Corrections and Improvements

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I would like to thank *Riccardo Plati*, *Dávid Uhrik*, and *Michael Weiss* for their numerous comments.

Chapter 3

page 35, line 10	...which implies that x_0 is not an $\in$-minimal element...
page 48, line 9	$\forall x\big(\exists z(z \in x) \to \ldots$
page 48, line 13	... we have $x_n \ni x_{n+1}$.
page 55, line -11 ff.	(a) (b) (c) instead of 1. 2. 3.
page 56, line -4	...any infinite ordinal...
page 62, line -2	$\ldots C \in \mathscr{W}_0 \ldots$
page 63, line 2	$\ldots C \in \mathscr{W}_0 \ldots$
page 69, line 4	$\lvert\operatorname{seq}(\kappa)\rvert = \kappa$

Chapter 5

page 118, line 3	Now, let $s_\alpha := f_0(M)$ and define $F_{\alpha+1} := F_\alpha \cup \{\langle \alpha, s_\alpha \rangle\}$.
page 127, line 14	$I_{n,k}$~~(X)~~

Chapter 6

page 141, line 21	$ax_0^{k_0}\cdots x_l^{k_l}$ where...
page 141, line -4	$x = \sum_{v \in B(x)} q_v^x \cdot v$
page 152, line -16	$p_u \vee p_{-u}$ instead of $p_u \vee \neg p_{-u}$
page 154, line 11	$\chi_A \cap \chi_B \supseteq \chi_{A\cup B}$

Chapter 7

page 185, line 2 ff.	Indeed, let $y \in [z]^{\sim}$, and let $\rho \neq \iota$ be such that $\rho(y) = y$ and ρ induces a proper cycle in $[z]^{\sim}$ (*i.e.*, the cycle starts and ends with y and the other points in the cycle are pairwise distinct).
page 186, line 2	... whenever σ has label ⓙ, $\varphi_m\sigma$ cannot get label ⓘ.

Chapter 8

page 201, line -8	$S \mapsto (k, E)$

page 202, line 8	...for some $m \in n$ ...
page 208, line 4	... we have $\pi a = \tau a$.
page 210, line 7	$f(s) := \{(m+l+1, s, 0), (m+l+1, s, 1)\}$
page 213, line 9	...in $\omega \setminus N_1$ instead of $\omega \setminus (N_1 \cup N_2)$
page 215, line 20 f.	

$$\begin{array}{rccc} \Psi_E : & \{S \subseteq A : \operatorname{supp}(S) = E\} & \longrightarrow & \mathscr{P}(\mathscr{P}(k)) \\ & S_0 & \longmapsto & \Big\{ I \subseteq k : \exists a \in S_0 \, (\vartheta_E(a) = \{\varphi_i(x) : i \in I\}) \Big\} \end{array}$$

page 215, line -7	... Ψ_E maps S to $\mathscr{P}(\mathscr{P}(k))$, and $l < 2^{2^k}$ encodes the set $\Psi_E(S)$ ...

Chapter 9

page 228, line -8	$\mathfrak{i} = \min\{|\mathscr{I}| : \mathscr{I} \subseteq [\omega]^\omega$ is maximal independent$\}$
page 229, line 11	$A_k := A_0 \cup \{t \cup \{k\} : t \in A_0\}$
page 229, line -10	$\{x \in [\omega]^\omega : x \in \mathscr{I}_0 \wedge f(x) = 1\} \cup \{(\omega \setminus y) \in [\omega]^\omega : y \in \mathscr{I}_0 \wedge f(y) = 0\}$
page 231, line 10 ff.	$g \in {}^\omega 2$ (four times).
page 231, line -1	$\bigcap I \setminus \bigcup J \supseteq \left(\bigcap I' \setminus \bigcup J'\right) \cap \bigcap_{n \in m} X_n^{g(n)} \,{}^*\!\supseteq \left(\bigcap I' \setminus \bigcup J'\right) \cap Y_g$
page 232, line 3	... and therefore $Z \cap (\bigcap I \setminus \bigcup J)$ is infinite.
page 232, line 19	... show that $\mathscr{F}'$ is a dominating ...
page 233, line -4	... for all $m \in A$ with $m \geq g_\xi(n)$ ...
page 236, line 3	... $\mathscr{A}_\xi \in \mathscr{E}$ ...

