COMPUTATIONS WITH PSEUDOHOLOMORPHIC CURVES

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1. Introduction

Symplectic topology is renowned for its invariants which are either incomputable or only conjecturally well-defined. For algebraic geometers this is something of an anaethema. The aim of this talk is to give a sketch of why we can prove anything in symplectic topology, using friendly words like "Dolbeaut cohomology" and "positivity of intersections". By the end of the talk we should be able to prove (amongst other things) that there is a unique line through two points in the (complex projective) plane. Everything I will say is a) due to Gromov [Gro85] and b) explained more thoroughly and clearly in the big book by McDuff and Salamon [MS04].

2. Some linear algebra

Recall that a symplectic vector space is a $2n$-dimensional vector space with a non-degenerate alternating form ω. The group $Sp_n(\mathbb{R})$ of symplectic matrices (which preserve this form) is a non-compact Lie group containing the unitary group $U(n)$ as a maximal compact subgroup. In fact, there is a homotopy equivalence $Sp_n(\mathbb{R}) \to U(n)$. Identifying the vector space with $\mathbb{R}^{2n} \cong \mathbb{C}^n$, this unitary group is the subgroup of symplectic matrices which additionally commute with the complex structure (multiplication by i). The point is that together ω and i define a metric by $g(v, w) = \omega(v, iw)$. The unitary group can then be thought of as the complex (or symplectic) isometries of this metric. This is known as the 3-in-2 property of the unitary group:

$$Sp_n(\mathbb{R}) \cap O(2n) = Sp_n(\mathbb{R}) \cap GL_n(\mathbb{C}) = GL_n(\mathbb{C}) \cap O(2n) = U(n)$$

If one considers the set $\mathcal{J}$ of all possible matrices J such that $J^2 = -1$ and such that $\omega(v, Jw)$ defines a (J-invariant) metric, it turns out to be a homogeneous space

$$\mathcal{J} = Sp_n(\mathbb{R})/U(n)$$

This is quite easy to see: if one picks a vector v, then v and Jv span a symplectic subspace, and passing to its symplectic orthogonal complement one obtains a lower-dimensional space and (by induction) symplectic coordinates on the whole space for which J and ω look like the standard complex and symplectic structures. Hence $Sp_n(\mathbb{R})$ acts transitively on $\mathcal{J}$ and the stabiliser is (by definition) $U(n)$.

The fact that $Sp_n(\mathbb{R})$ and $U(n)$ are homotopy equivalent implies that this homogeneous space of ω-compatible almost complex structures is contractible. It remains true that the space of ω-compatible almost complex structures (i.e. automorphisms J of the tangent bundle with $J^2 = -1$, $\omega(v, Jv) > 0$ and $\omega(Jv, Jw) = \omega(v, w)$) on a symplectic manifold (i.e. a manifold with a nondegenerate closed 2-form) is contractible and this is extremely useful for defining invariants of ω. These almost complex structures do not necessarily arise as multiplications for an underlying complex manifold structure. If they did, they'd be called *integrable.*

3. Invariants

In the standard complex structure on $\mathbb{CP}^2$ it's clear that there is a unique complex line through two points. This is an integrable complex structure and as such is very special. The moral of Gromov's work is that most things aren't integrable and, when we're wearing our symplectic topology hats, we shouldn't allow ourselves to be beguiled by the appearance of a beautiful integrable situation where everything is computable[1]. So we ask: if we perturb the complex structure in the space $\mathcal{J}$ of almost complex structures, does the same statement remain true? Are there still J-holomorphic curves homologous to lines through every pair of points? It is only through answering such questions that we will arrive at truly symplectic invariants. If something is true in every ω-compatible almost complex structure then it's certainly telling us something about ω rather than a particular complex structure. So, given a symplectic manifold (X, ω) we find ourselves with the following problem:

