Symplectic Mapping Class Groups: Motivation & Methods

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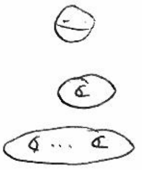
To a smooth manifold M, one associates its diffeomorphism group $\mathrm{Diff}(M)$. This is an infinite-dimensional Fréchet Lie group containing much information about the smooth topology of M. In fact, too much! Let's concentrate on

$$MCG(M) := \pi_0(\mathrm{Diff}^+(M))$$ the mapping class group of M.

Diff^+: orientation preserving

eg. In 2-d



M	$MCG(M)$
S^2	1
T^2	$SL(2,\mathbb{Z})$
Σ_g	known

These all have something in common. They are generated by Dehn twists: (theorem of Lickorish)

T^*S^1 $\rightsquigarrow$ T^*S^1

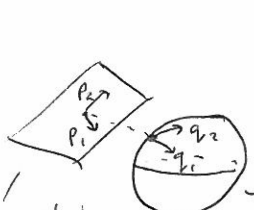
LOCAL MODEL OF A DEHN TWIST.

These can be chosen to preserve an area form on the surface

In higher dimensions (in particular 4D) little or nothing is known.

4D There is a Dehn twist on T^*S^2 which preserves the canonical symplectic form:

theorem/observation of Arnold



$$\omega_{can} = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$$

The zero-section $S^2 \subseteq T^*S^2$ is Lagrangian with respect to this symplectic form, ie. $\omega_{can}|_{0\text{-section}} = 0$.

Review: * A symplectic form on a 4-manifold is a closed 2-form ω ($d\omega = 0$) which is the square-root of a volume form ie. $\omega \wedge \omega = \mathrm{vol}$.

* $\iota : L \longrightarrow (M,\omega)$ is a Lagrangian embedding if $\iota^*\omega = 0$ (and $\dim L = \frac{1}{2}\dim M$).

* Theorem (Weinstein) A Lagrangian embedding $L \hookrightarrow (M,\omega)$ extends to an embedding $\phi : U \longrightarrow (M,\omega)$ where U is a neighbourhood of the 0-section in T^*L and $\phi^*\omega = \omega_{can}|_U$.

This implies that whenever we have a Lagrangian sphere $S^2 \hookrightarrow (M,\omega)$ we can implant a Dehn twist locally in (M,ω) which preserves ω.

Define $\mathrm{Symp}(M,\omega) = \{\phi \in \mathrm{Diff}(M) : \phi^*\omega = \omega\}$ (group of symplectomorphisms)

$$\boxed{MCG_s(M,\omega) = \pi_0(\mathrm{Symp}(M,\omega))}$$ SYMPLECTIC MAPPING CLASS GROUP

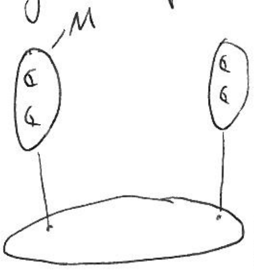
When is $MCG_s(M,\omega)$ generated by Dehn twists?

MOTIVATION

Algebraic Geometry

* A smooth projective variety $V \subseteq \mathbb{P}^N$ inherits a symplectic (Kähler) form from $\mathbb{P}^N$ (which has its $U(N+1)$-invariant Fubini-Study form).
* A family of smooth projective varieties M parametrised by some base manifold C is a smooth fibre bundle with structure group $\mathrm{Diff}(M)$.
* A Moser-type argument $\Rightarrow$ structure group reduces to $\mathrm{Symp}(M, \omega)$. Therefore we get a map $C \longrightarrow B\mathrm{Symp}(M,\omega)$

$$\Downarrow$$



$$\boxed{\begin{array}{l}\pi_1(C) \longrightarrow \pi_0(\mathrm{Symp}(M,\omega)) = MCG_s(M,\omega) \\ \text{So symplectic } MCG \text{ detects holes in "moduli spaces" of smooth projective varieties.}\end{array}}$$

eg. (Picard-Lefschetz) If $C = \mathbb{C}^*$ is the smooth part of an ODP degeneration over $\mathbb{C}$, we get a Dehn twist.

Gauge Theory

To M we can associate various gauge theoretic moduli spaces, eg. instanton moduli, $Y(M)$. These come with natural symplectic forms, natural in the sense that $\mathrm{Diff}(M)$ acts on $Y(M)$ symplectically, or at the very least we get a map

$$MCG(M) \longrightarrow MCG_s(Y(M), \omega)$$

There are interesting representations of MCG's of algebraic curves in terms of symplectic MCG's of associated moduli spaces of representations of $\pi_1(M)$.

Even these are little understood.

Often, a Dehn twist on M will correspond to a (fibred) Dehn twist on $Y(M)$, as observed by Seidel.

