Properties of Hesse derivatives of cubic curves

Sayan Dutta
Department of Mathematics and Statistics, IISER Kolkata, Mohanpur, West Bengal 741246, India
sd19ms148@iiserkol.ac.in

Lorenz Halbeisen
Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland
lorenz.halbeisen@math.ethz.ch

Norbert Hungerbühler
Department of Mathematics, ETH Zentrum, Rämistrasse 101, 8092 Zürich, Switzerland
norbert.hungerbuehler@math.ethz.ch

key-words: Elliptic curves, Hessian curves, geometry of cubic curves, halving formulae, discrete dynamical system

2020 Mathematics Subject Classification: **11G05** 37N99

Abstract

The Hesse curve or Hesse derivative $হ\Gamma_f$ of a cubic curve Γ_f given by a homogeneous polynomial f is the set of points P such that $\det(H_f(P)) = 0$, where $H_f(P)$ is the Hesse matrix of f evaluated at P. Also $হ\Gamma_f$ is again a cubic curve. We show that for a point $P \in হ\Gamma_f$, all the contact points of tangents from P to the curves Γ_f and $হ\Gamma_f$ are intersection points of two straight lines ℓ_1^P and ℓ_2^P (meeting on $হ\Gamma_f$) with Γ_f and $হ\Gamma_f$, where the product of ℓ_1^P and ℓ_2^P is the polar conic of Γ_f at P. The operator হ defines an iterative discrete dynamical system on the set of the cubic curves. We identify the two fixed points of this system, investigate orbits that end in the fixed points, and discuss the closed orbits of the dynamical system.

1 Introduction

We will work with cubic curves in the real projective plane $\mathbb{RP}^2$. Points $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \backslash \{0\}$ will be denoted by capital letters, the components with small letters, and the equivalence class by $[X] := \{\lambda X \mid \lambda \in \mathbb{R} \setminus \{0\}\}$. However, since we mostly work with representatives, we often omit the square brackets in the notation.

Let f be a homogeneous polynomial in the variables x_1, x_2, x_3 of degree 3. Then f defines the projective cubic curve

$$\Gamma_f := \left\{[X] \in \mathbb{RP}^2 \mid f(X) = 0\right\}.$$

The Hesse matrix of f is the symmetric 3×3 matrix $H_f = \left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right)$.

Observe that $\det(H_f)$ is again a homogeneous cubic polynomial. Therefore, we can define the *Hesse derivative* of Γ_f, denoted $হ\Gamma_f$[1], as the cubic curve

$$হ\Gamma_f := \Gamma_{\det(H_f)} = \left\{[X] \in \mathbb{RP}^2 \mid \det(H_f(X)) = 0\right\}.$$

The polar conic of Γ_f with respect to the pole P is given by the equation

$$\mathcal{C}_f(P): \ \langle X, H_f(P)X \rangle = 0 \tag{1}$$

[1] In order to denote the Hesse derivative of a cubic curve, we introduce the Bengali letter হ (pronounced "Haw"). As a fact we would like to mention that "Hesse" in Bengali means "to laugh"!

or equivalently

$$\mathcal{C}_f(P): \ \langle \nabla f(X), P \rangle = 0. \tag{2}$$

The equivalence of (1) and (2) is shown in [4]. It is clear from (2) that the contact points of the tangents from P to Γ_f are precisely the intersection points of $\mathcal{C}_f(P)$ with Γ_f (see Figure 1).

If there is no danger of confusion, we will omit the index and briefly write Γ instead of Γ_f. Moreover, we will use the notation H_Γ instead of H_f, and $\mathcal{C}_\Gamma(P)$ instead of $\mathcal{C}_f(P)$ if the polynomial f is determined by the context or if a general but unique polynomial is meant. We would like to mention that the Hesse derivative हΓ is also known as *Hessian curve*, denoted Hess(Γ) (see, e.g., [5, § 4.12, p. 111]. However, we prefer the notation हΓ because we want to interpret ह as an operator whose iterations we want to study. Whenever convenient, we will use x, y, z instead of x_1, x_2, x_3 for the coordinates. The figures below of the various projective curves show images of the curves in the affine plane $x_3 = 1$ embedded in $\mathbb{RP}^2$.

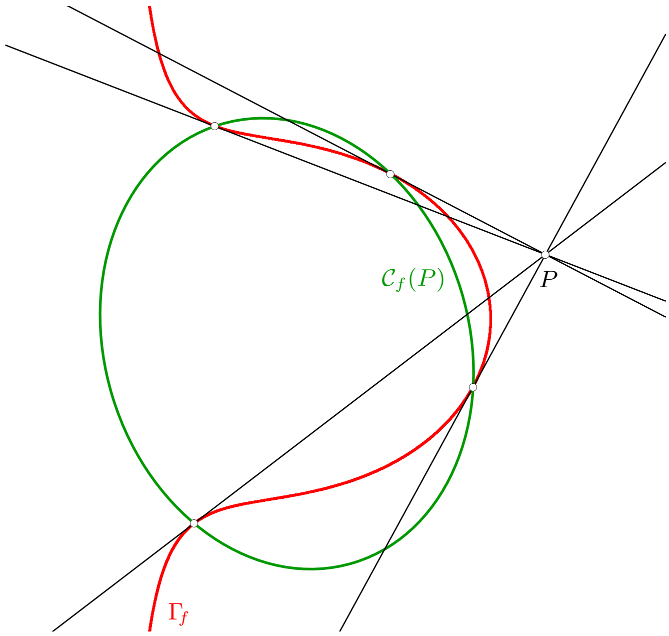

Figure 1: A cubic curve Γ_f and its polar conic $\mathcal{C}_f(P)$ with respect to the pole P.

