The Polish School of Mathematics between the world wars and its impact on Set Theory of the Reals

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The outline of the talk

- A historical introduction.
- Sierpiński's insights into sets of cardinality $\aleph_1$.
- Some consequences of the Continuum Hypothesis
- and their impact on future research.

A historical introduction

- In 1918, as World War I came to an end, Poland regained its independence after the period of 123 years of partitions conducted by the Habsburg monarchy, the Kingdom of Prussia, and the Russian Empire.
- Question: "How Poland (...) could achieve, within a relatively short period 1919–1939, a good international position in such fields of mathematics as functional analysis, topology, set theory, functions of a real variable, logic and foundations of mathematics (...)"? (W. Żelazko, "A short history of Polish mathematics").

The Polish School of Mathematics

- "The Polish school of mathematics" in a broader sense refers to all mathematical activities on the newly reintegrated territory of Poland between the two world wars, i.e., in the period of 1918-1939, carried out in Warsaw, Lvov (now Lviv in Ukraine), Cracow, Wilno (now Vilnius in Lithuania) and Poznań.
- In a more restricted sense it usually refers to the activities of two main research communities: the Lvov school lead by Stefan Banach and the Warsaw school lead by Wacław Sierpiński.

- The three “founding fathers”:
  - Wacław Sierpiński (1882–1969),
  - Zygmunt Janiszewski (1888–1920),
  - Stefan Mazurkiewicz (1888–1945).

Wacław Sierpiński (1882–1969)

- studied at the Imperial (Russian) University of Warsaw,
- received his doctorate at the Jagiellonian University in Cracow,
- worked (1908-1914) at the University of Lvov,
- spent the war in Moscow working with Nikolai Luzin,
- returned to Warsaw in 1918 and became a professor at the newly reactivated University of Warsaw,
- worked in the area of set theory, point set topology and number theory,
- published over 700 papers. He has 6345 mathematical descendants.

Zygmunt Janiszewski (1888–1920)

- studied in Zurich (the University of Zurich), Munich and Göttingen,
- received his doctorate in Paris (1911) under the supervision of Henri Lebesgue,
- for some time taught at the University of Lvov, working with Sierpiński,
- in 1918 was offered a professorship at the University of Warsaw,
- worked in topology, especially in the theory of continua,
- died untimely as a victim of the Spanish influenza pandemic.

Stefan Mazurkiewicz (1888–1945)

- studied in Cracow, Munich, Göttingen and Lvov,
- received his doctorate in Lvov (1913) under the supervision of Sierpiński,
- became a professor at the reborn University of Warsaw,
- worked in topology, especially the theory of continua, dimension theory and descriptive set theory,
- published more than 140 papers.

- Janiszewski wrote an article entitled "On the needs of mathematics in Poland" (1917) with a program aimed at "gaining an independent position of Polish mathematics" by:

  1. concentrating the efforts of most of the active mathematicians in the country in one discipline,
  2. founding a journal publishing papers devoted only to the chosen area of mathematics and written in internationally recognized languages.

- Janiszewski, Mazurkiewicz and Sierpiński founded Fundamenta Mathematicae (1920) – the first specialized journal devoted only to one area of mathematics, namely set theory and its applications.

Let us give a voice to Sierpiński himself ("The Warsaw School of Mathematics and the present state of mathematics in Poland", The Polish Review 4 no. 1-2, (1959), 51–63):

"In 1913 Stefan Mazurkiewicz came to Lvov to prepare for his doctorate under my direction. (...) In this same year I offered an assistantship at the Mathematical Seminar of Lvov University to Zygmunt Janiszewski, a doctor of the University of Paris, whose work was in the field of topology. In 1919, the three of us met as the first professors of mathematics in the re-born Polish University at Warsaw. There we decided to found a periodical dedicated to the theory of sets, topology, the theory of functions of a real variable and mathematical logic, which would publish studies in French, English, German and Italian. It was thus that Fundamenta Mathematicae (...) was born. "

And then Sierpiński continues (ibidem):

"When in 1919 Stefan Mazurkiewicz, Zygmunt Janiszewski and I were appointed professors of mathematics at the University of Warsaw, each of us, in adition to teaching other branches of mathematics, offered courses or seminars on the theory of sets, topology and the theory of functions of a real variable. Among the students of those days many were very capable, and some later became noted mathematicians. Among these were Kuratowski, Saks, Tarski, Knaster, Zygmund, Lindenbaum, Szpilrajn-Marczewski, Borsuk, Zarankiewicz, Eilenberg, Popruženko, Aronszajn, Mostowski, Charzynski and others. Some became professors or docents of our University and together we formed the Warsaw School of Mathematics ."