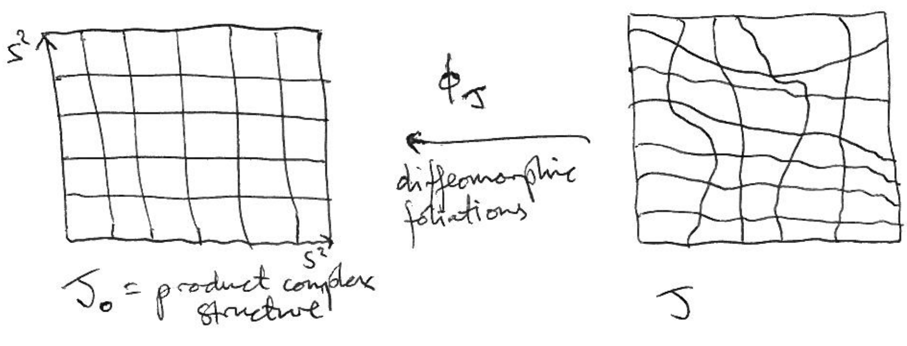
METHODS

Gromov's idea: Use "complex curves" for a eg complex structure J making ω Kähler. The space of such complex structures is complicated, so we weaken to non-integrable almost complex structures (a.c.s).

$$\mathcal{J}_\omega = \{ J \text{ an a.c.s. compatible with } \omega \} \text{ is contractible!}$$

Gromov showed there's a notion of pseudoholomorphic curves for a.c.s. which behave like complex curves in Kähler manifolds.

eg. **Theorem** (Gromov) Let $S^2 \times S^2$ be equipped with its product symplectic form $\omega = \omega_{S^2} \oplus \omega_{S^2}$ (ω_{S^2} is an area form on S^2). If J is an ω-compatible a.c.s. on $S^2 \times S^2$ then there are two foliations of $S^2 \times S^2$ by J-holomorphic spheres in the homology classes $S^2 \times \cdot$ and $\cdot \times S^2$ respectively. If A & B are leaves from the two foliations then A intersects B once transversely.



Theorem: (Gromov)

$$\mathrm{Symp}(S^2 \times S^2, \omega_{S^2} \oplus \omega_{S^2}) \simeq SO(3) \times SO(3) \rtimes \mathbb{Z}/2.$$

Proved using the holomorphic foliations from above.

What is the $\mathbb{Z}/2$ factor?

It comes from a Dehn twist in the Lagrangian sphere

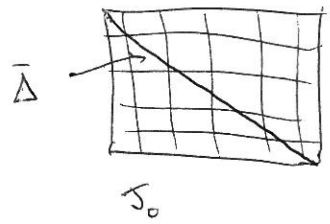
$$\bar{\Delta} = \{(x, -x) \in S^2 \times S^2\}$$

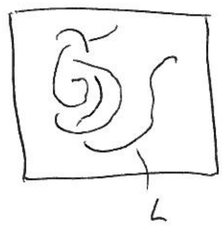
$\boxed{\text{In this case, } MCG_s(S^2 \times S^2, \omega) = \mathbb{Z}/2 \text{ is generated by Dehn twists.}}$

Question Can we classify Lagrangian spheres in $S^2 \times S^2$ up to isotopy (through Lagrangian spheres)?

Answer (Hind) Yes. Space $\mathcal{L}ag$ of Lagrangian spheres is path-connected.

Idea of proof: Need to find a $J \in \mathcal{J}_\omega$ s.t. the associated Gromov foliations intersect a given Lagrangian sphere L like the standard ones intersect $\bar{\Delta}$.





Find J by neck-stretching:

In limit $\lambda \to \infty$ a neighbourhood of L looks like ~~the~~ affine quadric with real part L. Foliations look like the rulings. (SFT)

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Blow-ups of $\mathbb{P}^2$ in n-points $\mathbb{D}_n$.

Theorem: (E.) For $\mathbb{D}_2, \mathbb{D}_3, \mathbb{D}_4$ fix a homology class containing a Lagrangian sphere and let $\mathcal{L}ag$ be the space of such homologous Lagrangian spheres. Then $\mathcal{L}ag$ is path-connected.

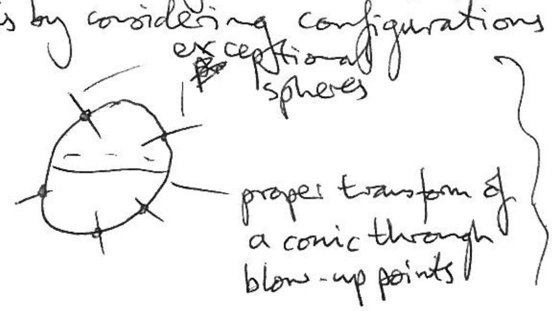
Idea of proof: Use a similar SFT argument to disjoin L from a divisor whose complement is symplectomorphic to a subset of T^*S^2 where Hind's argument can be adapted to prove path-connectedness for spaces of Lagrangian spheres. $\square$

Theorem: (Seidel) There are infinitely many homologous but non-isotopic Lagrangian spheres in $\mathbb{D}_5$!

Idea of proof: Algebraic geometry gives a family of $\mathbb{D}_5$'s parametrised by $\mathrm{Conf}(5)$. (configuration space of points in S^2)

This gives $\mathrm{Conf}(5) \to B\mathrm{Symp}(\mathbb{D}_5)$. Construct a map $B\mathrm{Symp}(\mathbb{D}_5) \to \mathrm{Conf}(5)$ such that $\mathrm{Conf}(5) \to B\mathrm{Symp}(\mathbb{D}_5) \to \mathrm{Conf}(5)$ is homotopic to the identity.

Do this by considering configurations of J-holomorphic curves



such configurations persist J-holomorphically for all J.

$\square$