Three Conics determine a Cubic

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Abstract

Given a cubic K. Then for each point P there is a conic C_P associated to P. The conic C_P is called the *polar conic* of K with respect to the *pole* P. We investigate the situation when three conics C_1, C_2, and C_3 are polar conics of K with respect to some poles P_1, P_2, and P_3, respectively. In particular we show that any three conics C_1, C_2, C_3 in general position determine a unique cubic K and three points P_1, P_2, P_3, such that C_1, C_2, C_3 are polar conics of K with respect to the three poles P_1, P_2, P_3. This can be seen as a higher degree variant of von Staudt's theorem.

1 Introduction

This work proceeds the paper [2], in which it is shown that two given conics C_0 and C_1 can always be considered as polar conics of some cubic K with respect to some poles P_0 and P_1. However, even though P_1 is determined by P_0, neither the cubic nor the point P_0 is determined by the two conics C_0 and C_1. This changes if we start with three conics C_1, C_2, C_3 in general position. In this situation, the cubic K as well as the poles P_1, P_2, P_3 are uniquely determined. This can be seen as a higher degree variant of von Staudt's theorem, which says that given three lines ℓ_1, ℓ_2, ℓ_3 and three points P_1, P_2, P_3 in perspective position determine a unique conic C such that the points P_i are the poles of the lines ℓ_i with respect to C (see [5, p. 135, §241]).

The setting in which we work is the same as in [2], but for the sake of completeness we recall our notation and terminology.

We will work in the real projective plane $\mathbb{RP}^2 = \mathbb{R}^3 \setminus \{0\}/\sim$, where $X \sim Y \in \mathbb{R}^3 \setminus \{0\}$ are equivalent, if $X = \lambda Y$ for some $\lambda \in \mathbb{R}$. Points $X = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \setminus \{0\}$ will be denoted by capital letters, the components with the corresponding small letter, and the equivalence class by $[X]$. However, since we mostly work with representatives, we often omit the square brackets in the notation.

Let f be a non-constant homogeneous polynomial in the variables x_1, x_2, x_3 of degree n. Then f defines a projective algebraic curve

$$C_f := \{[X] \in \mathbb{RP}^2 \colon f(X) = 0\}$$

of degree n. For a point $P \in \mathbb{RP}^2$,

$$Pf(X) := \langle P, \nabla f(X) \rangle$$

is also a homogeneous polynomial in the variables x_1, x_2, x_3. If the homogeneous polynomial f is of degree n, then C_{Pf} is an algebraic curve of degree $n-1$. The curve C_{Pf} is called the *polar curve* of C_f with respect to the *pole* P; sometimes we call it the *polar curve* of P with respect to C_f. In particular, when C_f is a cubic curve (*i.e.*, f is a homogeneous polynomial of degree 3), then C_{Pf} is a conic, which we call the *polar conic* of C_f with respect to the *pole* P, and when C_f is a conic, then C_{Pf} is a line, which we call the *polar line* of C_f with respect to the *pole* P (see, for example, Wieleitner [6]). Note that C_{Pf} is defined and can be a regular curve even if C_f is singular or reducible. For some historical background, for the geometric interpretation of poles and polar lines, for the iterated construction of polar curves, as well as for the analytical method used today, see Monge [4, §3], Bobillier [1], and Joachimsthal [3, p. 373], or [2].

2 Algebraic Curves and Multilinear Forms

Let C_f be a conic given by the non-constant homogeneous polynomial

$$f(x_1, x_2, x_3) \; := \sum_{1 \leq i \leq j \leq 3} c_{ij}\, x_i x_j.$$

Then, the symmetric matrix

$$T := \begin{pmatrix} c_{11} & c_{12}/2 & c_{13}/2 \\ c_{12}/2 & c_{22} & c_{23}/2 \\ c_{13}/2 & c_{23}/2 & c_{33} \end{pmatrix}$$

