Symplectic actions of mapping class groups on representation varieties of curves

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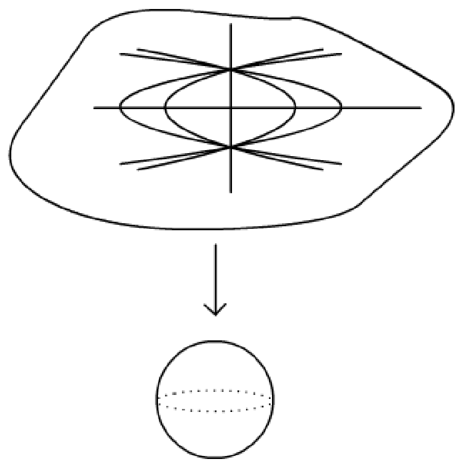
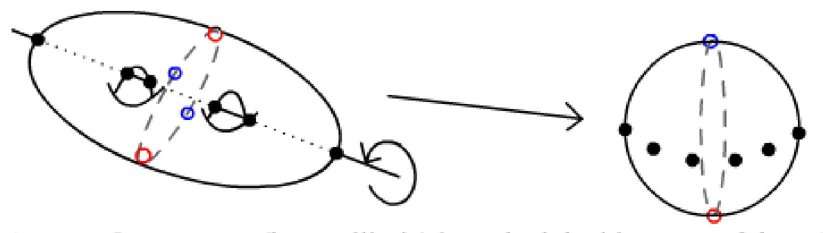
The Donaldson-Lefschetz picture



- Symplectic manifolds admit pencils of symplectic hypersurfaces, much like complex projective varieties. Some of the hypersurfaces are necessarily singular, but these singularities can be taken to be isolated nodes (represented as intersecting pairs of straight lines above).
- In algebraic geometry there is a notion of *vanishing cycle*, a homology class which is collapsed in a degenerating hypersurface, and *monodromy*, parallel transport of the homology of a hypersurface via the Gauss-Manin connection around a singular fibre.
- Symplectically, vanishing cycles are honest *Lagrangian* submanifolds (spheres, in fact) and monodromy is realised by *symplectomorphisms* of the fibres.

Simple example: branched double covers

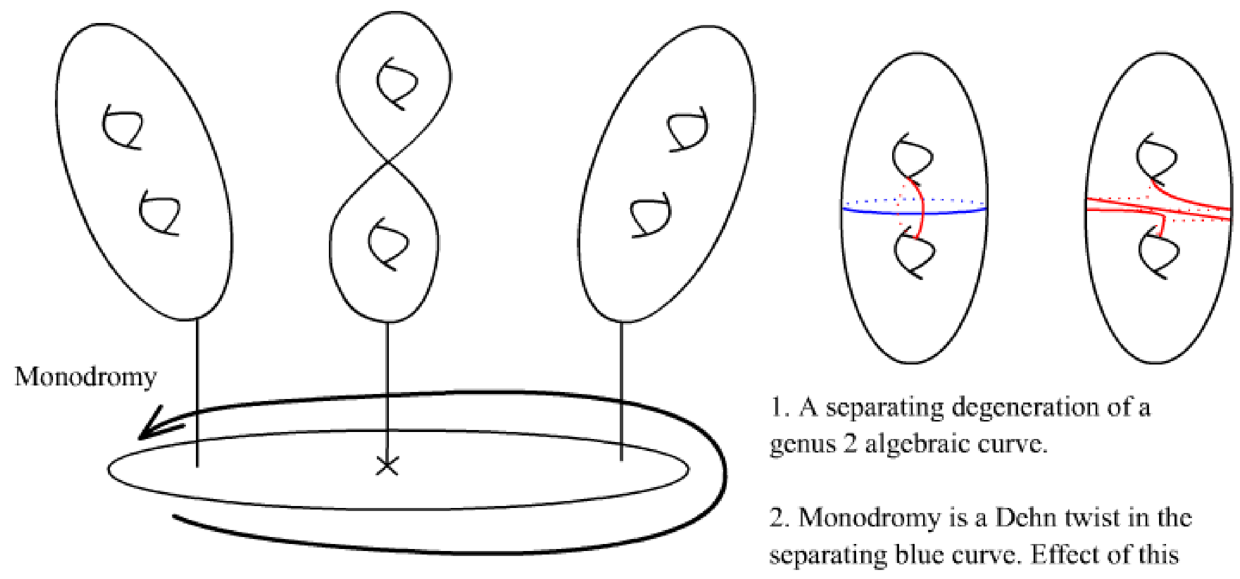
A simple example of a pencil is the branched cover of the Riemann sphere by a genus 2 Riemann surface (shown below). There are two points in a generic fibre of the map. The vanishing cycles are these pairs of points, thought of as 0-spheres, and monodromy around a critical point interchanges them, thought of as a *Dehn twist* in this vanishing cycle.