Chapter 10

page 245, line -4 f.	... such that $y_0 \notin C$ and $y_1 \in C$.
page 246, line 13	$[s, y_n]^\omega \cap C = \emptyset$
page 248, line 2	$D_\xi = \{y \in [\omega]^\omega : \forall z \in [\omega]^\omega (z \subseteq^* y \rightarrow [\emptyset, z]^\omega \cap C_\xi = \emptyset)\}$
page 248, line 6	... $x \in [\omega]^\omega$ ~~$\mathscr{A}_\xi$~~ ...

Chapter 11

page 266, line -15 f.	... ~~$x_{\alpha+1}$~~ x_α exists... (*twice*)

page 267, line 18 $x_0 := \bigcup\{x \cap I_{2m} : m \in \omega\}$ and $x_0 := \bigcup\{x \cap I_{2m+1} : m \in \omega\}$.

page 279, line 4 Now, since $f(D') \subseteq D''$ and $f(D'') \subseteq D' \cup (\omega \setminus D)$, this...

page 279, line 12 ff. ...but since $f(I_0') \subseteq I_0'' \cup (\omega \setminus I_0)$ and $f(I_0'') \subseteq I_0' \cup (\omega \setminus I_0)$, this is a contradiction to $f(\mathscr{U}) = \mathscr{U}$. So, $I_0 \notin \mathscr{U}$, which implies that $I_\omega \in \mathscr{U}$. Now, for each $n \in I_\omega$ there exists a least number $m_n \in I_\omega$ such that there are $k, k' \in \omega$ with $f^k(m_n) = f^{k'}(n)$. Let

$$I_\omega' := \Big\{ n \in I_\omega : \exists k, k' \in \omega \big(f^k(m_n) = f^{k'}(n) \ \wedge\ k + k' \text{ is odd} \big) \Big\}$$

and let $I_\omega'' := I_\omega \setminus I_\omega'$. Since the two sets I_ω' and I_ω'' are disjoint and their union is I_ω, either I_ω' or I_ω'' belongs to $\mathscr{U}$, but not both. Furthermore, we get $f(I_\omega') \subseteq I_\omega''$ and $f(I_\omega'') \subseteq I_\omega'$, which is again a contradiction to $f(\mathscr{U}) = \mathscr{U}$.

page 280, line 3 ... which shows that $\tilde{g}(\mathscr{U}) \supseteq \mathscr{V}$.

page 280, line -5 $\big\{ a' \in \omega : \{ b \in \omega : \langle a', b \rangle \in X_0 \} \notin \mathscr{V} \big\} \in \mathscr{U}$

page 281, line -1 ... for $y_Q := \pi_{\mathscr{U}}(Y_Q \cap D)$ ...

Chapter 14

page 324, line 2 ff. [throughout Chapter 14] $\mathbb{P} = (P, \leq)$

page 326, line -20 ...for any uncountable set...

page 327, line 7 ...the set $\{p \in \mathscr{F} : \mathrm{dom}(p) = K\}$ is countable...

page 327, line 16 Now we show that $|\mathscr{D}| < \mathfrak{c}$ cannot...

page 330, line -5 Let $\mathscr{F}_0 := \{\omega \setminus s : s \in [\omega]^{<\omega}$ ~~(ω)~~$\}$ ...

page 331, line -3 *For each* $\tilde{\mathscr{Y}} \in \mathrm{fin}(P_{\beta_0})$...

page 332, line 2 ...finite set $\tilde{\mathscr{Y}}_0 \in \mathrm{fin}(P_{\beta_0})$ ...