It is well known that the polar conic is the product of two projective lines, ℓ_1^P and ℓ_2^P iff the determinant of the Hesse matrix evaluated at P is equal to 0, i.e.,

$$\mathcal{C}_\Gamma(P) = \langle X, \ell_1^P \rangle \langle X, \ell_2^P \rangle \iff \det\left(H_\Gamma(P)\right) = 0.$$

In particular, we obtain the following result

Proposition 1. *If $P \in$ हΓ, i.e., the polar conic is the product of the two lines ℓ_1^P and ℓ_2^P, then the tangents from P to Γ touch Γ precisely at the points $\Gamma \cap \ell_1^P$ and $\Gamma \cap \ell_2^P$.*

2 Halving formulae for points on हΓ

We would now like to compute the contact points of the tangents from a point $P \in$ हΓ to the curve हΓ. By a suitable projective transformation, we may assume that the curve is of the form

$E_{a,b}$ defined by

$$E_{a,b}: \; y^2 = x^3 + a\,x^2 + b\,x \tag{3}$$

in the affine plane, where $a,\, b \in \mathbb{R}$.

Proposition 2. *Let $E_{a,b}$ be a non-singular elliptic curve over $\mathbb{C}$ defined by*

$$E_{a,b}: \; y^2 = x^3 + a\,x^2 + b\,x$$

where $a,\, b \in \mathbb{C}$, and let $P = (x_0, y_0)$ be a point on $E_{a,b}$.

Let

$$e_1 = \frac{-a + \sqrt{a^2 - 4b}}{2}, \qquad e_2 = \frac{-a - \sqrt{a^2 - 4b}}{2}.$$

and let

$$\gamma = \sqrt{x_0}, \qquad \alpha = \sqrt{x_0 - e_1}, \qquad \beta = \sqrt{x_0 - e_2}.$$

Then, $E_{a,b}$ is of the form

$$y^2 = x_0(x_0 - e_1)(x_0 - e_2)$$

and the x-coordinates of the contact points of the tangent of P with $E_{a,b}$, denoted by Q_1, Q_2, Q_3, Q_4 are

$$\begin{aligned}
x_{11} &= (\alpha + \gamma)(\beta + \gamma), \\
x_{12} &= (\alpha - \gamma)(\beta - \gamma), \\
x_{21} &= (\alpha + \gamma)(-\beta + \gamma), \\
x_{22} &= (\alpha - \gamma)(-\beta - \gamma).
\end{aligned}$$

Notice that for the points Q_i, $i = 1, 2, 3, 4$, we have $2 * Q_i = -P$, i.e., $\frac{P}{2} = -Q_i$, where $2 * Q_i = Q_i + Q_i$ is the usual elliptic curve operation on $E_{a,b}$ (see, e.g., [3]).

Proof. We obviously have $-e_1 - e_2 = a$ and $e_0 e_1 + e_1 e_2 + e_2 e_0 = b$.

To show that $x_{11}, x_{12}, x_{21}, x_{22}$ are the the x-coordinates of points $Q \in E_{a,b}$ such that $2 * Q = P$, it is enough to show that the x-coordinate of the point $Q_{ij} := (x_{ij}, y)$, where $i, j \in \{1, 2\}$ and $y = \sqrt{x_{ij}^3 + a\,x_{ij}^2 + b\,x_{ij}}$, is equal to x_0. Now, the x-coordinate x_{2ij} of the point $2 * Q_{ij}$ is given by the formula

$$x_{2ij} = \frac{x_{ij}^4 - 2b\,x_{ij}^2 + b^2}{4(x_{ij}^3 + a\,x_{ij}^2 + b\,x_{ij})} = \frac{(x_{ij}^2 - b)^2}{4x_{ij}(x_{ij}^2 + a\,x_{ij} + b)}.$$

Furthermore, we have $a = \alpha^2 + \beta^2 - 2\gamma^2$ and $b = (\alpha^2 - \gamma^2)(\beta^2 - \gamma^2)$, and if we write x_{ij}, a, b in terms of γ, α, β, it is not hard to verify that

$$(x_{ij}^2 - b)^2 = 4x_{ij}\gamma^2(x_{ij}^2 + a\,x_{ij} + b),$$

which shows that $x_{2ij} = x_0$. *q.e.d.*

3 Intersection of $\mathcal{C}_\Gamma(P)$ with हΓ for $P \in$ हΓ

In this section, we combine the results from Section 2 with the property that for every point $P \in$ हΓ, the polar conic of Γ with respect to the pole P is the product of two lines, ℓ_1^P and ℓ_2^P (see Proposition 1). In particular, we want to show that the two lines ℓ_1^P and ℓ_2^P intersect on the curve $\text{ह}\Gamma$ and that the points x_{ij}, $i,j=1,2$ correspond to the other intersection points of ℓ_1^P and ℓ_2^P with $\text{ह}\Gamma$ (see Figure 2).

For this, we start with the cubic curve

$$\Gamma_{a,b}:\ ax^3+3xy^2+3bx^2z-b^2z^3=0$$

with $a,b\in\mathbb{R}$, $b\neq 0$. For $\Gamma_{a,b}$, we get

$$H_{\Gamma_{a,b}}:=\begin{pmatrix} 6ax+6bz & 6y & 6bx \\ 6y & 6x & 0 \\ 6bx & 0 & -6b^2z \end{pmatrix}$$

and hence

$$\text{ह}\Gamma_{a,b}:\ y^2z=x^3+ax^2z+bxz^2.$$

In other words, $\text{ह}\Gamma_{a,b}=E_{a,b}$, as introduced in (3) in the previous section.

By definition, if $P=(x_0,y_0)\in\text{ह}\Gamma_{a,b}$, then $\det\big(H_{\Gamma_{a,b}}(P)\big)=0$, which implies that the conic section

$$\mathcal{C}_{\Gamma_{a,b}}(P):\ (ax_0+b)x^2+x_0y^2+2y_0xy+2bx_0x-b^2$$

can be written as the product of two lines ℓ_1^P and ℓ_2^P. In the following lemma, we will compute these two lines in terms of e_1 and e_2 defined in the previous section.

