SYMPLECTIC COBORDISM AND CONTACT 3-MANIFOLDS

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ABSTRACT. Certain properties of the symplectic cobordism relation for contact manifolds manifest themselves concretely in the geometry of contact 3-manifolds. I think this is illustrated brilliantly by the concave filling theorem and the tightness criterion which, when taken together, demonstrate the asymmetry of the relation of symplectic cobordism. I develop the basic theory of contact geometry in three dimensions and use it to prove the these two theorems. I conclude by making a heuristic sketch of the symplectic cobordism category.

1. INTRODUCTION

The integrability of smooth r-plane distributions is a key idea in differential geometry. In Lie theory, complete integrability gives us the correspondence between Lie subalgebras and Lie subgroups. In Riemannian geometry, the curvature tensor measures non-integrability of the Levi-Civita horizontal distribution and provides a local isometry invariant of the metric. It seems plausible that studying distributions for their own sake might be interesting and, indeed, contact geometry is the very interesting study of "maximally non-integrable" distributions of codimension 1. Essentially this means distributions with no top-dimension integral submanifolds, but for contact distributions on odd-dimensional manifolds there is a concise algebraic characterisation of this condition.

The fact that a distribution has codimension 1 means that it can be given locally as the kernel of a 1-form. I will be writing about *cooriented* contact distributions: those which are specified as the kernel of a global 1-form (the contact form) usually denoted α. The algebraic characterisation of the non-integrability condition on $2n+1$-manifolds is that

$$\alpha \wedge (d\alpha)^{\wedge n}$$

vanishes nowhere. To contrast this with the integrable case, Frobenius's theorem asserts that if the kernel of α were integrable its local sections would form a closed algebra under the Lie bracket. Thus the relation

$$d\alpha(V,W) = V\alpha(W) - W\alpha(V) - \alpha([V,W])$$

implies that $d\alpha|_{\ker\alpha} = 0$. On the other hand, our condition states that $d\alpha$ is a non-degenerate 2-form on $\ker\alpha$!

This non-degeneracy condition is reminiscent of the definition of a symplectic form and that suggests that we might look for contact distributions on codimension 1 submanifolds in symplectic manifolds. The nice thing about such "hypersurfaces of contact type" is that they provide interfaces along which to slice and glue symplectic manifolds, allowing for more adventurous symplectic surgery procedures than simple blow-ups and fibre sums.

Thinking about symplectic surgery naturally leads us to consider symplectic cobordisms. The relation of two manifolds being cobordant can be enriched by requiring that the cobordism is itself symplectic and that this symplectic structure is in some way compatible with specified contact distributions on either end. This new relation, written $\prec$, turns out to be reflexive and transitive but not symmetric. The asymmetry arises from the compatibility condition, which requires the local existence of a symplectic dilation, a vector field V for which the Lie derivative of the symplectic form satisfies $\mathfrak{L}_V\omega = \omega$. Since $\mathfrak{L}_{-V}\omega = -\omega$, there is no guarantee that we can have a symplectic dilation pointing the other way.

The geometric consequences of this asymmetry are striking and are the main subject of this essay. I will concentrate on the case of contact distributions on 3-manifolds. The two main theorems I will prove are

Theorem 1.1. *Let M be a 3-manifold and ζ a contact distribution (call the pair (M, ζ) a contact 3-manifold). Then*

- *(Concave filling theorem) There is a symplectic cobordism $(M, \zeta) \prec \emptyset$.*
- *(Tightness criterion) If there is a symplectic cobordism $\emptyset \prec (M, \zeta)$ then (M, ζ) does not contain an* overtwisted disc*, that is a Seifert surface D for an integral curve γ of ζ such that TD intersects ζ transversely along γ and such that D is a disc.*

In outline:

The first three sections discuss the contact geometry that we will use throughout the proofs of the main results. In particular, section 2 introduces basic definitions, section 3 discusses the theory of Legendrian knots and section 4 deals with overtwisted discs and contact manifolds which do not contain them.

Sections 5 and 6 develop the more advanced material that we will require on the construction of contact structures and the symplectic surgery process. Section 5 outlines an alternative useful viewpoint on contact geometry in three dimensions, the theory of open book decompositions. The two viewpoints are entirely equivalent, thanks to the beautiful work of Emmanuel Giroux, but we will only need the Thurston-Winkelnkemper construction in our subsequent arguments. Section 6 develops the methods of symplectic surgery due to Weinstein which will be our main tool in proving the main theorems.