Let (Σ, i) be a Riemann surface. Let χ be the (appropriate Sobolev completion of the) space of smooth maps $\Sigma \to X$ representing a homology class A. Let $\mathcal{J}$ be the (Banach) space of ($\mathcal{C}^\ell$) ω-compatible almost complex structures on X. Let $\mathcal{E}$ be the bundle over $\chi \times \mathcal{J}$ whose fibre over (u, J) is the space $\Omega^{0,1}(\Sigma, u^*TX)$ of $(0,1)$-forms (of a suitable Sobolev class) on Σ with values in u^*TX. There is a section $\overline{\partial}$ of $\mathcal{E}$ sending (u, J) to

$$\overline{\partial}_J u = \frac{1}{2}\left(du + J \circ du \circ i\right)$$

A pseudoholomorphic curve is a zero of that section, i.e. a map which is J-holomorphic for some J. We get a "universal moduli space" $\mathcal{M}^*$ of pseudoholomorphic curves (caveat: we omit those which factor through branched covers for technical reasons) which has a projection $\pi : \mathcal{M} \to \mathcal{J}$.

Theorem 3.1. *This universal moduli space $\mathcal{M}$ is a smooth Banach manifold and the projection π is Fredholm.*

In particular, at each point in $\mathcal{M}$, the tangent space to the fibre of π is finite-dimensional ($\ker D\pi$) and there is an associated finite-dimensional obstruction space $\mathrm{coker} D\pi$. The difference in dimension

$$\mathrm{ind} D\pi = \dim \ker D\pi - \dim \mathrm{coker} D\pi$$

is constant. This index is given by the formula:

$$\mathrm{ind} D\pi = 2n + 2\left\langle c_1(X), A \right\rangle \tag{1}$$

Now as human beings, we can only really do stuff with one complex structure at a time, i.e. look at the finite-dimensional slices of the universal moduli space. These slices are moduli spaces of J-holomorphic curves. They will not always be smooth manifolds, but by the Sard-Smale theorem they will be for a Baire set of regular almost complex structures (the regular values of π). A regular almost complex structure is one for which the obstruction cokernel vanishes. In this case,

[1] In Gromov's own words, "What fascinated me even more was the familiar web of algebraic curves in a surface emerging in its full beauty in the softish environment of general (nonintegrable) almost complex structures. (Integrability had always made me feel claustrophobic.)", from [Ber00]

the moduli space of J-holomorphic curves is a smooth manifold of dimension given by the formula 1.

It is out of these moduli spaces that one hopes to build a symplectic invariant. Different J could give different moduli spaces, but since $\mathcal{J}$ is contractible one might hope that connecting Js with paths would allow one to relate the moduli spaces canonically up to homotopy. In fact, if one looks at generic paths connecting regular values (again by Sard-Smale) one will obtain cobordisms of the moduli spaces at either end. So might the cobordism class of a moduli space at a regular almost complex structure provide a symplectic invariant of (X, ω)?

There are two problems here. One is in defining the invariant: since the moduli spaces and cobordisms one obtains are non-compact, the cobordism class is kind of useless. This can be rectified using Gromov's compactness theorem and thinking about bordism of evaluation pseudocycles instead of cobordism moduli spaces. The second problem is in computing the invariant. How on earth do we recognise a regular almost complex structure? Let's address this question first.

4. Regularity

4.1. **Integrable complex structures.** If J is integrable and u is a J-holomorphic sphere in (X, ω) then u^*TX is a complex vector bundle over $\mathbb{CP}^1$ and these are known to split as sums of line bundles: $u^*TX = L_1 \oplus L_2 \oplus \cdots \oplus L_k$.