Lemma 3. *The lines ℓ_1^P and ℓ_2^P are given by*

$$\begin{aligned}\ell_1^P:&\ ux+vy+wz=0\\ \ell_2^P:&\ rx+sy+tz=0\end{aligned}$$

where

$$\begin{aligned}u&=-x_0-\sqrt{(e_1-x_0)(e_2-x_0)}\\ v&=-\sqrt{x_0}\\ w&=e_1e_2\\ r&=-x_0+\sqrt{(e_1-x_0)(e_2-x_0)}\\ s&=\sqrt{x_0}\\ t&=e_1e_2\,.\end{aligned}$$

Proof. If we replace y_0 by $\sqrt{x_0^3+ax_0^2+bx_0}$, and a,b by $-e_1-e_2$ and e_1e_2, respectively, then we have:

$$\mathcal{C}_{\Gamma_{a,b}}:\ (e_1e_2-e_1x_0-e_2x_0)\,x^2+x_0\,y^2+2\sqrt{x_0^3-e_1x_0^2-e_2x_0^2+e_1e_2x_0}\,xy+2e_1e_2x_0\,x-e_1^2e_2^2$$

Now, in order to show that $\mathcal{C}_{\Gamma_{a,b}}=\ell_1^P\cdot\ell_2^P$, we just have to check that

$$\begin{aligned}ur&=-e_1e_2+e_1x_0+e_2x_0\\ us+vr&=-2\sqrt{x_0^3-e_1x_0^2-e_2x_0^2+e_1e_2x_0}\\ vs&=-x_0^2\\ ut+wr&=-2e_1e_2x_0\\ wt&=-e_1^2e_2^2\\ vt+ws&=0\end{aligned}$$

which is easy to see. *q.e.d.*

Before we compute the intersection points of ℓ_1^P and ℓ_2^P with the curve $E_{a,b}$, we show that the two lines intersect on the curve $E_{a,b}$.

Lemma 4. *Let $P=(x_0,y_0)$ be a point on $\text{इ}\Gamma_{a,b}$. Then the point $\ell_1^P \cap \ell_2^P =: S=(x_S,y_S)$ lies on the same curve $\text{इ}\Gamma_{a,b}=E_{a,b}$.*

Proof. Let us rewrite the lines as

$$\begin{aligned} \ell_1^P :\ y &= \frac{-u}{v}x+\frac{-w}{v} \\ \ell_2^P :\ y &= \frac{-r}{s}x+\frac{-t}{s} \end{aligned}$$

which shows that the intersection of the two lines is given as:

$$\begin{aligned} x_S &= \frac{tv-ws}{us-rv} = \frac{e_1e_2}{x_0} = \frac{b}{x_0} \\ y_S &= \frac{-u}{v}x+\frac{-w}{v} = -\frac{by_0}{x_0^2} \end{aligned}$$

Now, we use the fact that

$$y_0^2 = x_0^3 + ax_0^2 + bx_0$$

to show that

$$\left(\frac{b}{x_0}\right)^3 + a\left(\frac{b}{x_0}\right)^2 + b\left(\frac{b}{x_0}\right) = \left(\frac{-by_0}{x_0^2}\right)^2$$

which shows that $S \in E_{a,b}$. *q.e.d.*

Lemma 5. *The map $\text{इ}\Gamma_{a,b} \to \text{इ}\Gamma_{a,b},\ P=(x_0,y_0) \mapsto S=(x_S,y_S)$, is an involution.*

Proof. We just have to check that

$$\frac{b}{\frac{b}{x_0}} = x_0 \qquad \text{and} \qquad \frac{-by_0}{x_0^2} = \frac{-b\cdot\frac{-by_0}{x_0^2}}{\frac{b^2}{x_0^2}} = y_0$$

which is easy to see. *q.e.d.*

Now, we show that the other intersection points of ℓ_1^P and ℓ_2^P with $E_{a,b}$ are exactly the points x_{ij}, $i,j=1,2$ from Proposition 2.

Lemma 6. *Besides the point S, the intersection points of ℓ_1^P and ℓ_2^P with $E_{a,b}$ are exactly the points Q_i, $i=1,2,3,4$, with $2*Q_i=-P$. More precisely, the points x_{11} and x_{12} are on the line ℓ_1^P and the points x_{21} and x_{22} are on ℓ_2^P.*

Proof. To find the x-coordinate of the intersection points of ℓ_i^P with $E_{a,b}$ we eliminate y form the equations for $E_{a,b}$ and ℓ_1^P, and ℓ_2^P, respectively. The resulting equations are of degree 3 in x, but since we already know the root $\frac{b}{x_0}$, the problem reduces to quadratic equations

$$x^2 - 2\left(x_0 \pm \alpha\beta\right)x + e_1e_2 = 0.$$

The solutions are

$$x_0+\alpha\beta \pm \sqrt{\left(x_0+\alpha\beta\right)^2 - e_1e_2} \qquad \text{and} \qquad x_0-\alpha\beta \pm \sqrt{\left(x_0-\alpha\beta\right)^2 - e_1e_2}$$

and one checks easily that these expressions agree with the formulas for x_{ij} from Proposition 2.

It remains to show that the y-coordinates match as well. To see that, let us denote by A and B the points at which the tangents from P to $E_{a,b}$ meet $E_{a,b}$. So, for our claim to be true, we have

$$\begin{aligned} A + B &= -S \\ 2 * A &= -P \\ 2 * B &= -P \end{aligned}$$

which implies

$$2 * P = 2 * S$$

and hence it is enough to show that this is indeed the case. To do so, we note that a formula for doubling the point $P = (x_0, y_0)$ on $E_{a,b}$ is given by

$$2 * P = \left(\frac{\left(x_0^2 - e_1 e_2\right)^2}{4 y_0^2}, \frac{\left(x_0^2 - e_1 e_2\right)\left(e_1 e_2 - 2 e_1 x_0 + x_0^2\right)\left(e_1 e_2 - 2 e_2 x_0 + x_0^2\right)}{8 y_0^3} \right).$$

