Exercise 3.1.6. Let $c \in [0,\infty)$ and $\varepsilon \in (0,\infty)$. Let $N \in \mathbb{N}$ be such that $N\varepsilon \geq 1$.
Recall from Lemma 3.1.3 that Γ is monotone on $[\mathcal{C},\infty)$ for some $\mathcal{C} \in (0,\infty)$.

We have the following estimate:

$$\sum_{n=1}^{\infty} \frac{c^n}{\Gamma(n\varepsilon)} = \sum_{k=0}^{\infty}\sum_{n=1}^{N} \frac{c^{kN+n}}{\Gamma((kN+n)\varepsilon)}$$

$$= \sum_{k=0}^{\lceil \mathcal{C} \rceil}\sum_{n=1}^{N} \frac{c^{kN+n}}{\Gamma((kN+n)\varepsilon)} + \sum_{k=\lceil \mathcal{C} \rceil+1}^{\infty}\sum_{n=1}^{N} \frac{c^{kN+n}}{\Gamma((kN+n)\varepsilon)}$$

$$\leq \sum_{k=0}^{\lceil \mathcal{C} \rceil}\sum_{n=1}^{N} \frac{c^{kN+n}}{\Gamma((kN+n)\varepsilon)} + \sum_{k=\lceil \mathcal{C} \rceil+1}^{\infty} \frac{N\max(c^{(k+1)N}, c^{kN})}{\Gamma(k)}$$

$$= \sum_{k=0}^{\lceil \mathcal{C} \rceil}\sum_{n=1}^{N} \frac{c^{kN+n}}{\Gamma((kN+n)\varepsilon)} + N\max(c^N, c^{2N}) \sum_{k=\lceil \mathcal{C} \rceil+1}^{\infty} \frac{c^{(k-1)N}}{(k-1)!}$$

$$\leq \sum_{k=0}^{\lceil \mathcal{C} \rceil}\sum_{n=1}^{N} \frac{c^{kN+n}}{\Gamma((kN+n)\varepsilon)} + N\max(c^N, c^{2N})\, e^{c^N}.$$

$$< \infty$$

$\square$

__Exercise__ 3.1.7 Let $c \in [0,\infty)$ and $\varepsilon \in (0,\infty)$.

By the identity$^{\circledast}$ $B(x,y) = \dfrac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$ we have

$$\sum_{n=1}^{\infty} c^n \prod_{k=1}^{n} B(k\varepsilon,\varepsilon) = \sum_{n=1}^{\infty} c^n \prod_{k=1}^{n} \frac{\Gamma(k\varepsilon)\Gamma(\varepsilon)}{\Gamma((k+1)\varepsilon)}$$

$$= \sum_{n=1}^{\infty} [c\Gamma(\varepsilon)]^n \cdot \frac{\Gamma(n\varepsilon)\Gamma((n-1)\varepsilon)\cdot\ldots\cdot\Gamma(\varepsilon)}{\Gamma((n+1)\varepsilon)\Gamma(n\varepsilon)\cdot\ldots\cdot\Gamma(2\varepsilon)}$$

$$= \sum_{n=1}^{\infty} c^n (\Gamma(\varepsilon))^{n+1} \cdot \frac{1}{\Gamma((n+1)\varepsilon)}$$

$$= \frac{1}{c} \sum_{n=2}^{\infty} \frac{(\Gamma(\varepsilon)c)^n}{\Gamma(n\varepsilon)} < \infty \qquad \text{by Exercise 3.7.6.}$$

$\circledast$ Proof of identity: let $x, y \in (0,\infty)$, then

$$\Gamma(x)\Gamma(y) = \int_0^{\infty} s^{y-1} \int_0^{\infty} t^{x-1} e^{-(t+s)}\,dt\,ds$$

$$\overset{\boxed{r=t+s}}{=} \int_0^{\infty} s^{y-1} \int_s^{\infty} (r-s)^{x-1} e^{-r}\,dr\,ds$$

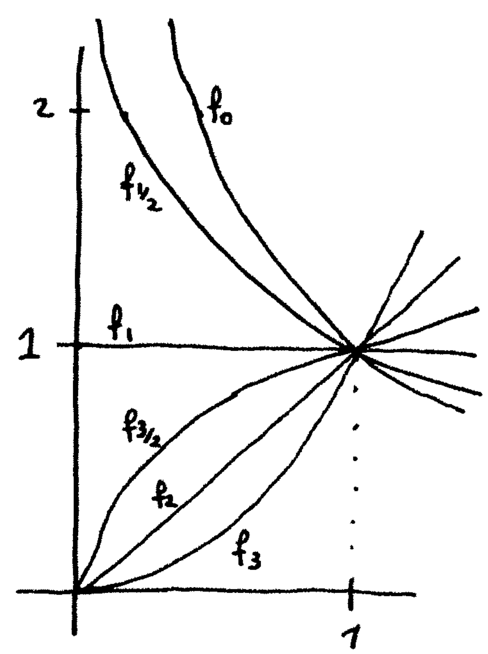
$$= \int_0^{\infty} e^{-r} \int_0^{r} s^{y-1}(r-s)^{x-1}\,ds\,dr$$

$$\overset{u=s/r}{=} \int_0^{\infty} e^{-r} r^{x+y-1} \int_0^{1} u^{y-1}(1-u)^{x-1}\,du\,dr$$

$$= \Gamma(x+y)\,B(y,x) = \Gamma(x+y)\,B(x,y).$$

Exercise 3.1.8 Observe that B is non-increasing in both parameters. Indeed, for $x, y \in (0,\infty)$ by definition we have

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$



plots of f_α where $f_\alpha(x) = x^{\alpha-1}$, $x \in (0,\infty)$

whence if $x_1 \geq x_2 > 0$ and $y_1 \geq y_2 > 0$ then (see picture)

$$t^{x_1-1}(1-t)^{y_1-1} \leq t^{x_2-1}(1-t)^{y_2-1}$$

for all $t \in [0,1]$.

It follows that

$$B(x_1,y_1) \leq B(x_2,y_2).$$

Thus for $c \in [0,\infty)$, $\varepsilon, \delta, \rho \in (0,\infty)$ we have:

$$\sum_{n=1}^{\infty} c^n \prod_{k=0}^{n-1} B(\varepsilon + k\delta, \rho) \leq \sum_{n=1}^{\infty} c^n \prod_{k=1}^{n} B(k\min(\varepsilon,\delta,\rho), \min(\varepsilon,\delta\rho))$$

$$< \infty \text{ by Exercise 3.1.7.}$$