has the property that a point X belongs to C_f (*i.e.*, $f(X) = 0$), if and only if $\langle X, T(X) \rangle = 0$. Thus, the conic C_f is represented by the matrix T. Since the expression $\langle X, T(Y) \rangle$ defines a bilinear form $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$, $(X, Y) \mapsto \langle X, T(Y) \rangle$, we can consider the matrix T also as a purely covariant tensor of rank 2 (*i.e.*, a tensor whose rank of covariance is 2 and whose rank of contravariance is 0). More precisely, if we consider the matrix T as a $(0,2)$-tensor, where for $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$ we define

$$T(X, Y) \quad := \sum_{1 \leq i,j \leq 3} a_{ij}\, x_i\, y_j,$$

then the expression $\langle X, T(X)\rangle = 0$ is equivalent to $T(X, X) = 0$. In order to obtain the coefficients of the $(0,2)$-tensor $T = (a_{ij})_{1\leq i,j\leq 3}$ from a conic C_f defined by a non-constant homogeneous polynomial f, we just set

$$a_{ij} := \frac{1}{2!} \cdot \frac{\partial^2 f}{\partial x_i \partial x_j} \quad \text{for all } 1 \leq i, j \leq 3.$$

The next result shows that this relation between a conic C_f and the corresponding $(0,2)$-tensor $T_f = (a_{ij})_{1\leq i,j\leq 3}$ can be generalised to algebraic curves of arbitrary degree.

Lemma 2.1. *Let Γ_f be an algebraic curve of degree d given by the non-constant homogeneous polynomial*

$$f(x_1, x_2, x_3) \;:= \sum_{1\leq i_1\leq\cdots\leq i_d\leq 3} c_{i_1\ldots i_d} \cdot x_{i_1} \cdot \ldots \cdot x_{i_d},$$

and let $T_f = (a_{i_1\ldots i_d})_{1\leq i_1,\ldots,i_d\leq 3}$, where

$$a_{i_1\ldots i_d} := \frac{1}{d!} \cdot \frac{\partial^d f}{\partial x_{i_1} \ldots \partial x_{i_d}} \quad \textit{for all } 1 \leq i_1, \ldots, i_d \leq 3.$$

Then T_f is a symmetric $(0,d)$-tensor and a point X is on the curve Γ_f if and only if

$$T_f(\underbrace{X, \ldots, X}_{d\text{-times}}) = 0\,.$$

Proof. Since for every rearrangement π of the sequence $\langle i_1, \ldots, i_d\rangle$ we have

$$\frac{\partial^d f}{\partial x_{i_1} \ldots \partial x_{i_d}} = \frac{\partial^d f}{\partial x_{\pi(i_1)} \ldots \partial x_{\pi(i_d)}} \quad \text{and therefore} \quad a_{i_1\ldots i_d} = a_{\pi(i_1)\ldots\pi(i_d)},$$

we get that the tensor T_f is symmetric. Furthermore, assume that the monomial $c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3}$ appears in f. Then $n_1 + n_2 + n_3 = d$ and

$$\frac{1}{d!} \cdot \frac{\partial^d (c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3})}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}} = \frac{n_1! \cdot n_2! \cdot n_3!}{d!} \cdot c_{n_1 n_2 n_3}.$$

Now, it is easy to see that the number of coefficients $a_{i_1\dots i_d}$ such that for $1 \le i \le 3$ the number i appears n_i-times in the sequence $\langle i_1, \dots, i_d\rangle$ is given by the trinomial coefficient

$$\binom{d}{n_1, n_2, n_3} = \frac{d!}{n_1! \cdot n_2! \cdot n_3!}.$$

This shows that for any point X we have $T_f(X, \dots, X) = 0$ if and only if $f(X) = 0$, or in other words, X is on the curve Γ. *q.e.d.*

