OPEN QUESTION I: SYMPLECTIC HYPERSURFACES IN $\mathbb{CP}^n$

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What do we know about the topology of codimension 2 symplectic submanifolds $\Sigma \subset X$ (otherwise known as symplectic hypersurfaces)? Basically nothing other than the adjunction formula

$$c(\nu\Sigma) \cup c(T\Sigma) = c(TX|_{\Sigma})$$

where c is the Chern character of a symplectic vector bundle.

The only symplectic hypersurfaces we know are Donaldson hypersurfaces. These are particular submanifolds which are Poincaré dual to $k[\omega]$ for some large integer k where $[\omega]$ denotes the cohomology class of the symplectic form (assumed to lie in the rational or integral lattice). They were constructed by Donaldson in his seminal paper, under the assumption that k is large enough and further examined by Auroux in his thesis. For sufficiently large k we know that the complement is Weinstein (in particular contracts onto an isotropic skeleton). This implies they satisfy the topological conclusions of the Lefschetz hyperplane theorem and this tells us rather a lot topologically.

I want to reduce attention to a very simple case. Take symplectic hypersurfaces in $\mathbb{CP}^n$. There every submanifold is Poincaré dual to some multiple d of the symplectic form, so maybe Donaldson submanifolds are not so special (no guarantees!). We say a symplectic hypersurface has degree d if it is in the homology class dH (where H is the homology class of a hyperplane). The adjunction formula gives us the formula

$$\mathrm{P.D.}(c_1(\Sigma)) = (n + 1 - d)H \cap \Sigma$$

for the first Chern class while $\mathrm{P.D.}[\omega] = H \cap \Sigma$ so we have the usual trichotomy

$$d < n + 1,\ d = n + 1,\ d > n + 1$$

between symplectically Fano, symplectic Calabi-Yau and symplectically general type manifolds by degree. At this point we comment that symplectically Fano 4-manifolds are classified up to deformation equivalence and so we can identify symplectic hypersurfaces of degree 1,2,3 in $\mathbb{CP}^3$ as being symplectomorphic to $\mathbb{CP}^2, S^2 \times S^2$ and a cubic surface respectively (to avoid using Seiberg-Witten theory in the proof, one can construct pseudoholomorphic curves in these by hand for degree 1,2 using pseudoholomorphic curves theory in $\mathbb{CP}^3$: see Hind's beautiful little paper. It almost feels like classical algebraic geometry!).

Indeed, if one wants to identify symplectic hypersurfaces in $\mathbb{CP}^3$ up to homeomorphism it suffices to show they're simply connected. This is because under this assumption the Chern classes determine the cohomology ring and therefore the homeomorphism type by Freedman's theorem. This begs the intriguing

Question 1. *Is a symplectic hypersurface in* $\mathbb{CP}^3$ *(more generally* $\mathbb{CP}^n$*) simply connected?*

All I can show is that the answer is yes for degree 1. The argument uses pseudoholomorphic curves and in particular positivity of intersections between (simultaneously pseudoholomorphic) curves and hypersurfaces and feels very like the arguments used by Hind in the aforementioned paper. I doubt that these methods will stretch much further. A new approach would be very interesting!