page 332, line -2 ff. Now, for each $x \in \mathscr{F}_{\beta_0}$ and $m \in \omega$, let

$$D_x := \big\{ (\langle s_{n_i} : i \in k+1 \rangle, X) \in P : x \in X \big\},$$

and

$$D_m := \big\{ (\langle s_{n_i} : i \in k+1 \rangle, X) \in P : m \in k+1 \big\}.$$

By Claim 2, for each $x \in \mathscr{F}_{\beta_0}$ and $m \in \omega$, the sets D_x and D_m are open dense subsets of P. Hence, since $|\mathscr{F}_{\beta_0}| < \mathfrak{c}$, the set

$$\mathscr{D} := \{ D_x \subseteq P : x \in \mathscr{F}_{\beta_0} \} \cup \{ D_m \subseteq P : m \in \omega \}$$

is of cardinality...

page 334, line -6 ...belongs to the dual ideal of the filter generated by $\mathscr{F}_{\eta|_{\beta_0}}$ ...

page 335, line 11 ...meets either infinitely many sets of $\mathscr{P}_{\beta_0}$ or has empty intersection with co-finitely many of them.

Chapter 15

page 342, line -16 $\mathbf{V}[G] = \{\emptyset\}$

page 350, line 8

$$\forall \langle \underset{\sim}{y}_2, s_2\rangle \in \underset{\sim}{x}_2\, \forall q \in P\,\big((q \geq s_2 \wedge q \Vdash_{\mathbb{P}} \underset{\sim}{y}_1 = \underset{\sim}{y}_2) \rightarrow q \perp r\big),$$

page 352, line -14 $\underset{\sim}{x}_1 := \big\{\langle \underset{\cdot}{\emptyset}, p\rangle, \langle \underset{\cdot}{\emptyset}, q\rangle\big\}\ldots$

page 353, line 12 …then there is a $\mathbb{P}$-name $\underset{\sim}{y}$ and a pair $\langle \underset{\sim}{y}, r\rangle \in \underset{\sim}{B}\ldots$

page 353, line 14 …and since $\underset{\sim}{y}[G] = \big\{\, \underset{\sim}{x}[G] : \exists q \in G\big(\langle \underset{\sim}{x}, q\rangle \in \underset{\sim}{y}\big)\big\}$

page 353, line -18 f. …then there is a $\mathbb{P}$-name $\underset{\sim}{y}$ and a pair $\langle \underset{\sim}{y}, r\rangle \in \underset{\sim}{B}\ldots$

page 353, line -3 f. …and since $p \in G$, for $y = \underset{\sim}{y}[G]$ we get $y \in \mathbf{V}[G]$. Hence…

page 360, line 6 ff. *Four times $\bigcup G$ instead of just G.*

page 361, line 18 ff. LEMMA 15.16. *If a forcing notion preserves cofinalities, then it preserves also cardinalities.*

page 361, line 20 f. *Proof.* ~~Since cofinalities are always cardinals, any forcing notion which preserves cardinalities must preserve cofinalities.~~
~~For the other direction,~~

page 362, line 15 ff. Since $p \in G$, for every $\alpha \in \lambda$, $G \cap D_\alpha \neq \emptyset$, and therefore, $\underset{\sim}{S}[G](\alpha) \in Y_\alpha$.

Chapter 16

page 371, line -3 If $\mathbf{V} \vDash$ ZFC …

page 372, line 1 … Let $\mathbf{V}$ be a model of ZFC …

page 372, line 8 … is equivalent to ψ, free$(\varphi_{\mathbf{0}}) \subseteq \ldots$

page 372, line -3 … reflects $\bar{\psi}$.