Proposition 4.1. *If $c_1(L_i) \geq -1$ for $i = 1, \ldots, k$ then J is regular (for the Fredholm problem of J-holomorphic curves in this homology class).*

Proof. It turns out in this case that the linearised Cauchy-Riemann operator is just the Dolbeault derivative, so the problem splits into looking at its cokernel on each line bundle L_i. This cokernel is $H^{0,1}_{\overline{\partial}}(\mathbb{CP}^1, L_i)$. Now

$$\begin{aligned} H^{0,1}_{\overline{\partial}}(\mathbb{CP}^1, L_i) &\cong \left(H^{1,0}_{\overline{\partial}}(\mathbb{CP}^1, L_i^*)\right)^* \\ &\cong \left(H^0(\mathbb{CP}^1, L_i^* \otimes K)\right) \end{aligned}$$

by Serre duality. The Kodaira vanishing theorem implies that this latter group vanishes if and only if $c_1(L_i^* \otimes K) < 0$, but

$$c_1(L_i^* \otimes K) = -c_1(L_i) - 2$$

which implies that the cokernel vanishes if and only if $c_1(L_i) \geq -1$. □

This is essentially due to positivity of intersections: a section of $L_i^* \otimes K$ intersects the zero-locus Σ with homological index $c_1(L_i^* \otimes K)$. If that number is negative then there can be no sections, as sections are subvarieties of the total space of the bundle and intersect Σ positively. Thus $c_1(L_i^* \otimes K) < 0$ implies the cokernel vanishes, without even mentioning Kodaira.

4.2. **Nonintegrable case.** In fact, one can still say something in the nonintegrable case in 4-dimensions.

Theorem 4.2 (Automatic transversality)**.** *Let J be an almost complex structure on X and u an immersed J-holomorphic sphere. Then J is regular for u is and only if $c_1(u^*TX) \geq 1$.*

More generally, if Σ is a Riemann surface with arbitrary genus and boundary components which map to totally real submanifolds (e.g. Lagrangian submanifolds) the Fredholm theory works and the automatic transversality theorem holds:

Theorem 4.3 (Automatic transversality)**.** *Let u be a simple (i.e. not factoring through a branched cover), embedded J-holomorphic curve with totally real boundary conditions. J is regular for u is*

$$\mu(E, F) + 2\chi(\Sigma) > 0$$

where $\mu(E, F)$ is the boundary Maslov index*, a kind of generalisation of the first Chern class. In fact, if $\partial\Sigma = \emptyset$, $\mu(E, F) = 2c_1(E)$ (E is the normal bundle of the curve, F the totally real subbundle tangent to the boundary condition).*

Of course, it becomes harder for J to be automatically regular for curves with high genus. The proof of the automatic transversality theorem is akin to the proof in the integrable case. The new geometric ingredient is the following theorem of McDuff (stated by Gromov):

Theorem 4.4 (Positivity of intersections)**.** *Let u and v be two simple J-holomorphic curves in an almost complex 4-manifold such that if $U \subset dom(u)$ and $V \subset dom(v)$ are open subsets of their respective domains, the images $u(U)$ and $v(V)$ are geometrically distinct then every intersection point of the images contributes a* positive *integer amount to the homological intersection of the homology classes represented by the submanifolds $u(U)$ and $v(V)$. This integer is 1 if and only if the intersection is transverse.*

The theorem is well-known for integrable complex structures J, but for non-integrable structures it requires some very detailed local analysis.

4.3. **Aside: Adjunction formula.** A useful fact in algebraic geometry is the adjunction formula. This has an analogue in the non-integrable case, which follows from the same local analysis as the positivity of intersections.

Theorem 4.5 (Adjunction inequality)**.** *Let $u : \Sigma \to X$ be a simple holomorphic curve in an almost complex 4-manifold (X, J) representing the homology class A. If we define the* self-intersection *of u to be the number*

$$\delta(u) = \#(a, b) \in \Sigma \times \Sigma : a \neq b, u(a) = u(b)\{\}$$

then

$$\delta(u) \leq A \cdot A - \langle c_1(X), A\rangle + \chi(\Sigma)$$

When $\delta(u) = 0$ (e.g. embedded curves) this becomes an inequality relating the Chern classes of the normal bundle to Σ, $TX|_{u(\Sigma)}$ and $T\Sigma$, hence the name. It is perhaps more useful for proving embeddedness of curves in a given homology class, as one can then apply the automatic transversality theorem to them without knowing much beyond the homology data. Remember that the version of automatic transversality we've stated only holds for immersed curves and embedded implies immersed!