On the other hand, since $S = \left(\frac{e_1 e_2}{x_0}, \frac{-e_1 e_2 y_0}{x_0^2}\right)$, we have

$$2 * S = \left(\frac{\left(\frac{e_1^2 e_2^2}{x_0^2} - e_1 e_2\right)^2}{4 \cdot \frac{e_1^2 e_2^2 y_0^2}{x_0^4}}, \frac{\left(\frac{e_1^2 e_2^2}{x_0^2} - e_1 e_2\right)\left(e_1 e_2 - 2 e_1 \cdot \frac{e_1 e_2}{x_0} + \frac{e_1^2 e_2^2}{x_0^2}\right)\left(e_1 e_2 - 2 e_2 \cdot \frac{e_1 e_2}{x_0} + \frac{e_1^2 e_2^2}{x_0^2}\right)}{-8 \cdot \frac{e_1^3 e_2^3 y_0^3}{x_0^6}} \right).$$

The equality $2 * P = 2 * S$ immediately follows by multiplying the numerator and denominator x-coordinate of $2 * S$ by $\frac{x_0^4}{e_1^2 e_2^2}$ and the y-coordinate by $\frac{-x_0^6}{e_1^3 e_2^3}$. *q.e.d.*

Theorem 7. *Let Γ be a cubic curve and let $P \in$ हΓ. Then, all the contact points of tangents from P to the curves Γ and हΓ are intersection points of ℓ_1^P and ℓ_2^P with Γ and हΓ. In addition, the intersection Q of ℓ_1^P and ℓ_2^P lies on हΓ (see Figure 2).*

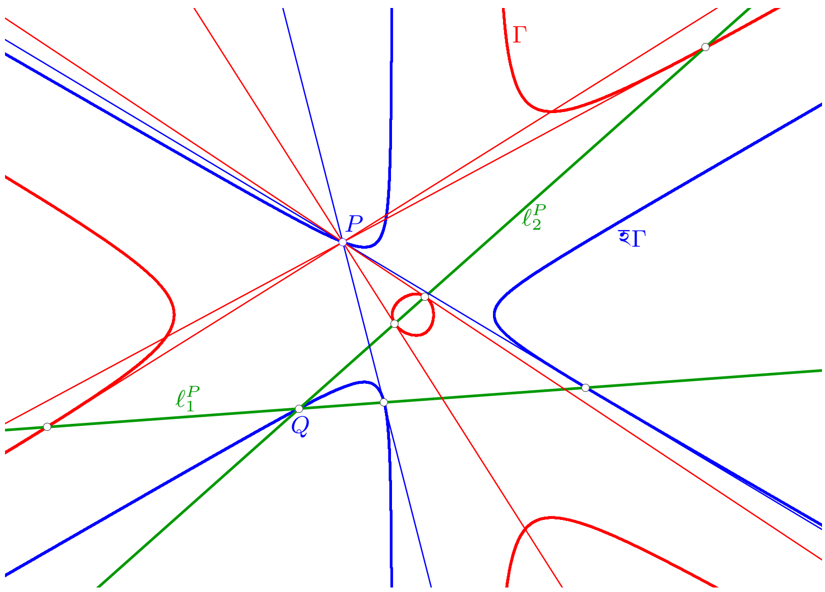

Figure 2: Illustration for Theorem 7 for a cubic curve Γ with the symmetry group of an equilateral triangle (see [1]). The Hesse derivative हΓ has the same symmetry. The curve Γ is given by $2\sqrt{3}x^3 + 9(\sqrt{3}+1)(x^2+y^2)z - 6\sqrt{3}xy^2 - 9z^3 = 0$ and has the property that ह$^2\Gamma = \Gamma$ (see Section 6).

4 Hesse Form of Cubic Curves

In this section, we consider a cubic curve in its Hesse form

$$\Gamma_c : \; x^3 + y^3 + z^3 + c\,xyz = 0$$

with $c \in \mathbb{R}$. Notice Γ_{-3} is a degenerate curve. Formally, we put

$$\Gamma_\infty : \; xyz = 0.$$

Lemma 8. *Let $c_0 \neq 0$. Then the Hesse derivative of Γ_{c_0} is $\text{ह}\Gamma_{c_0} = \Gamma_{c_1}$ where*

$$c_1 = -\frac{108 + c_0^3}{3c_0^2}.$$

The Hesse derivative of Γ_0 is $\text{ह}\Gamma_0 = \Gamma_\infty$, and the Hesse derivative of Γ_∞ is $\text{ह}\Gamma_\infty = \Gamma_\infty$.

Proof. We have

$$H_{\Gamma_{c_0}}(x, y, z) := \begin{pmatrix} 6x & c_0 z & c_0 y \\ c_0 z & 6y & c_0 x \\ c_0 y & c_0 x & 6z \end{pmatrix}.$$

This yields $\det H_{\Gamma_{c_0}}(x, y, z) = -6c^2(x^3 + y^3 + z^3) + 2(108 + c^3)xyz$, and the claim follows for $c_0 \in \mathbb{R} \setminus \{0\}$. The cases Γ_0 and Γ_∞ are also easily checked. *q.e.d.*

An immediate corollary is

Corollary 9. *Let $c_0 \neq 0$. Then, the $(n+1)$-th Hesse derivative of Γ_{c_0} is given by*

$$\text{ह}^{n+1}\Gamma_{c_0} : \; x^3 + y^3 + z^3 + c_{n+1}xyz = 0$$

where $c_{n+1} = -\frac{108 + c_n^3}{3c_n^2}$ for every $n \geq 0$, as long as $c_n \neq 0$.

5 Analysis of iterates

Motivated by Lemma 8, we consider the function

$$h(x) = \frac{a + x^3}{bx^2} \tag{4}$$

for $a \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$.

Lemma 10. *The function h defined in (4) has a pole at $x = 0$, and an oblique asymptote $y = \frac{x}{b}$. For $b \neq -1$*

$$\varphi := \sqrt[3]{\frac{a}{b-1}}$$

is the unique real fixed point of h. The function h has the unique critical point $\kappa := \sqrt[3]{2a}$ with critical value

$$h(\kappa) = \frac{3a^{1/3}}{2^{2/3}b}.$$

The proof is elementary.