In the last two sections, we get on to proving the main theorems. Section 7 will introduce the theory of Lefschetz fibrations and use them to prove the concave filling theorem. Section 8 will briefly recall a result from gauge theory about spheres in symplectic manifolds which is then used to deduce the tightness criterion as a corollary to the concave filling theorem.

* * *

Acknowledgements: I would like to thank my supervisor, Ivan Smith, for introducing me to symplectic and contact geometry and for pointing the way when there was too much choice. I would also like to thank Tomas Mrowka for his suggestion that I make use of the concave-filling theorem to reduce the tightness criterion to Taubes's non-vanishing result, rather than using the more intricate Seiberg-Witten theory for non-compact manifolds developed in [18].

2. Basic Contact Topology

Contact geometry studies 1-forms with non-integrable kernels. More precisely:

Definition 2.1. *Let X be a 3-manifold and α be a 1-form. If $(d\alpha)|_{\ker\alpha}$ is non-degenerate, or equivalently if $\alpha \wedge d\alpha \neq 0$, then we call α a* contact form*.*

A distribution $\zeta \subset TX$ we call a co-oriented contact structure *on X if it is the kernel of some global contact form. The pair (X, ζ) is then called a* contact manifold*.*

When considered a submanifold of the tangent bundle, a co-oriented contact distribution is oriented: everywhere there is a normal direction given by the *Reeb vector*, V, such that $\alpha(V) = 1$ and $d\alpha(V) = 0$. The contact form also defines an orientation on X itself, since $\alpha \wedge d\alpha$ is a non-vanishing 3-form. This orientation depends only on the underlying distribution. We call a contact structure ξ on an oriented 3-manifold $(X, \mathcal{O})$ *positive* if the orientation from the contact form agrees with $\mathcal{O}$.

The usual first example of a contact structure is the one given by the contact form $\alpha = dz - ydx$ on $\mathbb{R}^3$. Since $\alpha \wedge d\alpha = dx \wedge dy \wedge dz \neq 0$, this is a contact form. We will call $(\mathbb{R}^3, \ker\alpha)$ the *standard* contact structure (ζ_{std}) on $\mathbb{R}^3$.

One can see the non-integrability very explicitly here, as in figure 2.

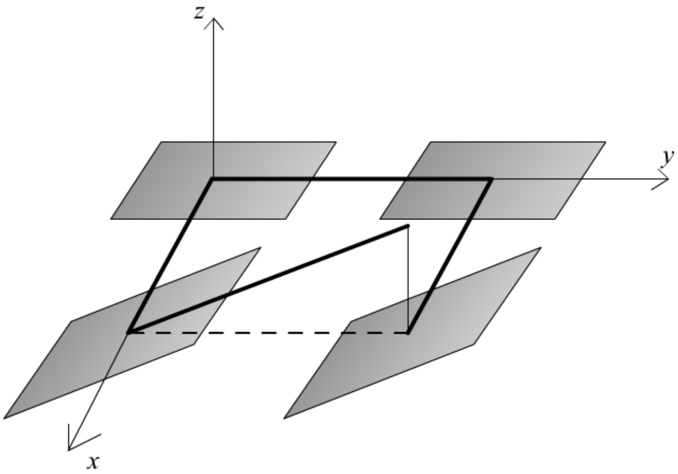


Figure 1. The nonintegrability of ζ_{std} on $\mathbb{R}^3$. Constrained to being tangent to the contact planes, we cannot embed an arbitrarily small square patch of $\mathbb{R}^2$.

Moreover, this is the local picture anywhere in a contact 3-manifold. This is a consequence of a neighbourhood theorem in contact geometry analogous to Darboux's theorem in symplectic geometry.

Theorem 2.2 (Darboux)**.** *Any point p in a contact 3-manifold (M, ξ) has a neighbourhood U which is diffeomorphic to $(\mathbb{R}^3, \zeta_{std})$ via a diffeomorphism ψ with $\psi_*\xi = \zeta_{std}$. Such a neighbourhood is called a* Darboux ball*.*

A diffeomorphism which pushes forward the contact structure on one manifold into the structure on its range is called a contactomorphism. A homotopy through contactomorphisms is called a contact isotopy.