4.4. **Example: The quadric surface, Q.** Through any point $p \in Q$ in the quadric surface there are two complex lines α_p and β_p. These give us two homology classes α and β in $H_2(Q;\mathbb{Z})$. The first Chern class is $c_1(Q) = 2\alpha + 2\beta$. For some nonintegrable J, the expected dimension of the moduli space of holomorphic curves in these homology classes is therefore

$$2n + 2\langle c_1(Q), \alpha\rangle = 2n + 2\langle c_1(Q), \beta\rangle = 4 + 4 = 8$$

Of course this is a moduli space of maps $\mathbb{CP}^1 \to Q$, and one must remember to divide out by the group of complex automorphisms of $\mathbb{CP}^1$, i.e. the 6-dimensional Möbius group. This leaves a 2-dimensional family of complex lines. The adjunction formula tells us that these curves will necessarily be embedded:

$$\delta \leq 0 - 2 + 2 = 0$$

and since $\langle c_1(Q), \alpha\rangle = \langle c_1(Q), \beta\rangle = 2 \geq 1$, these moduli spaces are automatically unobstructed *for any J*. This is a very well-behaved situation.

4.5. **Antiexample: The Hirzebruch surface H_n.** Let H_n be the n-th Hirzebruch surface. This is the projective compactification of a degree-n line bundle on $\mathbb{CP}^1$. The section at infinity is a holomorphic curve u and u^*TH_n splits with one factor being the normal bundle $\mathcal{O}(-n)$. By our earlier observation, this cannot be regular. Notice that for n even this surface is just diffeomorphic to $S^2 \times S^2$ (and hence to the quadric surface) but the symplectic form for the quadric surface (or the product form for $S^2 \times S^2$) are not compatible with this new complex structure. Instead there are other less symmetric Kähler forms like $\omega_{S^2} \oplus \lambda\omega_{S^2}$ for certain $\lambda > 1$.

5. Gromov Compactness

Now that we have some criteria for deciding when a complex structure is regular, we want to use our moduli spaces to construct symplectic invariants. In our example above, we know that the homology classes α and β contain smooth 2-d moduli spaces of embedded holomorphic spheres. By comparison with an explicit integrable complex structure, one might expect that the holomorphic curves in these homology classes foliate the whole manifold. This will indeed be the case.

5.1. **Derivatives bounded.** Let J_t be a sequence of almost complex structures which converge smoothly to some J_∞. Let u_t be a family of J_t-holomorphic curves in a fixed homology class.

Proposition 5.1. *Let $\infty \geq p > 2$. If $\sup_{S^2}(|du_t|_{L^p}) < \infty$ for all t then there is a subsequence u_{t_j} which converges to a J_∞-holomorphic curve u_∞.*

This is an application of elliptic regularity theory and the Arzela-Ascoli theorem. Unfortunately, we have no a priori bounds on the L^p norm of the derivative for $p > 2$. We do, however, have such a bound for the L^2 norm. For a smooth map $u : \Sigma \to X$, we define the energy

$$E(u) = \frac{1}{2}\int_\Sigma |du|^2 d\mathrm{vol}_\Sigma$$

and notice the following

$$\frac{1}{2}\int_\Sigma |du|^2 d\mathrm{vol}_\Sigma = \int_\Sigma |\overline{\partial}_J(u)|^2 d\mathrm{vol}_\Sigma + \int_\Sigma u^*\omega$$

where the norms are taken with respect to the metric $g_J(\cdot,\cdot) = \omega(\cdot, J\cdot)$. This can easily be verified in local coordinates, and immediately implies that the energy of a J-holomorphic curve is just $\langle u^*[\omega], [\Sigma]\rangle$ (as the Cauchy-Riemann term dies).

The moral is that although moduli spaces of holomorphic curves are not a priori compact, they might have good compactifications.