Remark: In our case, $b = -3$, we have

$$h(\varphi) = \varphi = -\sqrt[3]{\frac{a}{4}} = -\sqrt[3]{\frac{a^3}{4a^2}} = h(\kappa).$$

This case also gives us the crucial property for the partition of $\mathbb{R} \setminus \{\varphi\}$ into the intervals $N = (-\infty, \varphi)$ and $P = (\varphi, \infty)$. Namely, we have $x \in P \setminus \{0\}$ iff $h(x) \in N$, and $x \in N$ iff $h(x) \in P$. For the next two propositions, we will assume $b = -3$ and $a \neq 0$, and hence $\varphi \neq 0$.

Proposition 11. *Let $b = -3$ and $a \neq 0$. If we define*

$$h^{(n)} := \underbrace{h \circ h \circ \ldots h}_{n \text{ times}}$$

then, $y = \frac{x}{b^n}$ is an oblique asymptote of $h^{(n)}$. Furthermore, if κ_n is a critical point of $h^{(n)}$, we have

$$h^{(n)}(\kappa_n) = \varphi = -\sqrt[3]{\frac{a}{4}}.$$

Conversely, if $h^{(n)}(x) = \varphi$, then either $x = \varphi$ or $\frac{d}{dx} h^{(n)}(x) = 0$.

Proof. Since we already know the oblique asymptote of h from Lemma 10, we can now inductively argue that

$$\begin{aligned} \lim_{x \to \pm\infty} \left| h^{(n+1)}(x) - \frac{x}{b^{n+1}} \right| &= \lim_{x \to \pm\infty} \left| \frac{a + \left(h^{(n)}(x)\right)^3}{b\left(h^{(n)}(x)\right)^2} - \frac{\frac{x}{b^n}}{b} \right| \\ &= \lim_{x \to \pm\infty} \left| \frac{a}{b\left(h^{(n)}(x)\right)^2} + \frac{h^{(n)}(x) - \frac{x}{b^n}}{b} \right| = 0 \end{aligned}$$

using the fact that $\left(h^{(n)}(x)\right)^2 \to \infty$ for $x \to \pm\infty$. Note that in this part did we did not need the assumption $b = -3$.

For the next part, observe first that we have

$$h(x) = \varphi \iff (x - \varphi)(x - \kappa)^2 = 0.$$

This equation has only two solutions, namely $x_1 = \varphi$ and $x_2 = \kappa$. Now, by chain rule, we obtain

$$\begin{aligned} \frac{d}{dx} h^{(n)}(x) = \prod_{r=0}^{n-1} h'\left(h^{(r)}(x)\right) = 0 \qquad (5) \\ \iff h'\left(h^{(r)}(x)\right) = 0 \text{ for some } r \in \{0, 1, \ldots, n-1\} \\ \iff h^{(r)}(x) = \kappa \text{ for some } r \in \{0, 1, \ldots, n-1\}. \end{aligned}$$

So, it follows immediately that $\frac{d}{dx} h^{(n)}(x) = 0$ implies $h^{(n)}(x) = \varphi$.

Now we prove the converse, as stated in the lemma. For $n = 0$, the statement is trivially true. Assume that for some $n \geq 1$ we have that $h^{(n)}(x) = \varphi$ implies that either $x = \varphi$ or $\frac{d}{dx} h^{(n)}(x) = 0$. Then we have for $h^{(n+1)}(x) = \varphi$ that $h^{(n)}(x) = \kappa$ or $h^{(n)}(x) = \varphi$. On the other hand, $\frac{d}{dx} h^{(n+1)}(x) = h'(h^{(n)}(x)) \frac{d}{dx} h^{(n)}(x)$. If $h^{(n)}(x) = \kappa$, then the first factor in this product is zero and the derivative of $\frac{d}{dx} h^{(n+1)}(x)$ vanishes. If $h^{(n)}(x) = \varphi$, then, by induction, $x = \varphi$ or $\frac{d}{dx} h^{(n)}(x) = 0$, and again the derivative of $\frac{d}{dx} h^{(n+1)}(x)$ is zero. *q.e.d.*

Proposition 12. *Let χ_n be the number of critical points of $h^{(n)}$. Then, the sequence $\{\chi_n\}$ is given by*

$$\chi_{2r+1} = 2 \times 3^r - 1 \quad \textit{and} \quad \chi_{2r} = 3^r - 1$$

for all $r \geq 0$. This corresponds to OEIS A062318.

Proof. Without only carry out the case $\varphi < 0$. The proof for $\varphi > 0$ is essentially the same.

Let $N = (-\infty, \varphi)$ and $P = (\varphi, \infty)$. Observe first that for given $y \in N$ the equation

$$y = h(x) \quad \text{or equivalently} \quad x^3 + 3yx^2 - 4\varphi^3 = 0$$

has three distinct real roots. Indeed, the discriminant $\Delta = 27 \times 16\varphi^3 \left(y^3 - \varphi^3\right)$ is strictly positive for $y \in N$. Moreover, if $x, y \in N$ then the expression $x^3 + 3yx^2 - 4\varphi^3$ is strictly negative, hence the three solutions of $y = h(x)$ must lie in P. Hence, the preimage $h^{-1}(y)$ of a point in $y \in N$ has cardinality 3, and lies in P. Similarly, the preimage $h^{-1}(y)$ of a point in $y \in P$ has cardinality 1, and lies in N.