Concerning set theory of the reals the work of the following mathematicians from the above list was of particular importance:

- Sierpiński: AC and cardinal arithmetic, CH and its consequences, infinite combinatorics (almost disjoint sets, partition calculus), types of linear orders, measure and category, special sets of reals, descriptive set theory,
- Mazurkiewicz: the "Two Points Theorem", descriptive set theory,
- Kazimierz Kuratowski: the "Kuratowski-Zorn Lemma", special sets of reals, the general measure problem, descriptive set theory,
- Alfred Tarski: AC and cardinal arithmetic, cardinal exponentiation, infinite combinatorics, finite equidecomposability into congruent parts (the Banach-Tarski paradox).

A crash course in basic set theory

- Sets A and B are equinumerous or have the same cardinality ($|A| = |B|$) if there is a one to one function from A onto B.
- The cardinality of A is less or equal to the cardinality of B ($|A| \leq |B|$) if there is a one to one function from A into B.
- For any sets A and B either $|A| \leq |B|$ or $|B| \leq |A|$.
- The cardinalities of infinite sets are expressed with the help of cardinal numbers : $\aleph_0, \aleph_1, \ldots$.
- We write $|A| = \aleph_0$ if $|A| = |\mathbb{N}|$, where $\mathbb{N} = \{0, 1, 2, \ldots\ldots\}$. Then $|A| \leq |B|$ for every infinite B, i.e., $\aleph_0$ is the smallest infinite cardinal number.
- We write $|A| = \aleph_1$ if $|A| \leq |B|$ for every uncountable B, i.e., $\aleph_1$ is the smallest uncountable cardinal number.
- If T is infinite and $|A_x| \leq \aleph_0$ for each $x \in T$, then $|\bigcup_{x \in T} A_x| \leq |T|$.

Question (Cantor, 1878): is every infinite set of reals either equinumerous with $\mathbb{N}$ or equinumerous with $\mathbb{R}$?

The Continuum Hypothesis (CH) : YES, i.e., $|\mathbb{R}| = \aleph_1$.

- (Gödel, 1938): CH is consistent with the usual axioms (ZFC) of Set Theory. In ZFC one cannot prove that CH is false – we can add it to ZFC as a new axiom without increasing the risk of inconsistency.
- (Cohen, 1963): $\neg$CH is consistent with ZFC. In ZFC one cannot prove that CH is true – we can add $\neg$CH to ZFC as a new axiom.

How to imagine a set of cardinality $\aleph_1$?

It is usually done with the help of countable ordinal numbers but Sierpiński's ideas can be used to circumvent the use of them.

A family $\mathcal{L}$ of sets is a *chain* , if every sets A and B from $\mathcal{L}$ are comparable in the sense of inclusion: either $A \subseteq B$ or $B \subseteq A$.

Let $\mathcal{C}_1$ be *maximal* among all chains of *countable* subsets of $\mathbb{R}$, i.e.:

- all sets that form $\mathcal{C}_1$ are countable subsets of $\mathbb{R}$,
- $\mathcal{C}_1$ is a chain,
- $\mathcal{C}_1$ is maximal, i.e., $\mathcal{C}_1$ cannot be properly extended to a larger chain of countable subsets of $\mathbb{R}$.

The existence of $\mathcal{C}_1$ follows from the Hausdorff maximality principle, an earlier version of the Kuratowski-Zorn's lemma.

Let $\Omega_1 \subseteq \mathbb{R}$ be the union of all sets that form $\mathcal{C}_1$.

Then Ω_1 is uncountable; otherwise pick $x \in \mathbb{R} \setminus \Omega_1$ and properly extend $\mathcal{C}_1$ by adding $\Omega_1 \cup \{x\}$.

Theorem (Sierpiński)

Let $\mathcal{C}$ be a chain of countable subsets of $\mathbb{R}$ and let $X \subseteq \mathbb{R}$ be the union of all sets that form $\mathcal{C}$. If X is uncountable, then $|X| = \aleph_1$. In particular, $|\Omega_1| = \aleph_1$.

Proof.

Let B be an arbitrary uncountable set. If $|X| \leq |B|$, then we are done.
If $|B| \leq |X|$, then fix $T \subseteq X$ with $|B| = |T|$; in particular, T is uncountable.
For each $x \in X$ choose $A_x \in \mathcal{C}$ with $x \in A_x$.

- The key observation: $X = \bigcup_{x \in T} A_x$ $(*)$.

For suppose that $y \in X \setminus \bigcup_{x \in T} A_x$.

Then $A_y \not\subseteq A_x$ for any $x \in T$ (since $y \in A_y \setminus A_x$), hence $A_x \subseteq A_y$. It follows that $T \subseteq A_y$ – a contradiction (as T is uncountable).

