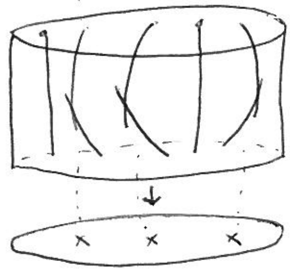
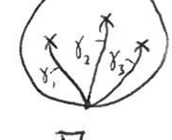
VANISHING CYCLES & MUTATIONS

OVERVIEW

Study exact Morse fibrations ...   ... by choosing a basis of vanishing paths ...   ... forming a directed A_∞-category of vanishing cycles ...   ... and forming its derived category.





Γ

Γ'

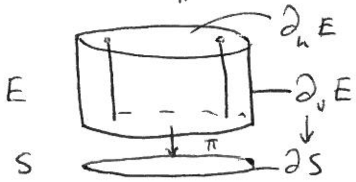
$$\begin{tikzcd} \rightsquigarrow \; \vec{Lag}\,\Gamma \arrow[r, squiggly, "H^0(Tw)"] \arrow[d, "\text{quiso}"] & D^b \vec{Lag}\,\Gamma \arrow[d] \\ \rightsquigarrow \; \vec{Lag}\,\Gamma' \arrow[r, squiggly] & D^b \vec{Lag}\,\Gamma' \end{tikzcd}$$

Changing basis by Hurwitz moves ...   ... induces quasi-isomorphisms ...   ... which become equivalences at the level of derived categories.

(1)

EXACT MORSE FIBRATIONS

* (M, ω, θ) is an exact symplectic manifold (with boundary) if $\omega = d\theta$.
* A compactly supported symplectomorphism ϕ of $M \setminus \partial M$ is called exact if
$$[\phi^*\theta - \theta] = 0 \in H^1(M, \partial M; \mathbb{R}).$$
* A submersion $E^{2n+2} \xrightarrow{\pi} S^2$ with a 2-form Ω and a 1-form Θ on E is called an exact symplectic fibration if:



* $d\Theta = \Omega \quad (d\Omega = 0)$
* $\Omega|_{\ker D\pi}$ is non-degenerate
* trivial at ∞, i.e. all data look like a product in a neighbourhood of $\partial_h E$.

Fibres are exact symplectic manifolds with boundary.

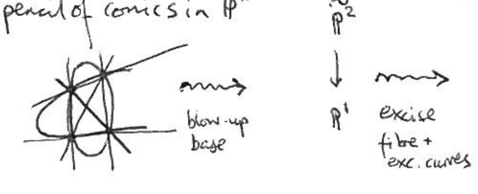
* An exact Morse fibration is defined in the same way, but π can have isolated quadratic singularities, i.e. in a neighbourhood of $(d\pi = 0)$ there are Ω-Kähler complex structures J_0 (and j_0 on a neighbourhood of $\pi(\partial\pi = 0)$) such that π is (J_0, j_0)-holomorphic and in local complex coordinates centred at critical points π takes the form $\pi(x_1, \ldots, x_{n+1}) = x_1^2 + \cdots + x_{n+1}^2$.

(2)

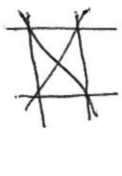
EXAMPLES

Examples come from Lefschetz pencils in algebraic geometry, after some tweaking (see "More VCM" section 3)

e.g. a pencil of conics in $\mathbb{P}^2$



generic fibre is



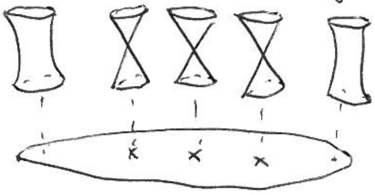
the three singular fibres.

+ 3 singular fibres.

e.g. $\mathbb{C}^{n+1} \xrightarrow{\pi'} \mathbb{C}$ Let $V \subseteq \mathbb{C}^{n+1}$ be given by $\sum_{i=1}^{n} x_i^2 = p(x_{n+1})$. Set $\pi = \pi'|_V$.

$\pi(x_1, \ldots, x_{n+1}) = x_{n+1}$

General fibre is a smooth affine quadric. $\deg p$ singular fibres.



$n = 3$, $\deg p = 3$.

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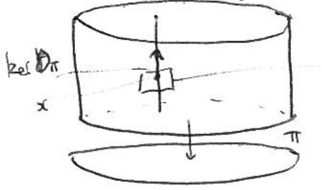
PARALLEL TRANSPORT

Define a horizontal distribution on E by $H_x = (\ker D\pi)_x^{\perp}$ where $\perp$ denotes Ω-orthogonal complement:



This allows us to lift vectors from S to vectors in E.