5.2. **Derivatives unbounded.** So what happens when the derivative $|du_t|$ blows up? Assume that the domain Σ is compact, so we can always find a point $p_t \in \Sigma$ with

$$|du_t(p_t)| = \sup_{\Sigma} |du_t| =: c_t$$

In local coordinates centred at that point, let's write

$$v_t(z) = u_t(p_t + z/c_t)$$

and we notice that in a ϵ/c_t-ball around p_t, $|dv_t|$ is bounded. The compactness theorem above was actually only a local theorem, so we can deduce the existence of a J_∞-holomorphic limiting curve $v : \mathbb{C} \to X$.

This is as good as useless to us until we realise that its energy is still bounded from above. Clearly, the energy of $u|_{B_{\epsilon c_t}(p_t)}$ is bounded by the (a priori bounded) energy of u. Now conformal invariance of the energy allows us to deduce that v_t also has uniformly bounded energy.

Why is this any more useful to us? Just as in complex analysis, there is a theorem on removal of singularities for J-holomorphic curves, but it relies on finiteness of energy.

Theorem 5.2 (Removal of singularities)**.** *A J-holomorphic curve $u : \mathbb{C} \setminus 0 \to X$ in a symplectic manifold (X, ω) with finite energy extends to a smooth map $u : \mathbb{C} \to X$.*

The proof uses some minimal surface theory, derivative bounds (like Schwarz's lemma) and isoperimetric stuff.

What does this imply for the limit curve v? It means that the map $v(1/z)$: $\mathbb{C} \setminus 0 \to X$ compactifies and thus v can be thought of as a J_∞-holomorphic sphere in X, a *bubble*. This process is called bubbling and happens all over the place. It was first observed for minimal surfaces, then for instantons. It crucially relies on the conformal invariance of energy.

It's a little more work to prove the full Gromov compactness theorem:

Theorem 5.3 (Gromov compactness)**.** *The limit of a sequence of holomorphic spheres in a fixed homology class A is a cusp-curve, that is a connected union of holomorphic spheres whose homology classes add up to A.*

One of the key points here is to show that no energy is lost (i.e. the homology classes add up).

This is not the cleanest or most precise statement of Gromov's compactness theorem, but setting up such a statement requires a lot of work: constructing moduli spaces of stable maps, equipping them with the (Hausdorff, separable) Gromov topology and showing they're compact. It's worth it, but we haven't got time. See chapters 5 and 6 of McDuff and Salamon.

5.3. **Back to the quadric surface.** In the example of the quadric surface, the homology classes α and β cannot be written as sums of homology classes which contain holomorphic curves. To see this, notice that if J is ω-compatible then a J-sphere u has positive ω-area $\int_{S^2} u^*\omega > 0$. But for the quadric surface, $[\omega] = c_1(Q)$ (as the symplectic form is induced by an anticanonical embedding), so the classes α and β have minimal ω-area (2) for spherical homology classes, so they cannot be divided up into cusp-curves. This implies that the moduli spaces of holomorphic curves in the classes α and β are necessarily compact.

5.4. **Examples of bubbles.** Let's now look at a situation where cusp-curves turn up. Take $\mathbb{CP}^2$ and blow it up at a point p. Let $\{\lambda_t\}_{t\in\mathbb{P}^1}$ be the set of complex lines through a point q. The lines which do not pass through p lift to smooth holomorphic curves in the blow-up. The line through q and p lifts to a cusp-curve, consisting of its proper transform and the exceptional divisor. Therefore this moduli space is noncompact, diffeomorphic to $\mathbb{C}$, but compactifies á la Gromov to S^2 by adding in the cusp-curve. This is only S^2 considered as a topological space. There is no natural smooth structure, though in this case one can noncanonically equip it with the smooth structure of S^2 by *gluing*.