Let us turn our attention now to polar curves. For this, we consider first polar curves of conics C_f with corresponding $(0, 2)$-tensor $T_f = (a_{ij})_{1\le i,j\le 3}$. Above we have seen that for a given point $P \in \mathbb{RP}^2$, a point X is on the polar curve $C_{Pf(X)}$ of C_f with respect to the pole P if and only if

$$Pf(X) := \langle P, \nabla f(X)\rangle = 0.$$

Now, for $P, X \in \mathbb{RP}^2$, a short calculation shows that $Pf(X) = 2 \cdot T_f(P, X)$, and hence, we get

$$Pf(X) = 0 \iff T_f(P, X) = 0.$$

Since T_f is symmetric, we have $T_f(P, X) = T_f(X, P)$, which shows that if X is a point on the polar curve of C_f with respect to the pole P, then P is a point on the polar curve of C_f with respect to the pole X. The next result shows that also this result can be generalised to algebraic curves of arbitrary degree.

Lemma 2.2. *Let Γ_f be an algebraic curve of degree d given by the non-constant homogeneous polynomial f, let T_f be the corresponding symmetric $(0, d)$-tensor, and let $P \in \mathbb{RP}^2$ be a point. Then*

$$Pf(X) = 0 \iff T_f(P, \underbrace{X, \dots, X}_{(d-1)\text{-}times}) = 0\,.$$

In particular, a point $X \in \mathbb{RP}^2$ is on the polar curve of Γ_f with respect to the pole P if and only if $T_f(P, X, \dots, X) = 0$.

Proof. Notice first that for $P = (p_1, p_2, p_3)$ and $X = (x_1, x_2, x_3)$ we have:

$$\begin{aligned} T_f(P, X, \dots, X) &= \sum_{j=1}^{3} p_j \cdot \left(\sum_{1\le i_2,\dots,i_d\le 3} a_{j\,i_2\dots i_d} \cdot x_{i_2} \cdot \ldots \cdot x_{i_d} \right) \\ &= \sum_{j=1}^{3} \sum_{1\le i_2,\dots,i_d\le 3} a_{j\,i_2\dots i_d} \cdot p_j \cdot x_{i_2} \cdot \ldots \cdot x_{i_d} \end{aligned}$$

Now, assume again that the monomial $c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3}$ appears in f. Then, for each $1 \le j \le 3$ we have

$$\frac{\partial(c_{n_1 n_2 n_3} \cdot x_1^{n_1} \cdot x_2^{n_2} \cdot x_3^{n_3})}{\partial x_j} = n_j \cdot c_{n_1 n_2 n_3} \cdot x_1^{n_1'} \cdot x_2^{n_2'} \cdot x_3^{n_3'},$$

where $n'_j = n_j - 1$ and $n'_i = n_i$ for $i \neq j$. Without loss of generality we assume that $j = 1$ and $n_1 \geq 1$. Now, it is easy to see that the number of coefficients $a_{1 i_2 \dots i_d}$ such that for $1 \leq i \leq 3$, the number i appears n_i-times in the sequence $\langle 1, \dots, i_d \rangle$ is given by the trinomial coefficient

$$\binom{d-1}{n_1 - 1, n_2, n_3} = \frac{(d-1)!}{(n_1-1)! \cdot n_2! \cdot n_3!} = \frac{n_1}{d} \cdot \frac{d!}{n_1! \cdot n_2! \cdot n_3!}.$$

This shows that for any points $P, X \in \mathbb{RP}^2$ we have

$$d \cdot T_f(P, X, \dots, X) = \big\langle P, \nabla f(X) \big\rangle,$$

in particular, we get

$$Pf(X) = 0 \iff T_f(P, X, \dots, X) = 0.$$

q.e.d.