page 373, line 3 f. $h_{n,i}\big(\langle x_1, \ldots, x_i\rangle\big) := \mu\big\{y \in V_{\alpha_{n+1}} : \forall x_{i+1} \in V_{\alpha_n} \exists y_{i+1} \cdots \forall x_k \in V_{\alpha_n} \exists y_k \ldots$

page 379, line -7 …the forcing notion $\mathbb{K}_0\ldots$

Chapter 17

page 384, line 19 ~~for each $a \in A$, $\{\alpha \in \mathcal{G} : \alpha a = a\} \in \mathscr{F}$~~

page 389, line -9 Let $\mathcal{G}$ be the group generated by automorphisms of $\mathbb{C}_\omega$ of the form α_{π_F, n_0}, *i.e.*,

$$\mathcal{G} = \big\langle \alpha_{\pi_F, n_0} : F \in \text{fin}(\omega) \wedge n_0 \in \omega \big\rangle.$$

page 397, line -8 f. …corresponds an automorphism α_π of $\mathbb{P}$ by stipulating

$$\alpha_\pi p\big(\pi\langle\bar{a},\xi\rangle,\eta\big) := p(\bar{a},\xi,\eta),$$

page 397, line -2 $\{\bar{H} : H \in \mathscr{F}_0\} \cup \big\{\operatorname{fix}_{\bar{\mathcal{G}}}(E) : E \in \operatorname{fin}(\bar{A}\times\kappa)\big\}$.

page 398, line 6 f. …*i.e.*, for every $\sigma \in \operatorname{sym}_{\mathcal{G}_0}(a)$, $\bar{\sigma} \subseteq \operatorname{sym}_{\bar{\mathcal{G}}}(a)$.

Chapter 19

page 338, line 1 …the function $H : \bigcup_{n\in\omega} {}^n2 \to \operatorname{fin}(\omega)$ …

Chapter 20

page 445, line 7 ff. …for some limit ordinal $\lambda \in \omega_1$ let

$$\bar{x} := \big\{y \in T'' : y < x\big\}.$$

For each $\bar{x}$ we add an extra node $w_{\bar{x}}$ to T'' and stipulate

$$z < w_{\bar{x}} \iff z < x \quad \text{and} \quad w_{\bar{x}} < z \iff x \leq z\,.$$

Roughly speaking, $w_{\bar{x}}$ is a node between $\{z \in T'' : z < x\}$ and x. Let

$$T''' := T'' \cup \{w_{\bar{x}} : x \in T'' \wedge \text{o.t.}(x) = \lambda\}$$

where $\lambda \in \omega_1$ is a limit ordinal. Notice that the root of T''' is $w_{\bar{x}_0}$, where x_0 is…

page 451, line -7 $$\cancel{f(}k_{i+1}\cancel{)} := \begin{cases} f(k_i) & \text{if } k_i \in A, \\ k_i & \text{otherwise.} \end{cases}$$

Chapter 21

page 460, line -5 H_{ω_1}

page 461, line 4 … GCH holds in the ground model and $|P| \leq \omega_1$, then $\chi = \omega_3$ …

Chapter 22

page 478, line 5 …model $\mathbf{V}[G|_\alpha]$, fix an arbitrary dense set $D \subseteq \operatorname{Fn}(\omega,2)$ and let $\underset{\sim}{D} \in \mathbf{V}\cancel{[G|_\alpha]}$…

page 478, line 12 … $T_{3,i,j} \geq T_2[s_{i,j}]$.

Chapter 23

page 486, line -14 … $T_{3,i,j} \geq T_2[s_{i,j}]$.

page 486, line -13 … $T_3[s_{i,j}] \in D_3$.

page 488, line 8 $\mathscr{T}[s] := T_0[s_0] \times \ldots \times T_{d-1}[s_{d-1}]$

page 489, line 8	$\left\|\mathscr{T}_i'(l_k)\right\| = 2^k$
page 493, line -16	$\delta_{\omega_1} := \bigcup_{\iota\in\omega_1} \delta_\iota$

