indices cyclically, we obtain the following result.

Proposition 4. *The points $Q_i, P_{i-2}, P_{i+2}, S_{i-2}, S_{i+2}$ lie on a common Möbius circle c_i for all $i \in \{1, \ldots, 5\}$.*

In Figure 1 the circle c_4 is drawn.

4 Computation of the points

It is instructive and useful for practical purposes to actually compute the points S_i and P_i. We continue to assume that $I = \infty$. Then, the blocks $h_1, \ldots, h_5$ are lines of the form

$$h_i : (Q_{i-1} - z)(\bar{Q}_{i+1} - \bar{z}) = (\bar{Q}_{i-1} - \bar{z})(Q_{i+1} - z).$$

The point S_i is the intersection of the lines h_{i-2} and h_{i+2}. Solving the corresponding linear system of the two equations yields

$$S_i = \frac{(Q_{i-2} - Q_{i+1})(Q_{i+2}\bar{Q}_{i-1} - Q_{i-1}\bar{Q}_{i+2}) - (Q_{i-1} - Q_{i+2})(Q_{i+1}\bar{Q}_{i-2} - Q_{i-2}\bar{Q}_{i+1})}{(Q_{i+2} - Q_{i-1})(\bar{Q}_{i-2} - \bar{Q}_{i+1}) - (Q_{i-2} - Q_{i+1})(\bar{Q}_{i+2} - \bar{Q}_{i-1})}.$$

By assumption, the blocks h_i intersect each other at $I = \infty$ (meaning they do not touch) so that the second intersection S_i of h_{i-2} and h_{i+2} is different from I. In particular, the denominator of S_i is different from 0 . The equation of the block k_i through the points S_i, Q_{i-1}, Q_{i+1} is then given by

$$(S_i - Q_{i+1})(Q_{i-1} - z)(\bar{S}_i - \bar{z})(\bar{Q}_{i-1} - \bar{Q}_{i+1}) = (\bar{S}_i - \bar{Q}_{i+1})(\bar{Q}_{i-1} - \bar{z})(S_i - z)(Q_{i-1} - Q_{i+1}).$$

The blocks k_{i-1} and k_{i+1} meet in Q_i and P_i. Solving the equation of the circle k_{i-1} for the variable $\bar{z}$, yields

$$\bar{z} = \frac{\bar{Q}_{i-2}(Q_{i-2} - Q_i)(\bar{Q}_i - \bar{S}_{i-1})(z - S_{i-1}) - \bar{S}_{i-1}(\bar{Q}_{i-2} - \bar{Q}_i)(Q_i - S_{i-1})(z - Q_{i-2})}{(Q_{i-2} - Q_i)(\bar{Q}_i - \bar{S}_{i-1})(z - S_{i-1}) - (\bar{Q}_{i-2} - \bar{Q}_i)(Q_i - S_{i-1})(z - Q_{i-2})}.$$

Similarly, solving the equation of the circle k_{i+1} for the variable $\bar{z}$, gives

$$\bar{z} = \frac{\bar{Q}_i(Q_i - Q_{i+2})(\bar{Q}_{i+2} - \bar{S}_{i+1})(z - S_{i+1}) - \bar{S}_{i+1}(\bar{Q}_i - \bar{Q}_{i+2})(Q_{i+2} - S_{i+1})(z - Q_i)}{(Q_i - Q_{i+2})(\bar{Q}_{i+2} - \bar{S}_{i+1})(z - S_{i+1}) - (\bar{Q}_i - \bar{Q}_{i+2})(Q_{i+2} - S_{i+1})(z - Q_i)}.$$