Remark: A contactomorphism $f : (M, \xi) \to (M', \xi')$ may not necessarily pullback the contact form α' for ξ' to the contact form α for ξ. However, $f^*\alpha' = g\alpha$ for some non-vanishing function g (write $v \in T_pM$ as $v + w$ with $v \in \ker\alpha$ and then $f^*\alpha'(v + w) = \alpha'(f_*v + f_*w) = \alpha'(f_*w)$ which is a multiple of $\alpha(w)$ since $\xi'_{f(p)}$ has codimension 1 in $T_{f(p)}M'$).

3. LEGENDRIAN KNOTS

3.1. **Front projection.** A knot K in a contact 3-manifold (M, ξ) is called Legendrian if $TK \subset \xi$.

Example 3.1 ($S^1 \times \mathbb{R}^2$)**.** *Consider the contact structure on* $S^1 \times \mathbb{R}^2$ *given by the contact form* $\cos(2\pi\phi)dx - \sin(2\pi\phi)dy$. *The core circle,* $S^1 \times \{0\}$ *is Legendrian.*

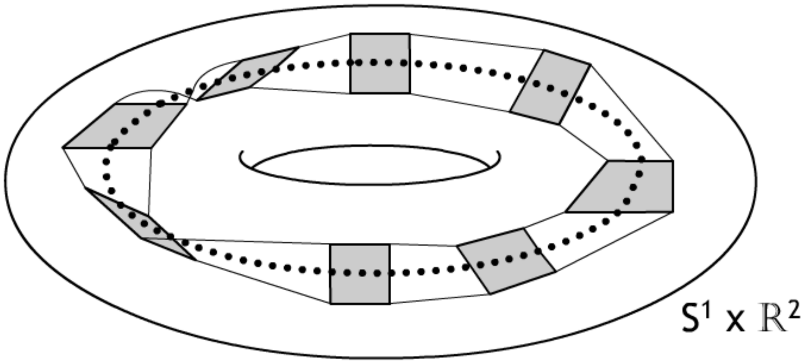
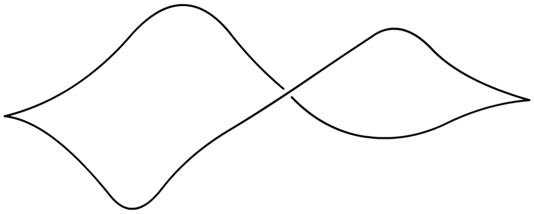


FIGURE 2. The Legendrian core circle in $S^1 \times \mathbb{R}^2$. Notice that the contact plane distribution twists once relative to the horizontal as it passes around the loop.

Consider now a Legendrian knot, $K : [0, 1] \to \mathbb{R}^3$, in $(\mathbb{R}^3, \zeta_{\mathrm{std}})$. The *front projection* of K is just its projection to the yz-plane, Π_{yz}. Since $dz - xdy$ is a contact form and the tangent field to K is contained in the contact distribution,

$$\frac{\dot{z}}{\dot{y}} = \frac{dz}{dy} = x \tag{1}$$

and hence the knot is never tangent to the vertical direction. If we assume our knot is generically embedded (that is, $\dot{y} = 0$ only at isolated points (*cusps*) where $\ddot{y} \neq 0$) then locally, near a cusp, we can reparametrise so that the cusp corresponds to $t = 0$ and $x(t)$ is linear in t. Then $y(t)$ is $\mathcal{C}^2$-close to a quadratic expression $y(0) + t^2\ddot{y}(0)$ and we can solve the differential equation 1 to see that near a cusp, K is $\mathcal{C}^2$ close to a curve of the form $(x(t), y(t), z(t))$ where x, y and z are respectively linear, quadratic and cubic in t. As one might expect, any Legendrian knot is smoothly isotopic to a generic one, so we are justified in drawing the following picture.



Notice that when two branches S_1 and S_2 of the front-projection cross at $P \in \Pi_{yz}$ with slopes $dz_1/dy_1(P) = s_1 < s_2 = dz_2/dy_2(P)$ respectively, S_1 must lie behind S_2 since by equation 1, $x_1 < x_2$.

Any picture satisfying the conditions that (i) there are no vertical tangencies, (ii) there are only semi-cubical cusps and (iii) at any self-crossing, the branches are ordered in the necessary way, will give a generic Legendrian knot: to find the x-coordinate, simply integrate equation 1.

Using this, we can isotope any *smooth* knot to a Legendrian knot. The yz-projection of a generic smooth knot can be continuously isotoped to satisfy i), ii) and iii) by replacing vertical tangencies with cusps and by rearranging bad crossing points so that the segment with greater slope passes in front of the other. This latter process may introduce new vertical tangencies, so isotoping these to cusps will give us an isotopy of the smooth knot to a Legendrian knot.