By $(*)$, $|X| \leq |T|$ but $|T| = |B|$ and we are done again. $\square$

We actually proved more:

Theorem (Sierpiński)

For any set X, the following are equivalent:

- $|X| = \aleph_1$,
- *X is uncountable and is the union of a chain of countable sets.*

Proof.

If $|X| = \aleph_1$, then $|X| = |\Omega_1|$, so X is the union of a chain of countable sets.

If $\mathcal{C}$ is any chain of countable sets and its union X is uncountable, then we may repeat the previous argument to get $|X| = \aleph_1$. □

Sierpiński in a monograph *"Hypothèse du continu"* (published in 1934) presented numerous consequences of CH among which he distinguished 11 statements, labelled P_1 to P_{11}, which are actually equivalent to CH.

Statement P_6 is what we have just considered:

Theorem (Sierpiński)

The following are equivalent:

- *the Continuum Hypothesis,*
- P_6*:* $\mathbb{R}$ *is the union of a chain of countable sets.*

In the rest of the talk we shall use this characterization in proofs of two consequences of CH (which are usually presented with the help of countable ordinals).

Theorem (Sierpiński)

CH is equivalent to statement P_1: the plane $\mathbb{R}^2$ is the disjoint union of sets A i B such that for any $x, y \in \mathbb{R}$, sets $\{y \in \mathbb{R} : (x, y) \in A\}$ and $\{x \in \mathbb{R} : (x, y) \in B\}$ are countable.

Proof of the necessity.

First assume CH and, by P_6, fix a chain $\{A_x : x \in \mathbb{R}\}$ of countable subsets of $\mathbb{R}$ with $x \in A_x$ for each $x \in \mathbb{R}$.
Let

$$A = \{(x, y) \in \mathbb{R}^2 : y \in A_x\}, \quad B = \mathbb{R}^2 \setminus A.$$

Clearly, $\mathbb{R}^2 = A \cup B$, $A \cap B = \emptyset$ and for each $x \in \mathbb{R}$ we have $\{y \in \mathbb{R} : (x, y) \in A\} = A_x$ is countable.
On the other hand for each $y \in \mathbb{R}$ we have

$$\{x \in \mathbb{R} : (x, y) \in B\} = \{x \in \mathbb{R} : y \notin A_x\} \subseteq A_y,$$

for if $y \notin A_x$, then $A_y \not\subseteq A_x$, hence $A_x \subseteq A_y$, so $x \in A_y$.
Consequently, $\{x \in \mathbb{R} : (x, y) \in B\}$ is also countable.

Proof of the sufficiency.

Now assume that $\mathbb{R}^2 = A \cup B$, where $A \cap B = \emptyset$ and for any x and y the sections $A_x = \{y \in \mathbb{R} : (x, y) \in A\}$ and $B^y = \{x \in \mathbb{R} : (x, y) \in B\}$ are countable.

Fix $T \subseteq \mathbb{R}$ with $|T| = \aleph_1$ and let $S = \bigcup_{y \in T} B^y$.

As $|B^y| \leq \aleph_0$ for all $y \in T$, we have $|S| \leq |T|$.

But $S = \mathbb{R}$. Otherwise, fix $x \in \mathbb{R} \setminus S$.

Now, if $y \in T$, then $x \notin B^y$, i.e., $(x, y) \notin B$, so $y \in A_x$, which shows that $T \subseteq A_x$ – a contradiction as A_x is countable.

Thus $|\mathbb{R}| \leq |T| = \aleph_1$, i.e., CH holds true.

□

Question (Banach): Does there exist a probability countably additive measure on $[0,1]$ which is zero on singletons and is defined for every $X \subseteq [0,1]$?

Theorem (Banach, Kuratowski, 1929)

CH implies that no such measure exists.

Let P be the set of all functions $f : \mathbb{N} \to \mathbb{N}$.

For $f, g \in P$ let us say that g *eventually dominates* f (notation: $f \preceq^* g$) if $\exists m \, \forall n \geq m \; f(n) \leq g(n)$.

Note that, by a diagonalization argument, if $A \subseteq P$ is countable, then there is $g \in P$ with $f \preceq^* g$ for any $f \in A$ (notation: $A \preceq^* g$).

Proof (Banach, Kuratowski).

By CH, $|P| = \aleph_1$. By Sierpiński's theorem, there is a chain $\{A_f : f \in P\}$ of countable subsets of P such that $f \in A_f$ for every $f \in P$. For each $f \in P$ choose $g_f \in P$ with $A_f \preceq^* g_f$.
Observe that for any uncountable $T \subseteq P$ there is no $g \in P$ with $\{g_f : f \in T\} \preceq^* g$ (otherwise, $\bigcup_{f \in T} A_f \preceq^* g$ which is impossible as $\bigcup_{f \in T} A_f = P$).
By re-indexing, we have $\{g_x : x \in [0,1]\} \subseteq P$ such that

(∗) for no uncountable T there is g with $g_x \preceq^* g$ for each $x \in T$.