A path in S, $c : [0,1] \to S$, defines parallel transport diffeos

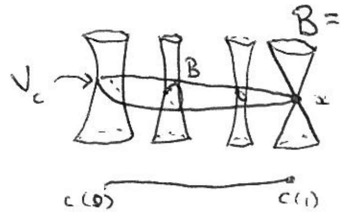
$$\rho_c : c^{-1}(0) \to c^{-1}(1)$$

In fact these are symplectic & if c is a contractible loop the map $\rho_c : c^{-1}(0) \to c^{-1}(1)$ is a Hamiltonian symplectomorphism (because $d\Omega = 0$. See McDuff–Salamon).

Caveats:
- Not obvious that ρ_c is well-defined for all times as fibres are non-compact. Triviality at ∞ helps us here.
- Not well-defined at singularities of π.

However, we can define the stable manifold ("vanishing thimble") associated to a path c with $c(1) = \pi(x)$, x a critical point:

$$B = \{ y : \lim_{t \to 1} \rho_c|_{[s,t]}(y) = x \}$$



Claim: B is a smoothly embedded ball. $\Omega|_B \equiv 0$.

$\boxed{\text{Define the vanishing cycle of } c \text{ to be the Lagrangian sphere } c^{-1}(0) \cap B = V_c.}$

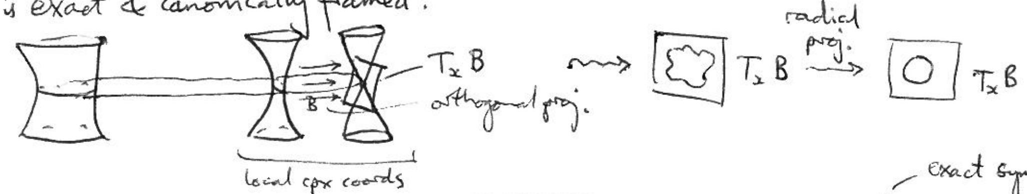
EXACT LAGRANGIANS & DEHN TWISTS

An exact Lagrangian submanifold $L \subseteq M$ is a Lagrangian s.t.

$$[\theta|_L] = 0 \in H^1(L;\mathbb{R}).$$

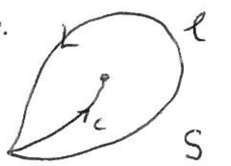
A framed Lagrangian sphere is a pair $(L, [f])$ where L is an exact Lagrangian and $f : S^n \to L$ is a diffeo. $f_1 \sim f_2$ if $f_2^{-1} f_1$ is isotopic to an element of $O(n+1) \subseteq \mathrm{Diff}(S^n)$.

A vanishing cycle is exact & canonically framed:



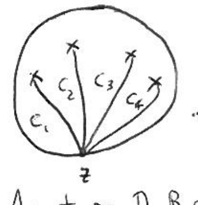
To a framed Lagrangian sphere, one can associate a Dehn twist $\tau_{(L,[f])} \in \mathrm{Symp}^e(M)$ (canonically up to isotopy) [exact symplectomorphisms]

Picard–Lefschetz theorem:



Let V_c be the vanishing cycle of c.
ρ_ℓ and τ_{V_c} are isotopic in $\mathrm{Symp}^e(E_{c(0)})$.

DISTINGUISHED BASES & HURWITZ EQUIVALENCE



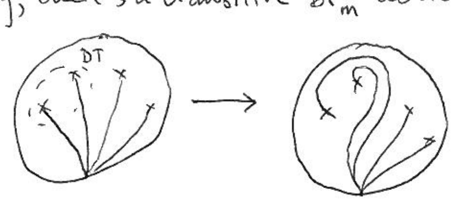
A distinguished basis of vanishing paths, connecting critical points to a point $z \in \partial S$. Ordered clockwise.

$\Gamma = (c_1, \dots, c_4)$ [Here there are $m = 4$ critical points]

Any two D.B.s are diffeomorphic rel$(\partial S \cup c_1(1) \cup \dots \cup c_m(1))$, since the complement of a D.B. is a disc.

Equivalently, there's a transitive Br_m-action on the set of D.B.s.

e.g.



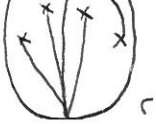
Question: What effect does this have on the vanishing cycles?