3.2. **Framings.** Let K be a knot in a 3-manifold, M. The normal bundle νK of K is homeomorphic to $S^1 \times D^2$ and a choice of trivialising homeomorphism $S^1 \times D^2 \to \nu K$ is called a framing of K. Thinking of νK as a complex line bundle it is clear that (up to homotopy) specifying a framing on K is equivalent to specifying a non-vanishing section σ of the normal bundle (or, equivalently, a section of the unit-circle subbundle $C(\nu K) \subset \nu K$). So we can classify framings of K up to homotopy as elements of $\pi_1(S^1) \cong \mathbb{Z}$.

Two framings σ_1 and σ_2 can be compared by looking at the homotopy class in $\pi_1(S^1)$ of the section $\sigma_1(x)\sigma_2(x)^{-1} : K \to C(\nu K)$. This corresponds to an integer, which we denote $\sigma_1 - \sigma_2$. We now list some canonical framings that arise for knots in contact 3-manifolds.

- *The Seifert framing* is a canonical framing for any knot K whose homology class is zero. Let Σ be a Seifert surface for K, that is, an embedded surface with $\partial\Sigma = K$. Then the vector field along K which points into Σ at each point defines a non-vanishing section of the normal bundle. To see that this framing is independent of the choice of Σ, suppose there were two Seifert surfaces Σ_1 and Σ_2. Form the space $\Sigma_1 \cup_{K\times\{0\}} K \times [0,1] \cup_{K\times\{1\}} \Sigma_2 \subset M \times I$ and smooth the corners. Endow this space with a metric which is flat along $K \times (0,1)$, so parallel transport from $K \times \{0\}$ to $K \times \{1\}$ along the obvious fibres yields an isomorphism of framings.
- *The contact framing* for a Legendrian knot, K, is the framing given by choosing a vector in the complement for the tangent vector of K in the contact 2-plane at each point. Note that since the contact structure is oriented (it is the kernel of a global 1-form) there is no ambiguity here.

We can define the *Thurston-Bennequin invariant* of a Legendrian knot to be the integer $\mathrm{tb}(K) = \sigma_{\text{contact}} - \sigma_{\text{Seifert}}$. This can be computed from the front-projection for a knot in $\mathbb{R}^3$. The following example will be of use in proving the tightness criterion.

Example 3.2. *Consider a Legendrian unknot, K, in a contact manifold (M, ξ) with $tb(K) = 0$ (such a K is called overtwisted) and push K off itself in the contact framing to get a second copy K'. In a small Darboux ball, B, consider the front projection of $(K \cup K') \cap B$. We can form the connected sum of these two knots (doing surgery localised in B) such that the result is a Legendrian knot. From considering the front projection in B we can calculate the change in the Thurston-Bennequin invariant. The effect is to add two cusps, and this increases the linking number of the contact push-off of $K\#K'$ with $K\#K'$ by 1 (see diagram). Taking the connected sum with another contact push-off of K with give a Legendrian unknot with Thurston-Bennequin invariant 2.*

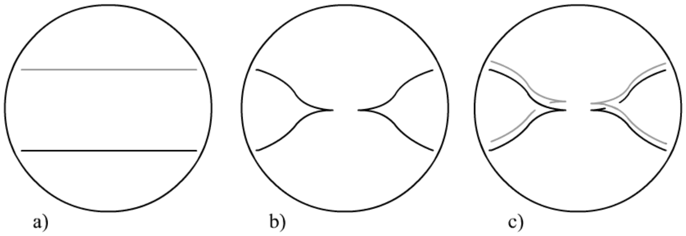


FIGURE 3. The connected sum of a knot and its contact pushoff performed inside a Darboux ball, a) the knot and its pushoff, b) the connected sum as a Legendrian knot, c) the contact pushoff of the connected sum.

4. TIGHT AND OVERTWISTED STRUCTURES

4.1. **Overtwisted discs.**

An important class of Legendrian knots are those Legendrian unknots with $\mathrm{tb}(K) = 0$. A Seifert surface for such a knot which is transverse to the contact distribution except at one point is called an *overtwisted disc* and any manifold which admits one of these is called overtwisted.