It is obvious how the iterated construction of polar curves is carried out: If, for example, $P, Q, R \in \mathbb{RP}^2$ are given and Γ_f is an algebraic curve of degree $d \geq 3$, then the polar curve of the polar curve of the polar curve of Γ_f with respect to the points P, Q, R, respectively, is given by the zeros of the $(0, d-3)$-tensor $T_f(P, Q, R, X, \dots, X)$. Notice that since T_f is symmetric, the order of P, Q, R is irrelevant. As a consequence, we obtain the following

Fact 2.3. *Let K be a cubic curve, let $P_1, P_2, P_3 \in \mathbb{RP}^2$, and for $1 \leq j \leq 3$ let T_j be the $(0,2)$-tensor of the polar conic of K with respect to the point P_j. Then for $1 \leq j_1, j_2 \leq 3$ we have*

$$T_{j_1}(P_{j_2}, X) = 0 \iff T_{j_2}(P_{j_1}, X) = 0,$$

in particular, if we consider the tensors T_j as 3×3-matrices, we obtain that

$$[P_{j_1}] = [(T_{j_2}^{-1} \cdot T_{j_1})\, P_{j_2}].$$

In the next section we show that three conics in general position determine a unique cubic. More precisely, given three different conics C_1, C_2, C_3 which satisfy two conditions, we show how to construct the unique cubic K such that for three points $P_1, P_2, P_3 \in \mathbb{RP}^2$ determined by the three conics, the conic C_j (for $1 \leq j \leq 3$) is the polar conic of K with respect to the pole P_j. The construction we provide in the next section proves the following result:

Theorem 2.4. *Let C_1, C_2, C_3 be three conics and let T_1, T_2, T_3 be the corresponding $(0,2)$-tensors given by 3×3-matrices. Assume that the matrices T_1, T_2, T_3 satisfy the following two conditions:*

(a) $T_3\, T_1^{-1}\, T_2 \;\neq\; T_2\, T_1^{-1}\, T_3$

(b) For all $P \in \ker\big(T_3\,T_1^{-1}\,T_2 - T_2\,T_1^{-1}\,T_3\big)$*, we have*

$$\det\begin{pmatrix} | & | & | \\ T_1P & T_2P & T_3P \\ | & | & | \end{pmatrix} \neq 0.$$

Then there are exactly three points P_1, P_2, P_3*, determined by the conics* C_1, C_2, C_3*, and a unique cubic curve* K*, such that for* $1 \leq j \leq 3$*,* C_j *is the polar conic of* K *with respect to the pole* P_j*.*

3 Constructing a Cubic from three Conics

Let C_1, C_2, C_3 be three conics and let T_1, T_2, T_3 be the corresponding $(0,2)$-tensors given by 3×3-matrices matrices T_1, T_2, T_3 which satisfy the conditions (a) and (b) of Theorem 2.4.

Example: Let C_1, C_2, C_3 be given by the following three non-constant homogeneous polynomials f_1, f_2, f_3, respectively:

$$\begin{aligned} f_1(X) &= x_1^2 + x_2^2 + 4x_1x_3 \\ f_2(X) &= 2x_1^2 + 2x_1x_2 + 2x_2^2 + 6x_1x_3 + 6x_2x_3 \\ f_3(X) &= x_1^2 + 6x_1x_2 + x_2^2 + 2x_1x_3 - 6x_2x_3 \end{aligned}$$

Figure 1 shows these three conics. Notice that all three conics meet in the origine, which is not excluded by the conditions (a) and (b), as we will see below. Notice also that one of the conics is a circle, which is not a restriction since we can transform any conic by a projective transformation into a circle.

Then the corresponding matrices are:

$$T_1 = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 0 \end{pmatrix} \qquad T_2 = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \\ 3 & 3 & 0 \end{pmatrix} \qquad T_3 = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & -3 \\ 1 & -3 & 0 \end{pmatrix}$$

It is easy to verify that the matrices T_1, T_2, T_3 satisfy condition (a), and since $\ker\big(T_3\,T_1^{-1}\,T_2 - T_2\,T_1^{-1}\,T_3\big) = [P]$ for $P = (\frac{6}{5}, -\frac{24}{5}, 1)$, condition (b) is also easily checked.