Answer: Use Picard–Lefschetz:

$$\boxed{V_{c_1'} = \tau_{c_2}^{-1} V_{c_1}}$$

Hurwitz moves

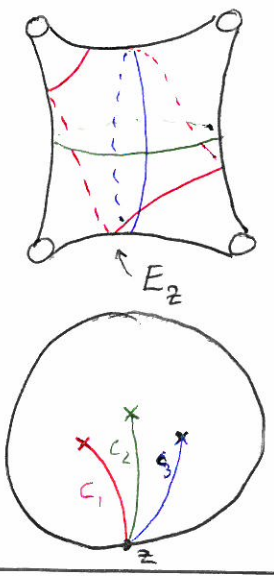
Seidel actually chooses generators



A configuration of framed Lagrangian spheres Γ is Hurwitz equivalent to another Γ' if they can be connected by isotopies & Hurwitz moves.

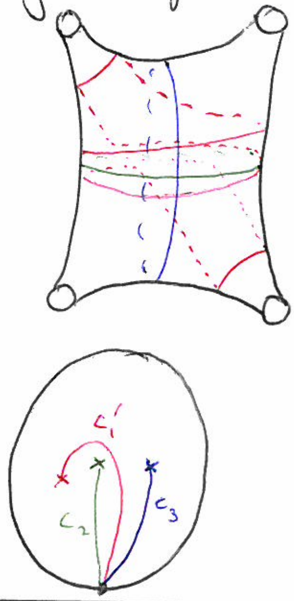
EXAMPLE

Pencil of quadrics in $\mathbb{P}^2$ from earlier gave an exact Morse fibration whose generic fibre was a thrice-punctured disc and having three singularities:



Hurwitz move

$$\Rightarrow \tau^{-1}_{V_{c_2}} V_{c_1} = V_{c_1'}$$



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MUTATION

Suppose we jump several weeks into the future and already know how to associate a triangulated category $D^b(A)$ to an exact Morse fibration and a choice of distinguished basis. How will $D^b(A)$ change if we change basis?

Def^n: Let $\mathcal{C}$ be a triangulated category (with finite-dimensional hom-spaces, $\mathrm{Hom}^*_{\mathcal{C}}(X,Y) = \bigoplus_{r\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(X, Y[r])$).
An exceptional collection $(X^1, \dots, X^m)$ in $\mathcal{C}$ is a family of objects s.t.

$$\mathrm{Hom}^*_{\mathcal{C}}(X^i, X^k) = \begin{cases} \mathbb{Z}/2 \cdot \mathrm{id}_X & i = k \\ 0 & i > k \end{cases}$$

A full exceptional collection is one which generates $\mathcal{C}$.

An exceptional collection is the analogue of a distinguished basis. In fact, the choice of distinguished basis will give us a canonical full exceptional collection in $D^b(A)$. There are operations (called mutations) analogous to Hurwitz moves c & r to transform one exceptional collection into another:

Ⓐ $$Y^i = \begin{cases} T_{X^1}(X^{i+1}) & i < m \\ X^1 & i = m \end{cases}$$

Ⓑ $$Z^i = \begin{cases} X^i & i < m-1 \\ T_{X^{m-1}} X^m & i = m-1 \\ X^{m-1} & i = m \end{cases}$$

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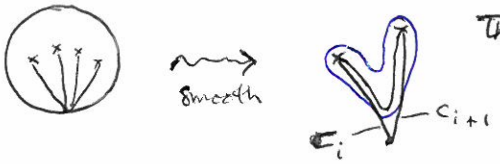
Comments: ① Given a distinguished basis, one can write down a set of generators $\sigma_1, \dots, \sigma_{m-1}$ for the Br_m-action:



The Dehn twist in the blue circle corresponds to σ_i.

However changing distinguished basis by σ_i changes the generators!

At the level of $D^b(A)$ there's no obvious global way to see Br_m, but given an exception collection one can still form Ⓐ & Ⓑ, so it's better to think of Hurwitz moves rather than a Br_m-action.

② $T_X Y$ is defined to be the cone on the evaluation map $ev: \mathrm{Hom}^*_{\mathcal{C}}(X,Y) \otimes X \to Y$

$$\begin{array}{ccc} \mathrm{Hom}^*_{\mathcal{C}}(X,Y)\otimes X & \xrightarrow{ev} & Y \\ & \nwarrow \quad \swarrow & \\ & T_X Y & \end{array}$$

Exercises! Check Ⓐ & Ⓑ do define exceptional collections.

Check $T_{X_i} T_{X_{i+1}} T_{X_i} = T_{X_{i+1}} T_{X_i} T_{X_{i+1}}$.

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