OPEN QUESTION III:
BORRELLI-LUTTINGER SURGERY IN DIMENSIONS 8 AND 16

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In dimension 4 there is a nice surgery which preserves the class of symplectic manifolds and which has been used extensively to provide interesting examples of symplectic 4-manifolds and beautiful theorems on Lagrangian tori. The surgery takes a Lagrangian torus with a specified circle γ inside, excises the torus and reglues it via a symplectomorphism of the punctured neighbourhood. The symplectomorphism takes a meridian (the unit normal bundle) of the torus and wraps it around a push-off of γ. It is entirely analogous to Dehn surgery on knots in 3-manifolds, but because the gluing map is symplectic we know that the surgered manifold admits a symplectic form.

The surgery was introduced by Luttinger who used it to prove that Lagrangian tori in $\mathbb{C}^2$ could not be knotted in any old way. The point is that if a symplectic manifold is standard at infinity (and has trivial H_2) then it is symplectomorphic to the standard $\mathbb{C}^2$. Luttinger surgery doesn't change H_2 and doesn't change infinity, so it gives back the standard $\mathbb{C}^2$. Topologically this implies something about the embedding (vanishing of the "self-linking" of the torus and even rules out certain smooth spun knotted tori from occuring as Lagrangians).

Borrelli observed that this surgery can actually be defined when you have a Lagrangian $S^1 \times S^k$ and the S^k is parallelisable. The unit normal bundle to such a Lagrangian is $S^1 \times S^k \times S^k$ and you can write down a diffeomorphism of the unit normal bundle which preserves the isotopy class of the contact structure (and hence can be isotoped to give a symplectomorphism of the punctured neighbourhood). He used this to prove vanishing self-linking number of Lagrangian $S^1 \times S^3$ or $S^1 \times S^7$ in $\mathbb{C}^4$ or $\mathbb{C}^8$ (respectively) which works because of the Eliashberg-Floer-Gromov-McDuff theorem that a symplectic manifold symplectomorphic to $\mathbb{C}^n$ at infinity is diffeomorphic to $\mathbb{C}^n$.

Question 1. *Is it possible to produce exotic symplectic manifolds diffeomorphic (but not symplectomorphic) to $\mathbb{C}^4$ or $\mathbb{C}^8$ by performing Borrelli-Luttinger surgery on Lagrangian $S^1 \times S^k$?*

Why is this a hard question? Well if you look at how people produce and detect exotic symplectic manifolds these days (and they do so quite a lot), they use symplectic homology (an invariant of Liouville domains). The manifolds we could construct by these surgeries would be symplectomorphic to $\mathbb{C}^n$ at infinity and hence they're Liouville domains, but an argument of Ivan Smith proves that they have vanishing symplectic homology. Therefore they can't contain exact Lagrangian submanifolds and they can't be distinguished from the standard $\mathbb{C}^n$ using symplectic homology.

Note something interesting. Auroux-Donaldson-Katzarkov produce interesting knotted (cuspidal) symplectic surfaces (in $\mathbb{CP}^2$) by finding Lagrangian annuli intersecting symplectic surfaces in a pair of circles ($S^1 \times S^0$) and "braiding" the surfaces along the annulus. Take the double cover of $\mathbb{CP}^2$ branched over the surface before and after braiding. These manifolds are related by performing Luttinger surgery on a Lagrangian torus in a cover of $\mathbb{CP}^2$ branched over the initial surface.

Now in higher dimensions, the analogy looks like it should be a symplectic 4-manifold inside an 8-manifold which intersects a Lagrangian $S^1 \times S^3$ along a Lagrangian $S^1 \times S^1$. Can one perform Luttinger surgeries ambiently? What phenomena arise? There are h-principles for symplectic submanifolds of codimension 4, but it's possible that performing Luttinger surgery will drastically alter the smooth topology of the 4-manifold.

Need references for Luttinger, Borrelli, McDuff et al, Seidel, Auroux et al