Now, for $n = 1$, the set of critical points of h is $C_1 = \{\kappa\} \subset P$. Let $S_1 := h^{-1}(C_1)$, and $S_k := h^{-1}(S_{k-1})$ for $k > 1$. For $n > 1$ can read of from equation (5) that the the set of critical points of $h^{(n)}$ is the set $C_n = C_{n-1} \cup S_{n-1}$. Observe that $S_n \subset N$ if n is odd, and $S_n \subset P$ if n is even. Hence we have $\operatorname{card} C_{2n} = \operatorname{card} C_{2n} + 3^{n-1}$ and $\operatorname{card} C_{2n+1} = \operatorname{card} C_{2n} + 3^n$. This corresponds to the sequence OEIS A062318. *q.e.d.*

Proposition 13. *Let Φ_n be the number of fixed points of $h^{(n)}$. Then, the sequence $\{\Phi_n\}$ is given by*

$$\Phi_{2r+1} = 1 \quad \textit{and} \quad \Phi_{2r} = 2\chi_{2r} - 1 = 2 \times 3^r - 3$$

for all $r \geq 0$.

Proof. Since h maps N to P and vice versa, the only fixed point of $h^{(2r+1)}$ is φ.

For the fixed points of $h^{(2r)}$, we begin by assuming without loss of generality that $a > 0$, $\varphi < 0$, and recalling that h has a pole at $x = 0$, it is decreasing in the intervals $(-\infty, 0)$, (κ, ∞) and increasing in $(0, \kappa)$. We also know that the only critical point of h is at κ and it has a local maximum there. So, for convenience, we define

$$\tilde{h}(x) := h(x + \varphi) - \varphi$$

i.e., we shift the point (φ, φ) to the origin. The number of fixed points of $h^{(n)}$ is the same as the number of fixed points of $\tilde{h}^{(n)}$. Consider the sets $\tilde{N} = (-\infty, 0)$, $\tilde{P} = (0, \infty)$, and $A = (0, \varphi), B = (\varphi, \varphi + \kappa), C = (\kappa, \infty)$. Then, $\tilde{h}$ maps $\tilde{N}$ bijectively to $\tilde{P}$, and A, B and C each bijectively to $\tilde{N}$. Hence $\tilde{h}$ maps $\tilde{N}, \tilde{P}$ to $\tilde{P}$ and tree copies of $\tilde{N}$. So after $2r$ iterations, the range of $\tilde{h}^{(2r)}$ consists of 3^r copies of $\tilde{N}$ and 3^r copies of $\tilde{P}$. Figure 3 shows schematically the behaviour of $\tilde{h}^{(2r)}$.

Observe that the oblique asymptote of $\tilde{h}^{(2r)}$ is given by $y = x/3^{2r}$, hence the line $y = x$ does not intersect the leftmost and the rightmost branch of the graph of $\tilde{h}^{(2r)}$. Hence the number of fixed points of $\tilde{h}^{(2r)}$ is $4 \cdot \frac{3^r - 1}{2} - 1 = 2 \times 3^r - 3$. *q.e.d.*

Analogous to the determination of the number of fixed points of $h^{(n)}$, the number of zeros is now calculated.

Proposition 14. *Let ρ_n be the number of zeros of $h^{(n)}$. Then, the sequence $\{\rho_n\}$ is given by*

$$\rho_{2r} = \rho_{2r+1} = 3^r$$

for all $r \geq 0$.

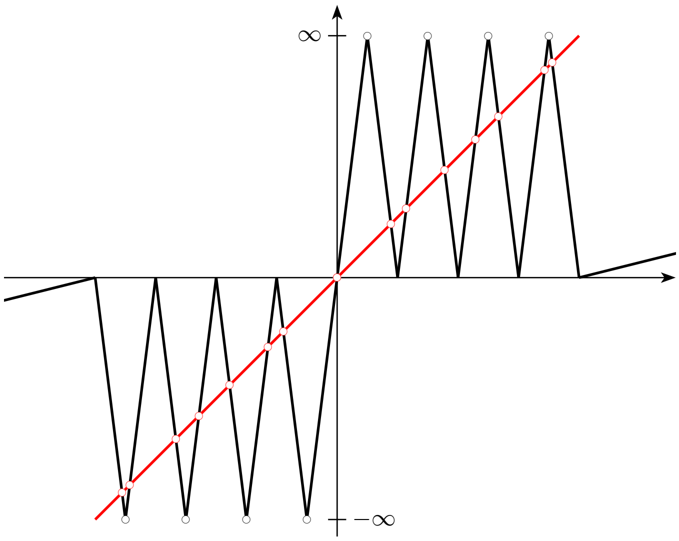

Figure 3: Schematic profile of the function $\tilde{h}^{(2r)}$: We have $(3^r-1)/2$ spikes on the positive x-axis and $(3^r-1)/2$ on the negative x-axis.

Proof. The number of zeros of $h^{(n)}$ equals the number of solutions of the equation $\tilde{h}^{(n)} = -\varphi$. We continue to assume that φ is negative. If $n = 2r$ is even, then each spike on the positive x-axis contributes two solutions, and the rightmost branch another one. So, the number of solutions of $\tilde{h}^{(2r)} = -\varphi$ is $2 \cdot \frac{3^r-1}{2} + 1 = 3^r$, as claimed.

Similarly, for $n = 2r+1$, the function $\tilde{h}^{(2r+1)}$ exhibits $(3^r-1)/2$ spikes in the second quadrant, is negative on the positive x-axis, and approaches the asymptote $-x/3^{2r+1}$. Therefore, we also find $2 \cdot \frac{3^r-1}{2} + 1 = 3^r$ zeros of $\tilde{h}^{(2r+1)}$. *q.e.d.*

6 Loops and chains of Hesse derivatives

We return to considering the curves in Hesse form, i.e.,

$$\Gamma_c : \; x^3 + y^3 + z^3 + c\,xyz = 0, \qquad \Gamma_\infty : \; xyz = 0 \tag{6}$$

for $c \in \mathbb{R}$. So, in this section, while referring to the function h from the previous section, we will assume $a = 108$, $b = -3$, $\varphi = -3$, $\kappa = 6$ (see Lemma 8 and Lemma 10).

We will begin with the following observation.