Remark 5.4. *Gluing is the process of taking a cusp curve and finding a nearby smoothing. This is fine under certain simplicity and transversality assumptions, but when one tries to use it to equip moduli spaces with smooth structures one runs into problems. The noncanonical nature of gluing means that it's well-nigh impossible to check compatibility for coordinate charts between different strata of the Gromov compactification with corners.*

6. Gromov-Witten invariants

Now that we have compactness, we can get invariants as follows. Look at moduli spaces of holomorphic curves with marked points (to be more specific, we'd have to set up the language of stable maps) and define *evaluation maps*, which take a curve $u : \Sigma \to X$ and a point $p \in \mathrm{dom}(u)$ to the point $u(p) \in X$. This defines a chain, but noncompactness of the moduli space means that we don't necessarily get a cycle this way. We want to get a homology class.

The cunning trick is to define a new homology theory whose chains are "pseudocycles", i.e. maps from noncompact manifolds (like moduli spaces) into X whose images have good compactness properties: the closure of the image of an evaluation map consists of cusp-curves by Gromov's compactness theorem. More precisely,

Definition 6.1. *A d-dimensional pseudocycle in X is a smooth map*

$$f : V \to X$$

for an oriented d-manifold V, whose image has compact closure and such that the limit set

$$\Omega_f = \bigcap_{K\subset V} \overline{f(V \setminus K)}$$

has dimension at most $d-2$. A cobordism W with $\partial W = -V_0 \cup V_1$ and a map $F : W \to X$ which interpolates between two pseudocycles is called a bordism of pseudocycles if Ω_F has dimension at most $d-1$.

The resulting homology theory is equivalent to ordinary singular homology. Now under suitable transversality conditions on the intersections of pseudocycles of complementary dimension one can define a bordism-invariant notion of intersection. It is this which allows one to construct Gromov-Witten invariants (in nice cases), where the bordisms are understood to come from generic variations of complex structure.

6.1. **Example: Lines in projective space.** In the standard complex structure, which is regular (exercise!), the moduli space of holomorphic lines in $X = \mathbb{CP}^2$ with two marked points defines a Gromov-Witten pseudocycle in $X \times X$ by its evaluation map. In fact, there is no bubbling. Noncompactness only arises when two points come together (forming a "ghost bubble" in the moduli space of stable maps - this is a subtlety we have overlooked). Nonetheless, the diagonal has codimension 2, so the intersection number of this pseudocycle with a (generic) point in $X \times X$ is 1, independently of bordism and hence independently of the almost complex structure.

Theorem 6.2. *There is a unique J-holomorphic sphere in the line class through any two distinct points of $\mathbb{CP}^2$.*

Proof. Uniqueness follows from positivity of intersections. Existence is certainly true for generic pairs of points. The noncompactness I mentioned above is an artefact of the language of stable maps: if we want a curve through p and q then take generic pairs (p_i, q_i) which do lie on holomorphic curves and such that $p_i \to p$, $q_i \to q$. The limit of (a subsequence) of these lines passes through p and q and is a smooth holomorphic curve by Gromov compactness. □

7. Holomorphic discs and Lagrangians

Recall that the Fredholm theory which gave us a Riemann-Roch formula for the dimension of our moduli spaces worked when the holomorphic discs had *totally real* boundary conditions, i.e. the boundary lies on a submanifold with no complex tangencies. If one is allowing J to vary over almost complex structures which are ω-compatible, one can only allow submanifolds which are totally real for all such J. This is (one) reason we use Lagrangian boundary conditions.

Lemma 7.1. *A Lagrangian submanifold L in (X, ω) is totally real for any ω-compatible almost complex structure.*

Proof. Suppose L had a complex tangency $\pi \subset T_pL$ for some J. Since J is ω-compatible, $\omega|_\pi > 0$, hence $\iota^*\omega$ is nonvanishing at p where $\iota : L \to X$ is the inclusion. This contradicts the fact that L is Lagrangian. □

In our discussion of Gromov compactness, we only talked about bubbling of holomorphic spheres. What would happen if the derivative was to blow up at a point on the boundary for a sequence of holomorphic discs? The same bubbling process would now produce a disc bubble with boundary on the Lagrangian. This phenomenon turns out to be codimension 1 in the moduli space and therefore contributes to the anomaly that makes Lagrangian Floer theory go wrong.

References

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