Let us turn back to our general construction and construct the three points P_1, P_2, P_3: By Fact 2.3, the points P_1, P_2, P_3 satisfy the following three necessary conditions

$$T_2\,P_1 = T_1\,P_2, \qquad T_3\,P_2 = T_2\,P_3, \qquad T_1\,P_3 = T_3\,P_1,$$

which is equivalent to

$$(T_1^{-1}\,T_2)P_1 = P_2, \qquad (T_2^{-1}\,T_3)P_2 = P_3, \qquad (T_3^{-1}\,T_1)P_3 = P_1,$$

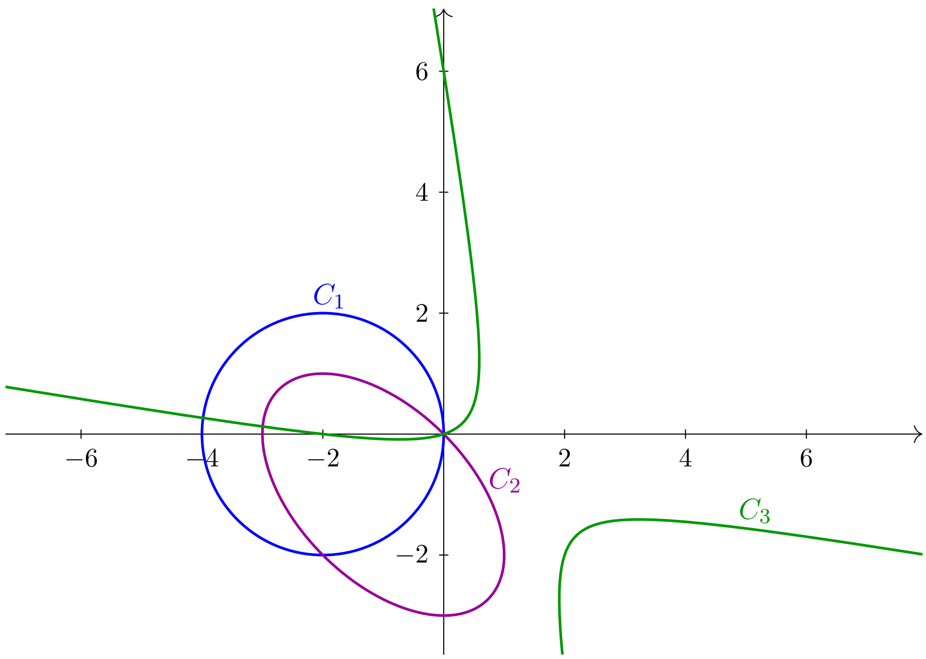

Figure 1: The three conics C_1, C_2, and C_3 of the example.

and implies that P_1 satisfies

$$(T_3^{-1}T_1)(T_2^{-1}T_3)(T_1^{-1}T_2)P_1 = P_1. \tag{1}$$

Since the matrices T_j are symmetric, for $M := T_3T_1^{-1}T_2$ we have $M^T = T_2T_1^{-1}T_3$. So, equation (1) is equivalent to $MP_1 = M^TP_1$, which is equivalent to $(M - M^T)P_1 = 0$. Now, condition (a) ensures that $M \neq M^T$ and since $(M - M^T)$ is a non-zero, real, anti-symmetric 3×3-matrix, it has exactly one eigenvalue equal to zero. In fact, if

$$A = \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$$

is an anti-symmetric matrix, then the eigenvalues of A are 0 and $\pm i\sqrt{a^2+b^2+c^2}$ and an eigenvector to the eigenvalue 0 is $(c, -b, a)^T$.

Hence, the pole P_1 is uniquely determined by equation (1), and we obtain $P_2 = (T_1^{-1}T_2)P_1$ and $P_3 = (T_1^{-1}T_3)P_1$. Before we proceed, let us compute the points P_1, P_2, P_3 in our example.

Example: With respect to T_1, T_2, T_3 we get $P_1 = (\frac{6}{5}, -\frac{24}{5}, 1)$, $P_2 = (-\frac{27}{5}, -\frac{27}{5}, 3)$, and $P_3 = (\frac{39}{5}, -\frac{21}{5}, -10)$, which correspond to the affine points $\bar{P}_1 = (\frac{6}{5}, -\frac{24}{5})$, $\bar{P}_2 = (-\frac{27}{15}, -\frac{27}{15})$, and $\bar{P}_3 = (-\frac{39}{50}, \frac{21}{50})$, respectively. Figure 2 shows the conics with their poles.

