Exercise 2.5.12.

Take $E = \mathbb{N}$ with $d_E(n,m) = \begin{cases} 0 : n = m \\ 1 : n \neq m \end{cases}$

for $n, m \in \mathbb{N}$. Observe that this is indeed a metric space. One easily checks that

$$\omega_{id_E}(h) = \begin{cases} 0 : h \in [0,1) \\ 1 : h \in [1,\infty]. \end{cases}$$

Exercise 2.5.15

Take $F = \mathbb{R}$ with $d_F(x,y) = |x-y|$, $x, y \in F$.
Take $E = (0,1) \cup (3,4)$ with $f(x) = x$, $x \in E$ and $d_E(x,y) = |x-y|$, $x, y \in E$.

Observe that $\omega_f(2) = 1$

and that f is uniformly continuous. The closure of E is given by $[0,1] \cup [3,4]$ and $\tilde{f} : [0,1] \cup [3,4]$, $\tilde{f}(x) = x$ satisfies $\tilde{f}|_E = f$.

However, $\omega_{\tilde{f}}(2) = d(f(3), f(1)) = 2$.

Exercise 2.5.24

Suppose A is symmetric, $\lambda \in \sigma_p(A)$. Let $v \in H$ be such that $Av = \lambda v$ and $v \neq 0$. We thus have:

$$\lambda \langle v, v \rangle = \langle v, \lambda v \rangle = \langle v, Av \rangle = \langle Av, v \rangle$$
$$= \langle \lambda v, v \rangle = \bar{\lambda} \langle v, v \rangle$$

whence $\lambda = \bar{\lambda}$ (because $\langle v, v \rangle \neq 0$).
It follows that $\lambda \in \mathbb{R}$.

On the other hand, suppose A is a linear diagonal operator and $\sigma_p(A) \subseteq \mathbb{R}$. It follows that there exists an index set I, $(e_i)_{i \in I} \subseteq H$ orthonormal, and $(\lambda_i)_{i \in I} \subseteq \mathbb{R}$ such that

$$\mathcal{D}(A) = \left\{ v \in H : \sum_{i \in I} |\langle v, e_i \rangle|^2 |\lambda_i|^2 < \infty \right\}$$

and $Av = \sum_{i \in I} \lambda_i \langle v, e_i \rangle e_i$.

Let $u, v \in \mathcal{D}(A)$. It follows that

$$\begin{aligned}
\langle Av, u \rangle &= \Big\langle \sum_{i \in I} \lambda_i \langle e_i, v \rangle e_i, u \Big\rangle \\
&= \sum_{i \in I} \lambda_i \langle v, e_i \rangle \langle e_i, u \rangle \\
&= \Big\langle v, \sum_{i \in I} \lambda_i e_i \langle e_i, u \rangle \Big\rangle \\
&= \langle v, Au \rangle
\end{aligned}$$

whence A is symmetric.

Exercise 2.5.30

Let $(e_i)_{i \in \mathbb{N}}$ be an orthonormal basis for a separable Hilbert space H. Define $A \in \mathcal{L}(H)$ by

$$Av = \sum_{i=1}^{\infty} \frac{1}{i} \langle v, e_i \rangle e_i.$$

Observe that $\mathcal{D}(A) = H$ and that e.g. $x_n = \sum_{i=1}^{n} e_i$, $n \in \mathbb{N}$, defines a Cauchy sequence in $(\mathcal{D}(A), \|A(\cdot)\|_H)$ that is not convergent.

(After all, for $n, m \in \mathbb{N}$, $n \leq m$ it holds that

$$\| A(x_n - x_m) \|_H = \sum_{i=n+1}^{m} (1/i)^2$$

$$\leq 1/n - 1/m \;).$$