A genus 2 curve as a (hyperelliptic) branched double cover of the sphere: fibres of the cover are pairs of points. Singular fibres are *branch points*, shown in black, where the two points coalesce.

Genus 2

Complex algebraic curves have many interesting types of diffeomorphism. The group they form is called the mapping class group and is very complicated. It's relatively easy to construct degenerations of the curve realising these diffeomorphisms as monodromies: the figure below illustrates how a *separating Dehn twist* might arise.



1. A separating degeneration of a genus 2 algebraic curve.

2. Monodromy is a Dehn twist in the separating blue curve. Effect of this twist on the red curve is shown.

This diffeomorphism is particularly interesting. It induces a non-trivial automorphism of the fundamental group, but since the separating curve is nullhomologous, the twist has no effect on homology.

So far, this is all very topological. The fact that our curve is a complex algebraic variety means we have the extra structure of a symplectic form. In this low dimension this gives us virtually no extra information: the group of *symplectomorphisms* (maps preserving the symplectic structure) is homotopy equivalent to the group of diffeomorphisms. In higher dimensions, that's not true any more.

A lot of the problems I've been considering involve trying to use knowledge of diffeomorphisms on curves to get information about symplectomorphisms on higher dimensional symplectic manifolds which arise naturally in gauge theory and algebraic geometry as moduli spaces of objects (vortices, flat connections, Hilbert schemes, stable bundles) on curves.

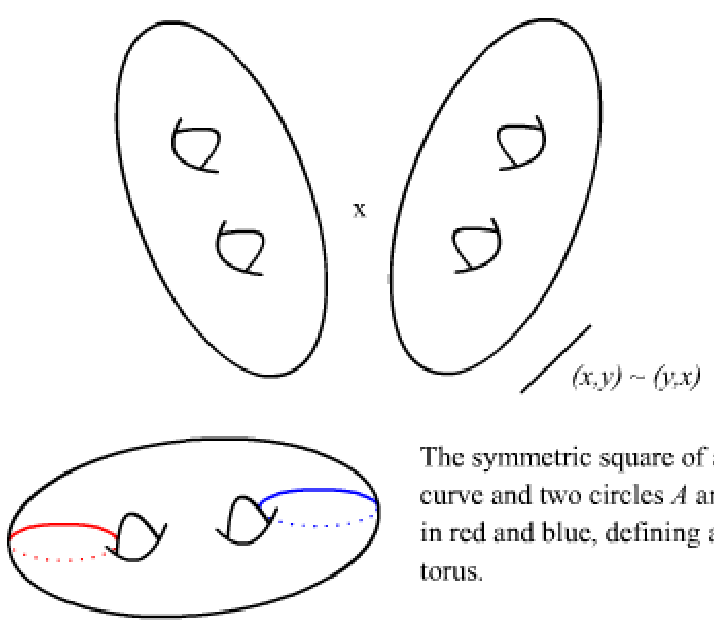
Jacobians and symmetric products

Jac(Σ_2)

We saw earlier that the separating Dehn twist is invisible on the level of homology of the curve. The Jacobian torus Jac(Σ_2) of the algebraic curve Σ_2 is an associated symplectic 4-torus which classifies representations of . That is to say, for each loop in the curve, it gives a "holonomy" - a matrix in $U(1)$ - unchanged by small deformations of the loop and which behaves well with respect to concatenation of loops.