Lemma 15. *The geometric interpretation of the property $x \in P \iff h(x) \in N$ is that the curve Γ_c has two components when $x < -3$ and only one component when $x > -3$. In particular, if Γ has one component, then* हΓ *has two components, and if Γ has two component, then* हΓ *has one components.*

Proof. We begin by noting that Γ_c is unchanged under the transformation $(x, y) \mapsto (y, x)$, and hence it is symmetric with respect to the line $y = x$. Now, let us calculate the number of intersection points of Γ_c with the line $y = x$. This gives us the equation

$$2x^3 + cx^2 + 1 = 0$$

the discriminant of which is

$$-27 \times 4 - 4c^3$$

and hence the equation has three real roots iff $c < -3$. *q.e.d.*

The operator হ defines via $\Gamma \mapsto \text{হ}\Gamma$ an iterative discrete dynamical system on the set of the cubic curves (6) in Hesse form. The dynamics is given by Lemma 8. The system has exactly two fixed points, namely Γ_{-3} and Γ_∞. We are now interested in orbits of a given length which end in one of the fixed points, and in closed orbits of a given length. The former we call *Hesse chains*, the latter *Hesse loops*. So, a Hesse chain is given by

$$\text{হ}^n\Gamma_{c_0} = \Gamma_{c_n} = \Gamma_{-3} \qquad \text{or} \qquad \text{হ}^n\Gamma_{c_0} = \Gamma_{c_n} = \Gamma_\infty$$

where we call the minimal n with this property the length of the chain. Similarly, a Hesse loop is given by

$$\text{হ}^n\Gamma_{c_0} = \Gamma_{c_n} = \Gamma_{c_0}$$

where the minimal $n > 0$ with this property is the length of the loop.

The number of Hesse chains ending in Γ_{-3} of length n is easy to calculate as shown in the following Lemma.

Lemma 16. *If* $\text{চ}_n^{(-3)}$ *denotes the number*[2] *of Hesse chains ending in* Γ_{-3} *of length* n*, then*

$$\text{চ}_{2r} = \text{চ}_{2r-1} = 3^{r-1}$$

for $r \geq 1$.

Proof. From Proposition 12, it is clear that

$$\text{চ}_{2r}^{(-3)} = \chi_{2r} - \chi_{2r-1} = 3^{r-1}$$

and

$$\text{চ}_{2r-1}^{(-3)} = \chi_{2r-1} - \chi_{2r-2} = 3^{r-1}$$

hence completing the proof. *q.e.d.*

Lemma 17. *For any positive* $B \in \mathbb{R}$*, there exists* $c > B$ *and* $c < -B$ *and an* $n \in \mathbb{N}$ *such that* $\text{হ}^n\Gamma_c = \Gamma_{-3}$.

Proof. Observe first that $\Gamma_6 = \Gamma_{-3}$. Also note that for $c \geq 6$ the solution $\bar{c}$ of the equation $h(\bar{c}) = c$ satisfies $\bar{c} < -3c$. On the other hand, for $c \leq -6$, the largest of the tree solutions $\bar{c}$ of the equation $h(\bar{c}) = c$ satisfies $\bar{c} > -3c - 1$. Hence by backward iteration and choosing always the solution with the largest absolute value, we can construct an orbit ending in Γ_{-3} and starting at some Γ_c with $c > B$ or $c < -B$. *q.e.d.*

Similarly as before, we consider the Hesse chains ending in Γ_∞.

Lemma 18. *If* চ_n^∞ *denotes the number of Hesse chains of length* n *ending in* Γ_∞*, then*

$$\text{চ}_{2r}^\infty = \text{চ}_{2r-1}^\infty = 3^{r-1}$$

for $r \geq 1$.

[2] We again find ourselves at a loss of expressions to notate the "ch" sound used in "chain" as such an alphabet is not present in English, Latin or Greek script. We will use this excuse to use another Bengali alphabet, namely চ, pronounced as *"chaw"*.

Proof. Since $\text{ह}\Gamma_0 = \Gamma_\infty$, the number of Hesse chains ending in Γ_∞ of length n is the number of zeros of $h^{(n-1)}$. Therefore the claim follows from Lemma 14. *q.e.d.*

Now we turn our attention towards Hesse loops.

Proposition 19. *The only Hesse loop of odd length is the trivial loop* $\text{ह}\Gamma_{-3} = \Gamma_{-3}$.

Proof. This follows immediately from Lemma 15. *q.e.d.*

Now, we want to determine the number of Hesse loops of length n for even n. We start with the following observation.

Proposition 20. *For every even* n*, there is at least one Hesse loop of length* n.

Proof. Recall that Φ_n denotes the number of fixed points of $h^{(n)}$ (see Proposition 13). Now, note that the value of Φ_n already includes the trivial fixed point -3. So let $\Phi'_n := \Phi_n - 1$ denote the number of non-trivial fixed points of $h^{(n)}$. Furthermore, for any r, a fixed point of $h^{(r)}$ is also a fixed point of $h^{(mr)}$ for any m. Also, any loop of length r consists of r elements and contributes this number to Φ'_n. It is enough to show that the quantity

$$\Phi'_{2r} - \sum_{k=1}^{r-1} \Phi'_{2k}$$

is strictly positive. This follows from the fact that for $r > 1$, we have

$$\sum_{k=1}^{r-1} \Phi'_{2k} = \sum_{k=1}^{r-1} \left(2 \times 3^k - 4\right) = 3^r - 4r + 1,$$

and this is indeed strictly smaller than $\Phi'_{2r} = 2 \times 3^r - 4$. *q.e.d.*

Remark. The above calculation also shows that we must at least have

$$\left\lceil \frac{1}{2r}\left(\Phi'_{2r} - \sum_{k=1}^{r-1} \Phi'_{2k}\right) \right\rceil = \left\lceil \frac{3^r - 5}{2r} \right\rceil + 2$$

loops of length $2r$.

