Dr Halbeisen

MODULES 110PMA003 & 110PMA107

Department of Pure Mathematics

Solutions, Week 2

The pdf-file you may download from
`http://www.math.berkeley.edu/~halbeis/4students/zero.html`

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5. By distributivity, for any number q we have $\sum_{n=0}^{\infty} q \cdot a_n = q \cdot \sum_{n=0}^{\infty} a_n$.

   (a) $$\sum_{n=0}^{\infty} \frac{7}{9 \cdot 2^n} = \sum_{n=0}^{\infty} \frac{7}{9} \cdot \frac{1}{2^n} = \frac{7}{9} \cdot \underbrace{\sum_{n=0}^{\infty} \frac{1}{2^n}}_{=2} = \frac{7}{9} \cdot 2 = \frac{14}{9}$$ [5]

   (b) $$\sum_{n=1111}^{\infty} \frac{1}{2^n} = \sum_{k=0}^{\infty} \frac{1}{2^{1111+k}} = \sum_{k=0}^{\infty} \frac{1}{2^{1111}} \frac{1}{2^k} = \frac{1}{2^{1111}} \underbrace{\sum_{k=0}^{\infty} \frac{1}{2^k}}_{=2} = \frac{2}{2^{1111}} = \frac{1}{2^{1110}} = 2^{-1110}$$ [5]

   (c) $$\sum_{n=0}^{1110} \frac{1}{2^n} = \underbrace{\sum_{n=0}^{1110} \frac{1}{2^n} + \sum_{n=1111}^{\infty} \frac{1}{2^n}}_{= \sum_{n=0}^{\infty} \frac{1}{2^n} = 2} - \underbrace{\sum_{n=1111}^{\infty} \frac{1}{2^n}}_{= 2^{-1110}} = 2 - 2^{-1110}$$ [5]

   (d) $$\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2}^n} = \left(1 - \frac{1}{\sqrt{2}} + \frac{1}{2} - \frac{1}{\sqrt{2} \cdot 2} + \frac{1}{4} - \frac{1}{\sqrt{2} \cdot 4}\right) =$$
   $$\underbrace{\left(1 + \frac{1}{2} + \frac{1}{4} + \dots\right)}_{= \sum_{n=0}^{\infty} \frac{1}{2^n}} - \underbrace{\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2} \cdot 2} + \frac{1}{\sqrt{2} \cdot 4} + \dots\right)}_{= \frac{1}{\sqrt{2}} \cdot \sum_{n=0}^{\infty} \frac{1}{2^n}} = 2 - \frac{2}{\sqrt{2}}$$ [5]

6. (a) For two complex numbers w and z, remember the trick to write $\frac{w}{z}$ as $\frac{w}{z} \cdot \frac{\bar{z}}{\bar{z}}$.

   $z = \frac{(7-i)}{(4+i3)} = \frac{(7-i)}{(4+i3)} \cdot \frac{(4-i3)}{(4-i3)} = \frac{(7-i)(4-i3)}{(4+i3)(4-i3)} = \frac{(28-3-i21-i4)}{(16+9+i12-i12)} = \frac{(25-i25)}{25} = (1-i)$ [10]

   (b) Let $z = (-12 + i5)$:

   $|z| = \sqrt{12^2 + 5^2} = \sqrt{169} = 13$; $\overline{z} = (-12 - i5)$; $-z = \big(-(-12) - i5\big) = (12 - i5)$;
   $-\overline{z} = -(-12 - i5) = (12 + i5)$; $\overline{-z} = \overline{(12 - i5)} = (12 + i5)$.
   To show these numbers in an Argand diagram should not cause any problems.

   [10]

7. The set of complex numbers for which $|z| \leq 1$ *and* $\mathrm{Im}(z) \geq 0$ is the "upper half of the unit disc with centre 0". [10]

8. $E(i\varphi) := \sum_{n=0}^{6} \frac{(i\varphi)^n}{n!} = \big(1 + i\frac{\varphi}{1} - \frac{\varphi^2}{2} - i\frac{\varphi^3}{6} + \frac{\varphi^4}{24} + i\frac{\varphi^5}{120} - \frac{\varphi^6}{720}\big)$.

(a) For $\varphi = \frac{\pi}{2}$ we get $E(i\frac{\pi}{2}) = (-0.00089 + i \cdot 1.00452)$ and therefore $\big|E\big(i\frac{\pi}{2}\big)\big| = 1.00907$ [10]

(b) For $\varphi = \frac{\pi}{4}$ we have $E(i\frac{\pi}{4}) = (0.70710 + i \cdot 0.70714)$. Thus, $\sqrt{2} \cdot E(i\frac{\pi}{4}) = 1.41421 \cdot (0.70710 + i \cdot 0.70714) = (0.99999 + i \cdot 1.00005)$ and $\big|\sqrt{2} \cdot E(i\frac{\pi}{4})\big| = 1.41424$. [10]

(c) It is not hard to see that for any φ we have $E(-i\varphi) = \overline{E(i\varphi)}$. Thus, $E\big(-i\frac{\pi}{4}\big) = \overline{(0.70710 + i \cdot 0.70714)} = (0.70710 - i \cdot 0.70714)$ and $\big|E\big(-i\frac{\pi}{4}\big)\big| = 1.00002$. [10]

9. First note that if z^n is a real number, then $\overline{z}^n = \overline{z^n} = z^n$.

(a) $(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) \cdot (-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = (\frac{1}{4} - \frac{3}{4} - i\frac{\sqrt{3}}{4} - i\frac{\sqrt{3}}{4}) = (\frac{1}{2} - i\frac{\sqrt{3}}{2})$;
$(\frac{1}{2} - i\frac{\sqrt{3}}{2})(-\frac{1}{2} + i\frac{\sqrt{3}}{2}) = (\frac{1}{4} + \frac{3}{4} - i\frac{\sqrt{3}}{4} + i\frac{\sqrt{3}}{4}) = 1$.
Since $(-\frac{1}{2} - i\frac{\sqrt{3}}{2})$ is the conjugate of $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})$, we get $(-\frac{1}{2} - i\frac{\sqrt{3}}{2})^3 = (-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = 1$, and of course, since 1 is neutral with respect to multiplication, we have $1^3 = 1$. [10]

(b) First notice that $(\sqrt{3} + i\sqrt{3})^4 = \big((\sqrt{3} + i\sqrt{3})^2\big)^2$.
$(\sqrt{3} + i\sqrt{3})(\sqrt{3} + i\sqrt{3}) = (3 - 3 + i3 + i3) = (0 + i6)$, and $(0 + i6)(0 + i6) = -36$.
Since we have $(-z)^4 = z^4$, we get $(-\sqrt{3} - i\sqrt{3})^4 = (\sqrt{3} + i\sqrt{3})^4 = -36$. Further we have that $(\sqrt{3} - i\sqrt{3})$ and $(-\sqrt{3} + i\sqrt{3})$ are the conjugates of $(\sqrt{3} + i\sqrt{3})$ and $(-\sqrt{3} - i\sqrt{3})$ respectively, so we get $(\sqrt{3} - i\sqrt{3})^4 = (-\sqrt{3} + i\sqrt{3})^4 = -36$. [10]