For $n, k \in \mathbb{N}$ let $T_{n,k} = \{x \in [0,1] : g_x(n) = k\}$.
Suppose that $m : \mathcal{P}([0,1]) \to [0,1]$ is a probability σ-additive measure with $m(T) = 0$ for any countable $T \subseteq [0,1]$.
For each n choose k_n such that $m(T_{n,0} \cup \ldots \cup T_{n,k_n}) \geq 1 - \frac{1}{2^{n+2}}$, which means that $g_x(n) \leq k_n$ for "many" (in the sense of m) x's.
Consequently, letting $T = \bigcap_n \bigcup_{k \leq k_n} T_{n,k}$, we have $m(T) > 0$ (so T must be uncountable) but $g_x(n) \leq k_n$ for each $x \in T$ and every n, contradicting (∗) (as $g_x \preceq^* g$, where $g(n) = k_n$ for each n). □

Proof (Kenneth Kunen).

By CH and Sierpiński's theorem, establishing its equivalence with statement P_1, there is a set $A \subseteq [0,1]^2$ with all vertical sections $A_x = \{y \in [0,1] : (x,y) \in A\}$ countable and all horizontal sections $A^y = \{x \in [0,1] : (x,y) \in A\}$ co-countable.

It is easy to show that A, being the union of a countable collection of graphs of functions from $[0,1]$ to $[0,1]$, belongs to the σ-algebra of subsets of $[0,1]^2$, generated by abstract rectangles of the form $C \times D$ for $C,\ D \subseteq [0,1]$.

Suppose that $m : \mathcal{P}([0,1]) \to [0,1]$ is a probability σ-additive measure with $m(T) = 0$ for any countable $T \subseteq [0,1]$.

Then A is measurable with respect to the product measure $m \otimes m$.

But $m(A_x) = 0$ and $m(A^y) = 1$ for every $x, y \in [0,1]$ which contradicts the Fubini theorem for $m \otimes m$.

□

The theorem of Banach and Kuratowski was a remarkable result which caused great interest and became the starting point for further research in various directions.

- The work of **Stanisław Ulam** (1933), one of the most famous students of Kuratowski, on the *abstract measure problem:* is there an infinite set X and a probability countably additive measure m on X which is zero on singletons and is defined for every $A \subseteq X$?

  Ulam proved that if X and m are as above then:

  - if m has no atoms, then there is an extension of the Lebesgue measure on $[0, 1]$ to a σ-additive measure defined on $\mathcal{P}([0, 1])$ and there is a weakly inaccessible cardinal $\kappa \leq |\mathbb{R}|$ (in particular: no such measure exists under CH),
  - if m has an atom, then there is a *measurable* (in particular: strongly inaccessible) *cardinal* $\kappa \leq |X|$, i.e., such an uncountable cardinal κ that there is a κ-complete free ultrafilter over κ.

  So it turned out that the positive answer to the abstract measure problem cannot be given in ZFC but Ulam's work was a very important contribution to the origins of the study of *large cardinals* - one of the main branches of set theory.

More impact on future research

- The work on the question of whether given any countable collection of subsets of $[0, 1]$ we can include all of them in the domain of a σ-additive extension of the Lebesgue measure (Timothy Carlson (1984): the answer "yes" is relatively consistent with ZFC).

- The work on a combinatorial property of the matrix $\langle T_{n,k} : n, k \in \mathbb{N}\rangle$, formulated by Banach and Kuratowski, which implies that the sets $\{T_{n,k} : n, k \in \mathbb{N}\}$ cannot all be measured by a single probability σ-additive measure vanishing on singletons (Tomek Bartoszyński and Lorenz Halbeizen (2003): the existence of a BK-Matrix is independent of ZFC+$\neg$CH; in particular, it is *not equivalent* to CH).

- Connections with the research on special subsets of the reals (in particular, universal measure zero sets [Marczewski] and K-Lusin sets of Bartoszyński and Halbeisen).

A final remark

The Sierpiński's representation of a set of cardinality $\aleph_1$ with the help of a chain of countable subsets of $\mathbb{R}$ with uncountable union somewhat resembles
the Dedekind's construction of the reals: the set of Dedekind cuts can also be identified with a chain of countable sets, namely the chain of all non-empty proper initial segments of $\mathbb{Q}$.

Thank you for your attention!