Proposition 21. *If* Λ_n *denotes the number of Hesse loops of length* n*, then the sequence* $\{\Lambda_{2r}\}$ *is strictly increasing.*

Proof. Since Φ'_{2r} includes the two elements of the only 2-loop, the value of Λ_{2r} can be at most

$$\left\lfloor \frac{\Phi'_{2r} - 2}{2r} \right\rfloor = \left\lfloor \frac{3^r - 3}{r} \right\rfloor$$

and hence

$$\left\lceil \frac{3^r - 5}{2r} \right\rceil + 2 \le \Lambda_{2r} \le \left\lfloor \frac{3^r - 3}{r} \right\rfloor$$

for $r > 1$. So, to prove that the sequence $\{\Lambda_{2r}\}$ is strictly increasing, it is enough to show that

$$\left\lceil \frac{3^r - 5}{2r} \right\rceil > \left\lfloor \frac{3^{r-1} - 3}{r - 1} \right\rfloor.$$

This follows from

$$\frac{3^x - 5}{2x} > \frac{3^{x-1} - 3}{x - 1}$$

which is true for $x \geq 3$ as then we have

$$3^{x-1} \cdot x + 5 + x > 3^x$$

which completes the proof for $r \geq 3$.

The cases $r = 1, 2$ can be checked by hand. *q.e.d.*

We close this discussion by an explicite formula for the number of loops of length $n = 2r$.

Theorem 22. *The number of loops of length* $2r$ *is*

$$\Lambda_{2r} = \frac{1}{2r} \sum_{d|r} \mu\left(\frac{r}{d}\right) \Phi'_{2d}$$

where $\Phi'_{2d} = 2 \times 3^d - 4$*, and* μ *is the Möbius function.*

Proof. Let the even divisors of $2r$ be $d_1 = 2, d_2, \ldots, d_k = 2r$. Since each loop of length d_m contains exactly d_m elements, the total number of fixed points $\neq -3$ of $h^{(2r)}$ is given by

$$\Phi'_{2r} = \sum_{\substack{d|2r \\ d \text{ even}}} d \cdot \Lambda_d.$$

The even divisors of $2r$ are twice the divisors of r. Hence we may write

$$\Phi'_{2r} = \sum_{d|r} 2d \cdot \Lambda_{2d}.$$

Using the Möbius inversion formula, we obtain

$$2r \cdot \Lambda_{2r} = \sum_{d|r} \mu\left(\frac{r}{d}\right) \Phi'_{2d}$$

hence completing the proof. *q.e.d.*

The sequence (Λ_{2r}) starts as follows:

$$\Lambda_2 = 1,\ \Lambda_4 = 3,\ \Lambda_6 = 8,\ \Lambda_8 = 18,\ \Lambda_{10} = 48,\ \Lambda_{12} = 116,\ \Lambda_{14} = 312,\ \Lambda_{16} = 810,\ \ldots$$

The Hesse loop of length 2 is shown in Figure 2.

7 Hesse derivatives of other normal forms

So far, we just considered Hesse derivatives of cubic curves in Hesse form, i.e., of curves Γ_c. The reason was that the Hesse derivative of a curve in Hesse form is again a curve in Hesse form, which is in general not the case for curves, for example, in Weierstrass normal form (WNF).

Below, we first provide curves in WNF such that their Hesse derivatives are also in WNF, and then we provide cubics with a D_3-symmetry, whose Hesse derivatives have same symmetry — like the cubics in Figure 2.

Curves in Weierstrass normal form

Let $\Gamma_c:\ x^3+y^3+z^3+c\,xyz=0$ be a cubic curve in Hesse form. Then, as described in [2, Sec. 3], by a projective transformation, the curve Γ_c, where

$$c=-\frac{2q^3+1}{q^2}$$

can be transformed to the curve

$$\Gamma_{a,b}: y^2=x^3+a\,x^2+b\,x$$

where

$$b=\frac{(q-1)^3}{q+q^2+q^3}\qquad\text{and}\qquad a=\frac{b^2-6b-3}{4}\,.$$

For example, for $c_0=3(\sqrt{3}-1)$ we obtain

$$q_0=-\frac{\sqrt{3}+1}{2},\qquad b_0=3+2\sqrt{3},\qquad a_0=0\,,$$

and for $c_1=-\frac{108+c_0^3}{3c_0^2}=-3(\sqrt{3}+1)$ we obtain

$$q_1=\frac{\sqrt{3}-1}{2},\qquad b_1=3-2\sqrt{3},\qquad a_1=0\,.$$

Notice that the curves Γ_{a_0,b_0} and Γ_{a_1,b_1} form a Hesse loop of length 2.

Curves in D_3-symmetric form

In [1, Sec. 2], a D_3-symmetric form of cubic curves was introduced and it was shown how to transform a curve in WNF with a projective transformation into a curve in D_3-symmetric form. Now, with a similar projective transformation we can transform any curve $\Gamma_c:\ x^3+y^3+z^3+c\,xyz=0$ in Hesse form directly into the curve

$$x^3-3\,xy^2+\frac{\sqrt{27}(c-6)}{2(c+3)}\,(x^2+y^2)-\frac{\sqrt{27}}{2}=0\,,$$

where the latter curve is D_3-symmetric (like the curves in Figure 2).

References

[1] Andrin Halbeisen, Lorenz Halbeisen, and Norbert Hungerbühler. Configurations on elliptic curves. *Innov. Incidence Geom.*, 19(3):111–135, 2022.

[2] Lorenz Halbeisen and Norbert Hungerbühler. An elementary approach to Hessian curves with torsion group $\mathbb{Z}/6\mathbb{Z}$. *Int. Electron. J. Pure Appl. Math.*, 13(1):1–30, 2019.

[3] Lorenz Halbeisen and Norbert Hungerbühler. Constructing cubic curves with involutions. *Beitr. Algebra Geom.*, 63(4):921–940, 2022.

[4] Lorenz Halbeisen, Norbert Hungerbühler, and Vera Stalder. Three conics determine a cubic. *submitted*, 2023.

[5] Audun Holme. *A royal road to algebraic geometry.* Springer, Heidelberg, 2012.