MINOR CORRECTIONS AND IMPROVEMENTS

page 14, line 16	Let φ, φ_1, φ_2, φ_3, and ψ ...
page 16, line -6	...is equal to the formula $\forall\nu\,\varphi_j$, where ν is a variable which does not occur free in any non-logical axiom of T.
page 41, line 9	subset instead of subsets
page 120, line 1	... $2^{\mathfrak{m}} \leq \operatorname{seq}(\mathfrak{m})$ ...
page 120, line 14	... $2^{\mathfrak{m}} \cdot 2^{\aleph_0} \leq \operatorname{seq}(\mathfrak{m}+\aleph_0)$ ...
page 126, line 2 f.	...such that ~~we have~~ $\varphi(U', X)$.
page 143, line -2 f.	~~which shows that V_{α_0} can be well-ordered in the case when α_0 is a successor ordinal.~~
page 154, line 8	add a space: of$G$
page 154, line -16	add a space: $\}$.Notice
page 185, line 3 ff.	...is as above. So, $\rho\sigma_y(x_0) = \sigma_y(x_0)$ and therefore $\sigma_y^{-1}\rho\sigma_y(x_0) = x_0$. Consequently we have $\sigma_y^{-1}\rho\sigma_y = \vartheta^n$, and therefore $\rho = \sigma_y\vartheta^n\sigma_y^{-1}$. Thus, since ρ induces a proper cycle, this implies $y \in \{x_0, \dots, x_k\}$.
page 198, line -9	The Ordered Mostowski Model instead of "Ordered Mostowski Models".
page 211, line -3	Fraïssé-limit
page 213, line 18	Fraïssé-limit
page 215, line -11	$\Psi: \mathscr{P}(A)$ ...
page 231, line -9	... J are arbitrary finite, disjoint subfamilies...
page 234, line -11	add a space: shatter$x$
page 323, line 2,...	$(P, \leq)$
page 324, line -7	$\mathscr{D} \subseteq \mathscr{P}(P)$
page 325, line 17	~~In other words, $\mathsf{MA}(\kappa)$ holds for each cardinal $\kappa < \mathfrak{c}$~~

page 343, line -6 ff.

$$\mathrm{up}(\underset{\sim}{x},\underset{\sim}{y}) := \{\langle \underset{\sim}{x}, \mathbf{0}\rangle, \langle \underset{\sim}{y}, \mathbf{0}\rangle\}$$

and

$$\mathrm{op}(\underset{\sim}{x},\underset{\sim}{y}) := \{\langle\{\langle \underset{\sim}{x}, \mathbf{0}\rangle\}, \mathbf{0}\rangle, \langle\{\langle \underset{\sim}{x}, \mathbf{0}\rangle, \langle \underset{\sim}{y}, \mathbf{0}\rangle\}, \mathbf{0}\rangle\}.$$

page 343, line 10 — Replace everywhere $\underset{\sim}{G}$ with $\underset{\cdot}{G}$, and cancel in the index the definition of $\underset{\sim}{G}$.

page 345, line -7 — In order to show the second part of this proof (G is $\mathbb{P}$-generic) one needs FACT 15.7.

page 357, line 4 ff. — Axiom of Foundation: Let $G \subseteq P$ be $\mathbb{P}$-generic over $\mathbf{V}$. With respect to G, we define a rank-function $\mathrm{rk}_G : \mathbf{V}^{\mathbb{P}} \to \Omega$ by stipulating

$$\mathrm{rk}_G(\underset{\sim}{z}) := \bigcup\Big\{\mathrm{rk}_G(\underset{\sim}{y}) + 1 : \exists p \in G\big(\langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{z}\big)\Big\}.$$