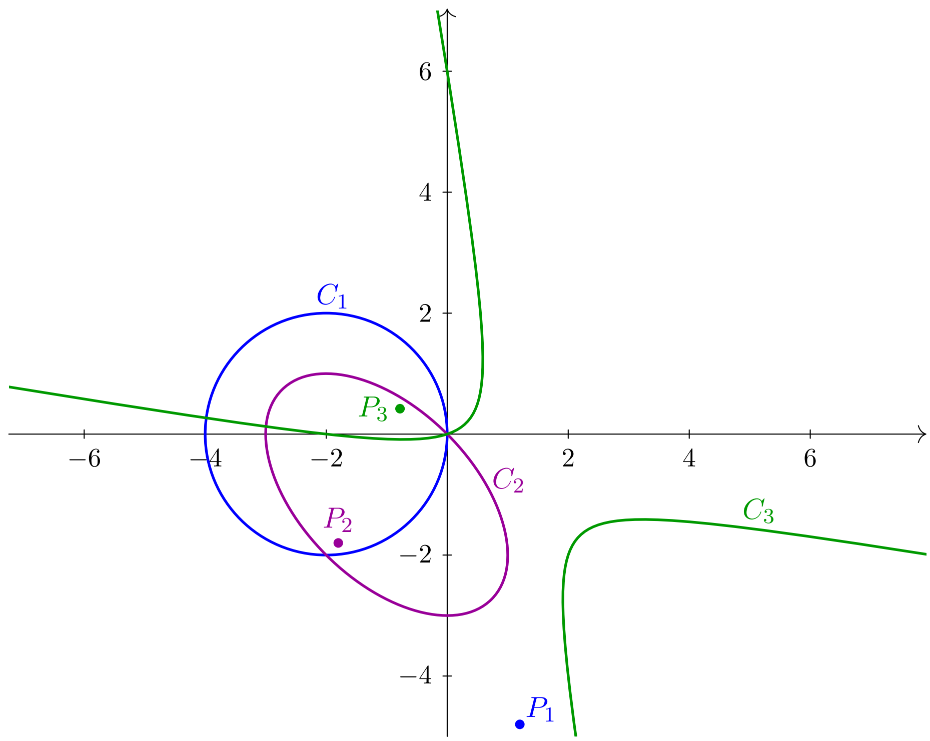

Figure 2: The three conics C_1, C_2, C_3 of the example with the three poles P_1, P_2, P_3.

The goal of our construction is to find a $(0,3)$-tensor T_K of a cubic K, such that we have

$$T_K(P_j, X, X) = T_j(X, X) \quad \text{for } 1 \le j \le 3.$$

Since by condition (b), the points P_1, P_2, P_3 are not incident with a projective line, we may choose $\{P_1, P_2, P_3\}$ as a new basis. In other words, for $\tilde{P}_1 = (1,0,0)$, $\tilde{P}_2 = (0,1,0)$, and $\tilde{P}_3 = (0,0,1)$, we map $P_i \mapsto \tilde{P}_i$ (for $1 \le i \le 3$), For $1 \le i \le 3$, let $T_i = (a^i_{jk})_{1 \le j,k \le 3}$ and let $\tilde{T}_i$ be the $(0,2)$-tensors (*i.e.*, the conics $\tilde{C}_i$) in this new basis. Since for any $1 \le i,j,k \le 3$ we have $T_i(P_j, P_k) = T_i(P_k, P_j) = T_j(P_k, P_i)$, we also have

$$\tilde{T}_i(\tilde{P}_j, \tilde{P}_k) = \tilde{T}_i(\tilde{P}_k, \tilde{P}_j) = \tilde{T}_j(\tilde{P}_k, \tilde{P}_i). \tag{2}$$

