1.2 Integers: inverse elements

To each natural number n we introduce an **inverse element** $(-n)$ and define

$$n + (-n) := 0\,.$$

Since 0 is the neutral element with respect to addition, we obviously have $(-0) = 0$. Now, since $0 = 0 + 0 = \underbrace{n + (-n)}_{=0} + \underbrace{m + (-m)}_{=0} = (n + m) + (-n) + (-m)$, we get

$$(-n) + (-m) = \big(-(n+m)\big)\,,$$

and therefore, the addition is also defined for these new inverse elements.

We also write $n - m$ instead of $n + (-m)$, which is in fact the definition of the **subtraction**.

The set of **integers** is $\{\dots, -3, -2, -1, 0, 1, 2, 3\}$ and is denoted by $\mathbb{Z}$.

Multiplication

First we define the **multiplication** "$\cdot$" for natural numbers in a similar way as we have done it for the addition:

(i) For any natural number n we define $1 \cdot n := n$ (the number 1 is called the **neutral element** with respect to multiplication)

(ii) For natural numbers n and m we define $n \cdot (m+1) := (n \cdot m) + n$.

Some calculations: In the following we use the basic laws of calculation (these laws will be explained below). Let x and y be integers.

1. $0 \cdot x$?
   $0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x$, and therefore, by adding $(-0 \cdot x)$ on both sides,

   $$\underbrace{0 \cdot x + (-0 \cdot x)}_{=0} = 0 \cdot x + \underbrace{0 \cdot x + (-0 \cdot x)}_{=0}\,,$$

   and therefore, $0 = 0 \cdot x + 0$ which implies $0 = 0 \cdot x$.

2. $(-1)\cdot x\,?$
   $$\big((-1)\cdot x\big)+x \underset{\substack{\uparrow\\ \text{1 is neutral}}}{=} \big((-1)\cdot x\big)+1\cdot x \underset{\substack{\uparrow\\ \text{distributive law}}}{=} \big((-1)+1\big)\cdot x \underset{\substack{\uparrow\\ \text{(-1)+1=0}}}{=} 0\cdot x \underset{\substack{\uparrow\\ \text{by 1.}}}{=} 0.$$
   So, $\big((-1)\cdot x\big)+x = 0 = (-x)+x$, which implies $\big((-1)\cdot x\big) = (-x)$.

3. $\big(-(-x)\big)\,?$ *(What is the inverse of an inverse element?)*
   By definition, $(-x)+\big(-(-x)\big) = 0 = (-x)+x$, which implies $\big(-(-x)\big) = x$.

4. $(-x)\cdot(-y)\,?$ *(What is the product of two inverse elements?)*
   $$(-x)\cdot(-y) \underset{\substack{\uparrow\\ \text{by 2.}}}{=} \big((-1)\cdot x\big)\cdot\big((-1)\cdot y\big) \underset{\substack{\uparrow\\ \text{commuta-}\\ \text{tive law}}}{=} \underbrace{\big((-1)\cdot(-1)\big)}_{\underset{\substack{\uparrow\\ \text{by 2.}}}{=} (-(-1)) \underset{\substack{\uparrow\\ \text{by 3.}}}{=} 1}\cdot(x\cdot y) = 1\cdot(x\cdot y) = x\cdot y.$$

5. $(-x)\cdot y\,?$
   $$(-x)\cdot y \underset{\substack{\uparrow\\ \text{by 2.}}}{=} \big((-1)\cdot x\big)\cdot y \underset{\substack{\uparrow\\ \text{associa-}\\ \text{tive law}}}{=} (-1)\cdot(x\cdot y) \underset{\substack{\uparrow\\ \text{by 2.}}}{=} \big(-(x\cdot y)\big).$$

6. $(-x)+(-x)\,?$
   $$(-x)+(-x) \underset{\substack{\uparrow\\ \text{by 2.}}}{=} \big((-1)\cdot x\big)+\big((-1)\cdot x\big) \underset{\substack{\uparrow\\ \text{distribu-}\\ \text{tive law}}}{=} (-1)\cdot(x+x) \underset{\substack{\uparrow\\ \text{1 is neutral}}}{=}$$
   $$(-1)\cdot(1\cdot x+1\cdot x) \underset{\substack{\uparrow\\ \text{distribu-}\\ \text{tive law}}}{=} (-1)\cdot\big((1+1)\cdot x\big) \underset{\substack{\uparrow\\ 1+1=2}}{=} (-1)\cdot(2\cdot x) \underset{\substack{\uparrow\\ \text{by 2.}}}{=} -2\cdot x.$$

In $\mathbb{Z}$, not every element has an inverse element with respect to multiplication (for $x\in\mathbb{Z}$, an inverse element w.r.t. multiplication would be a number y such that $x\cdot y=1$). In fact, just 1 and (-1) have inverse elements with respect to multiplication, all other elements, like 2 or (-5) don't have inverse elements with respect to multiplication.

However, a set like the integers together with the two operations addition and multiplication, with the inverse operation "$-$" (with respect to addition) and with the two neutral elements 0 and 1 (0 w.r.t. addition and 1 w.r.t. multiplication), such a set is called a **ring**, or more precisely, a **commutative ring** (since the multiplication is commutative). So, $(\mathbb{Z},0,+,-,1,\cdot)$ is a ring. Later we will also see a non-commutative ring, namely the ring of matrices.