## 1.4.42

Let $(H, \langle\cdot,\cdot\rangle_H, \|\cdot\|_H)$ be an $\mathbb{R}$-Hilbert space

Let $A: D(A) \subseteq H \to H$ be a linear diagonal operator

Prove that $A$ is symmetric, that is

$$\langle Ax, y\rangle_H = \langle x, Ay\rangle \quad \forall x, y \in D(A).$$

**Proof:**

Since $A$ is diagonal, there exists $\mathbb{B} \subset H$ ONB and $\lambda : \mathbb{B} \to \mathbb{R}$ s.t

$$Ax = \sum_{b \in \mathbb{B}} \lambda_b \langle x, b\rangle_H \, b$$

Further, it holds $y = \sum_{b \in \mathbb{B}} \langle y, b\rangle_H \, b$

By a result from the lecture notes we know that there exists a sequence $\{b_n\}_{n=1}^{\infty} \subset \mathbb{B}$ s.t for all $b \in \mathbb{B} \setminus \{b_n\}_{n=1}^{\infty}$ it holds that $|\langle x, b\rangle| = |\langle y, b\rangle| = 0$

Therefore

$$\langle Ax, y\rangle = \left\langle \sum_{n=1}^{\infty} \lambda_{b_n} \langle x, b_n\rangle b_n, y \right\rangle$$

$$= \sum_{n=1}^{\infty} \langle x, b_n\rangle \lambda_{b_n} \langle b_n, y\rangle$$

by continuity of the inner product

$$= \left\langle x, \sum_{n=1}^{\infty} \lambda_{b_n} \langle b_n, y\rangle b_n \right\rangle$$

$$= \langle x, Ay\rangle \qquad \square$$

1.4.43

The point spectrum is given by

$$\sigma_p(A) = \{ \lambda \in \mathbb{C} : Ker( A - \lambda Id) \neq \{0\} \}$$

To show: If $A$ is diagonal, then $\sigma_p(A) = Im(\lambda)$

"$\supset$": For every $b \in \mathbb{B}$ it holds

$$Ab = \sum_{b' \in \mathbb{B}} \lambda_{b'} \underbrace{\langle b, b' \rangle}_{= 0 \text{ if } b \neq b'} b' = \lambda_b b$$

$$\Rightarrow b \in Ker(A - \lambda_b Id) \rightarrow \lambda_b \in \sigma_p(A) \quad \forall b \in \mathbb{B}.$$

This proves $Im(\lambda) \subset \sigma_p(A)$.

"$\subset$": Let $\mu \in \sigma_p(A)$, then $\exists\, x_\mu \in D(A)$ such that

$$A x_\mu = \mu \cdot x_\mu \quad , \quad x_\mu \neq 0$$

Consider the norm $\| A x_\mu - \mu x_\mu \|^2 \overset{!}{=} 0$ (by the choice of $x_\mu$)

$$\Rightarrow \| A x_\mu - \mu x_\mu \|^2 = \| \sum_{n=1}^{\infty} (\mu - \lambda_{b_n}) \langle x_\mu, b_n \rangle b_n \|^2$$

$$= \sum_{n=1}^{\infty} |\mu - \lambda_{b_n}|^2 \, |\langle x_\mu, b_n \rangle|^2 \overset{!}{=} 0$$

$$\Rightarrow \forall n \in \mathbb{N} \text{ it holds } |\mu - \lambda_{b_n}|^2 \, |\langle x_\mu, b_n \rangle|^2 \overset{!}{=} 0$$

Since $x_\mu \neq 0$ $\exists\, \hat{n} \in \mathbb{N}$ such that $\mu = \lambda_{b_{\hat{n}}}$.

$$\Rightarrow \mu \in Im(\lambda) \quad \Rightarrow \quad \sigma_p(A) \subset Im(\lambda)$$

$\square$

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**1.4.42:**

To show: Diagonal operators are densely defined