Now, let $T_{\tilde{K}} = (\tilde{a}_{ijk})_{1 \le i,j,k \le 3}$ be a $(0,3)$-tensor defined by stipulating

$$\tilde{a}_{ijk} := \tilde{T}_i(\tilde{P}_j, \tilde{P}_k) \quad \text{for } 1 \le i,j,k \le 3.$$

Then, by equation (2), the tensor $T_{\tilde{K}}$ is symmetric and has the property that for $1 \le i \le 3$,

$$T_{\tilde{K}}(\tilde{P}_i, X, X) = \tilde{T}_i(X, X).$$

For the corresponding cubic $\tilde{K}$ we therefore have that $\tilde{C}_i$ is the polar conic of $\tilde{K}$ with respect to the pole $\tilde{P}_i$. Thus, the re-transformed cubic K has the property that the conics C_1, C_2, C_3 are the polar conics of K with respect to the poles P_1, P_2, P_3, respectively.

Example: In our example, $\tilde{K}$ in the affine plane is given by

$$-2192-2919x+264x^2+122x^3-1557y+3384xy+198x^2y+3726y^2-81xy^2-81y^3=0,$$

and finally, the sought cubic K is

$$-13x^3-66x^2y-27x^2-216xy-39xy^2-27y^2-22y^3.$$

Figure 3 shows the cubic K together with the three polar conics C_i with respect to their three poles P_i. Recall that the lines connecting P_i and the points of intersection of K with the polar curve C_i are tangent to K.

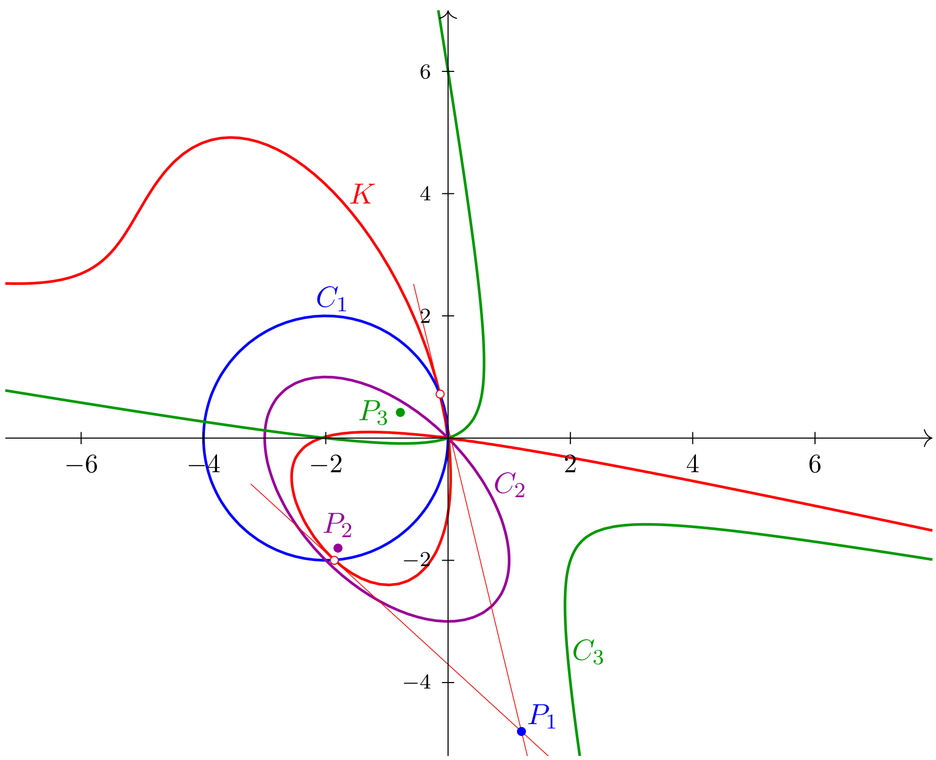

Figure 3: The cubic K together with the three poles P_1, P_2, P_3 and the three polar conics C_1, C_2, C_3 of the example. The tangents from P_1 to K are also displayed.

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