From this description, we can identify Jac(Σ) with $\mathrm{Hom}(\pi_1(\Sigma), U(1)) \cong H^1(\Sigma, U(1))$. Since this moduli space is built out of homology, it does not see the separating Dehn twist. However, if we modify it slightly, it will.

Sym$^2(\Sigma_2)$



The symmetric square of a genus 2 curve and two circles A and B, drawn in red and blue, defining a Lagrangian torus.

The second symmetric product of Σ_2 with itself is a one-point blow-up of the Jacobian torus (this follows from the Riemann-Roch theorem applied to the Abel-Jacobi map, for those in the know).

Proposition 1. *The mapping class group of Σ_2 acts on Sym$^2\Sigma_2$ by symplectomorphisms and this action does not factor through the action of the mapping class group on the homology.*

This can be seen by the existence of homology 3-spheres (certain surgeries on the trefoil knot - see Ozsváth and Szabó (2003)) admitting genus 2 Heegaard splittings but distinguished from the honest 3-sphere by their Heegaard-Floer invariant. This invariant is derived by looking at intersections of a *Lagrangian torus* in the symmetric product with its image under the symplectomorphism. The torus arises from a suitable choice of two disjoint circles A and B on the curve and looking at points $[(x, y)]$ in the symmetric product where $x \in A$, $y \in B$.

The twisted representation variety

Another highly nontrivial moduli space comes from considering twisted representations

$$\pi_1(\Sigma_2) \to SU(2)$$

(the twisting means that the relation $[A, C][B, D] = 1$ in $\pi_1(\Sigma_2)$ is replaced by $[A, C][B, D] = -1$). This is another symplectic moduli space and the mapping class group (or some central extension of it to compensate for twisting) acts here.

Specifically for genus 2 curves, the hyperelliptic branched cover allows us to describe this space more explicitly. Newstead (1968) showed that it was isomorphic to a complete intersection of two quadrics in $\mathbb{CP}^5$:

$$\begin{aligned} x_0^2 + \ldots + x_5^2 &= 0 \\ \lambda_0 x_0^2 + \ldots + \lambda_5 x_5^2 &= 0 \end{aligned}$$

Explicitly, the λ_k which arise as coefficients here are the (black) critical values of the branched cover from the earlier diagram. Can we see the action of the separating Dehn twist (or some central lift of it) here? The standard line of attack here is to compute some *Floer homology group* associated to the symplectomorphism, invariant under Hamiltonian deformations. A localisation argument with respect to the involution

$$(x_0, x_1, x_2, x_3, x_4, x_5) \mapsto (x_0, x_1, x_2, -x_3, -x_4, -x_5)$$

computes this to have the same rank as the Floer homology group of the identity (namely 8). Our next line of attack is to localise with respect to the involution $x_5 \mapsto -x_5$, whose fixed locus is much bigger - a Del Pezzo surface. It is known (Seidel (2003)) that on this particular Del Pezzo surface, the separating twist can be seen symplectically, but his argument doesn't compute its Floer homology. If we could find a lower bound on the rank of the Floer homology group of an iterate of our separating twist, we'd know it was symplectically nontrivial.

References

1. Newstead (1968) "Stable bundles of rank 2 and odd degree over a curve of genus 2" Topology, 7.
2. Ozsváth, P. and Szabó, Z. (2003) "On the Floer homology of plumbed 3-manifolds" Geometry and Topology, 7.
3. Seidel, P. (2003) "Lectures on four-dimensional Dehn twists" preprint

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