Let $\underset{\sim}{x}$ and $\underset{\sim}{y}$ be two $\mathbb{P}$-names. First, we show that $\underset{\sim}{x}[G] = \underset{\sim}{y}[G]$ implies $\mathrm{rk}_G(\underset{\sim}{x}) = \mathrm{rk}_G(\underset{\sim}{y})$. To see this, assume that $\alpha = \mathrm{rk}_G(\underset{\sim}{y}) < \mathrm{rk}_G(\underset{\sim}{x})$. By definition of rk_G, there is a name $\underset{\sim}{z}$ with $\alpha \le \mathrm{rk}_G(\underset{\sim}{z})$ and $\underset{\sim}{z}[G] \in \underset{\sim}{x}[G]$. Since $\alpha \le \mathrm{rk}_G(\underset{\sim}{z})$, we have $\underset{\sim}{z}[G] \notin \underset{\sim}{y}[G]$, and hence, $\underset{\sim}{x}[G] \neq \underset{\sim}{y}[G]$.
Now, let

$$\alpha_0 := \bigcap\Big\{\mathrm{rk}_G(\underset{\sim}{y}) : \exists p \in G\big(\langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x}\big)\Big\}.$$

Then there is a $\mathbb{P}$-name $\underset{\sim}{y}_0$ such that $\underset{\sim}{y}_0[G] \in \underset{\sim}{x}[G]$ and $\mathrm{rk}_G(\underset{\sim}{y}_0) = \alpha_0$, which implies that $\underset{\sim}{x}[G] \cap \underset{\sim}{y}_0[G] = \emptyset$.

page 361, line 1 — **collapses** $\boldsymbol{\kappa}$ [bold] and **preserves** $\boldsymbol{\kappa}$ [bold]

page 362, line 13 — This is because whenever $q_1 \Vdash_{\mathbb{P}} \underset{\sim}{S}(\underset{\cdot}{\alpha}) = \underset{\cdot}{\gamma_1}$ and $q_2 \Vdash_{\mathbb{P}} \underset{\sim}{S}(\underset{\cdot}{\alpha}) = \underset{\cdot}{\gamma_2}$, where $\gamma_1 \neq \gamma_2$, then $q_1 \perp q_2$.

page 362, line 6 ff. — Replace p with p_0 on line 6, 7, 9, 10, 15.

page 364, line 17 f. — . . .countable union of ~~at most~~ countable sets of ordered pairs. . .

page 364, line 17 f. — . . .countable union of ~~at most~~ countable sets of ordered pairs. . .

page 372, line -5 — . . .(b), we refine the construction in the proof of (a). By . . .

page 378, line 5 — . . .is a countable transitive model in $\mathbf{V}$, $\mathbf{N}[G] \vDash \Phi_0$, and if $p_0 \Vdash_{\mathbb{P}} \varphi$, then $\mathbf{N}[G] \vDash \varphi$.

page 378, line 20 — . . .then $\mathbf{N}[G] \vDash \Phi_0 + \varphi$.

page 391, line -3 — add a space: containsa

page 407, line 8 f. — $\underset{\cdot}{q}$ instead of $\underset{\sim}{q}$ and q, respectively.

page 428, line -5 — . . . let $\mathbb{P}_{\omega_2} = \langle \mathbb{Q}_\alpha : \alpha \in \omega_2\rangle$ . . .

page 465, line 2 ff. — Since $\hat{\mathscr{U}}$ is generated by $\mathscr{U}$, for each $n \in \omega$ there is an $x'_n \in \mathscr{U}$ such that $x'_n \subseteq x_n$. Then define $A := \{f(x'_n) : n \in \omega\}$ and notice that $y \subseteq^* x'_n \subseteq x_n$.

page 478, line 12	. . .there is some $\langle q_0, q_1\rangle \in (G\|_\alpha \times G(\alpha)) \cap E$.
page 492, line -11	A general form of the Δ-System-Lemma (see Kunen, Thm. 1.6, p. 49) is needed here.
page 493, line 3	$\ldots \leq \omega_2 \cdot \omega_2 \ldots$
page 493, line -8	. . . P-point—and in particular every Ramsey ultrafilter—in . . .
page 551, line -6	. . . P-point in $\mathbf{V}[G\|_\delta]$, for some $\delta \in \omega_2$.