Equating the resulting expressions yields a quadratic equation in z. However, since $z = Q_i$ is a solution, the equation reduces to a linear one for the second solution $z = P_i$. One finds

$$\begin{aligned} P_i \ = & \Big[(Q_{i+2} - Q_{i-1})\big((Q_{i+2} - Q_i)Q_{i-2}\bar{Q}_{i+1} - (Q_{i-2}Q_{i+2} - Q_iQ_{i+1})\bar{Q}_{i-2}\big) \\ & + (Q_{i-2} - Q_{i+1})\big((Q_i - Q_{i-2})Q_{i+2}\bar{Q}_{i-1} + (Q_{i-2}Q_{i+2} - Q_{i-1}Q_i)\bar{Q}_{i+2}\big)\Big] \\ & \Big/ \Big[(Q_{i+2} - Q_{i-1})\big((Q_{i+2} - Q_i)\bar{Q}_{i+1} - (Q_{i-2} - Q_i - Q_{i+1} + Q_{i+2})\bar{Q}_{i-2}\big) \\ & + (Q_{i-2} - Q_{i+1})\big((Q_i - Q_{i-2})\bar{Q}_{i-1} + (Q_{i-2} - Q_{i-1} - Q_i + Q_{i+2})\bar{Q}_{i+2}\big)\Big]. \end{aligned}$$

We now obtain a second proof of Theorem 1: Indeed, the cross ratio $\delta = \frac{P_1 - P_3}{P_2 - P_3} \cdot \frac{P_2 - P_4}{P_1 - P_4}$ simplifies to

$$\frac{(Q_5(\bar{Q}_2 - \bar{Q}_4) + Q_2(\bar{Q}_4 - \bar{Q}_5) + Q_4(\bar{Q}_5 - \bar{Q}_2))(Q_5(\bar{Q}_3 - \bar{Q}_1) + Q_3(\bar{Q}_1 - \bar{Q}_5) + Q_1(\bar{Q}_5 - \bar{Q}_3))}{(Q_5(\bar{Q}_1 - \bar{Q}_4) + Q_1(\bar{Q}_4 - \bar{Q}_5) + Q_4(\bar{Q}_5 - \bar{Q}_1))(Q_5(\bar{Q}_3 - \bar{Q}_2) + Q_3(\bar{Q}_2 - \bar{Q}_5) + Q_2(\bar{Q}_5 - \bar{Q}_3))}.$$

The first factor of the denominator can be written as

$$Q_5(\bar{Q}_2 - \bar{Q}_4) + Q_2(\bar{Q}_4 - \bar{Q}_5) + Q_4(\bar{Q}_5 - \bar{Q}_2) = (\bar{Q}_2Q_4 + Q_2\bar{Q}_5 + \bar{Q}_4Q_5) - \overline{(\bar{Q}_2Q_4 + Q_2\bar{Q}_5 + \bar{Q}_4Q_5)}.$$

In fact all factors of the numerator and of the denominator are of the form $u - \bar{u}$, and hence $\delta \in K$, which implies that P_1, P_2, P_3, P_4 lie on a circle. By shifting the index by one, we again find all points $P_1, \ldots, P_5$ on a common circle.

We conclude by the following observation. We have already seen in Section 3 that $P_i \neq P_{i\pm1}$. Now, the equation $P_{i-1} = P_{i+1}$ simplifies to

$$\begin{aligned} & (Q_{i-1} - Q_{i+1})(Q_{i-1} - Q_{i+2})(Q_{i+1} - Q_{i-2})\big(Q_{i-2}(\bar{Q}_{i+2} - \bar{Q}_i) + Q_{i+2}(\bar{Q}_i - \bar{Q}_{i-2}) + Q_i(\bar{Q}_{i-2} - \bar{Q}_{i+2})\big) \times \\ & \times \big(Q_i(\bar{Q}_{i+1} - \bar{Q}_{i-1}) + Q_{i-2}(\bar{Q}_{i-1} - \bar{Q}_{i+2}) + Q_{i+1}(\bar{Q}_{i+2} - \bar{Q}_i) + Q_{i-1}(\bar{Q}_i - \bar{Q}_{i-2}) + Q_{i+2}(\bar{Q}_{i-2} - \bar{Q}_{i+1})\big) = 0. \end{aligned}$$

The first three factors are clearly different from 0. The next factor is 0 if and only if Q_i, Q_{i-2}, Q_{i+2} lie on a line, which would mean $h_{i-1} = h_{i+1}$ which is excluded. The last factor is cyclically symmetric in i, hence $P_{i-1} = P_{i+1}$ for some i implies $P_{i-1} = P_{i+1}$ for all i. This leads to the following result.

Proposition 5. *Either the points P_i are pairwise distinct or the points P_i collapse to one single point.*

References

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