One nice way of visualising overtwisted discs uses the *characteristic foliation.* Note that for generic surfaces $\Sigma \subset (M, \xi)$ the intersection $T\Sigma \cap \xi$ is one-dimensional outside a finite set where it is zero-dimensional. The integral curves of this distribution give a foliation of Σ called the characteristic foliation. The characteristic foliation of a generic overtwisted disc therefore has exactly one singularity in its interior and the leaves approach the Legendrian boundary as a limit cycle (see figure 4.1).

It is a strong condition on the global geometry of a contact 3-manifold to be overtwisted, so strong that overtwisted contact structures are completely classified:

Theorem 4.1 (Eliashberg [4]). *Let ζ be a 2-plane distribution on a 3-manifold, M. Then there is a unique isotopy class of overtwisted contact structures on M homotopic to ζ.*

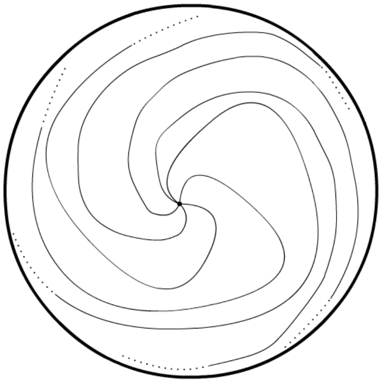


FIGURE 4. The characteristic foliation on a generic overtwisted disc. There is exactly one singularity and the boundary is a Legendrian unknot.

A contact structure which is not overtwisted is said to be *tight*. Bennequin [2] showed that the standard contact structure on $\mathbb{R}^3$ is tight. His proof uses an inequality for Legendrian knots in $(\mathbb{R}^3, \zeta_{\text{std}})$ which Eliashberg later extended to all tight contact 3-manifolds,

Theorem 4.2 (Eliashberg [6])**.**

$$tb(K) \leq -\chi(\Sigma)$$

for a Legendrian knot K with Seifert surface Σ in a tight contact 3-manifold, where $\chi(\Sigma)$ is the Euler characteristic of Σ.

(In fact this is only a weak version of the inequality, but it is all we will need). Clearly this inequality cannot hold for every Legendrian knot in an overtwisted manifold, since an overtwisted disc is a $\chi = 1$-Seifert surface for the Legendrian boundary K and $\text{tb}(K) = 0$.

4.2. **Tight structures on B^3 and S^3.** Tight structures are still poorly understood in general, though for the 3-sphere the situation is completely worked out.

Theorem 4.3. *There is only one isotopy class of tight contact structures on S^3.*

To deduce this we need a result of Eliashberg [6],

Theorem 4.4. *If ξ_0 and ξ_1 are tight contact structures on B^3 which agree on ∂B^3 then they are isotopic via an isotopy which fixes ∂B^3.*

The proof proceeds by noting that ξ_0 and ξ_1 are isotopic on a small Darboux ball U and by extending this isotopy via *taming functions* over the annulus $B^3 \setminus U$. These methods would take too long to discuss in full, so we omit the proof. However, uniqueness of tight contact structures on S^3 follows as a simple corollary:

Proof of theorem 4.3. Let ξ_0 and ξ_1 be tight contact structures on S^3. Pick a Darboux neighbourhood, U, of a point in S^3. On this ball we know the contact structures are isotopic to the standard one, so without loss of generality we can assume they agree on U. Then $S^3 \setminus U$ is a 3-ball and $\xi_0|_{\partial U} = \xi_1|_{\partial U}$. □

In later work, we will need the

Corollary 4.5. *Two tight contact structures* ξ_0 *and* ξ_1 *on a genus g handlebody which agree on the boundary are isotopic.*

Although this is a direct corollary to 4.4 the theory of convex surfaces is required to see it. It is also true that

Theorem 4.6 (Eliashberg [5])**.** *There is a unique isotopy class of symplectically fillable contact structures on* S^3*.*

Symplectically fillable structures will be defined in section 6.2. This theorem is significantly harder to prove and uses holomorphic curve techniques. See (corollary 5.3 [5]). These preliminary results form the basis of our proof of the concave filling theorem and the tightness criterion.

5. Open book decompositions

5.1. **Monodromy.** Let us consider oriented fibre bundles over S^1. Since $S^1 = [0,1]/(0 \sim 1)$ and $[0,1]$ is contractible, any oriented fibre bundle $F \to E \to S^1$ is isomorphic to $F \times [0,1]/(x,0) \sim (h(x),1)$ for some $h \in \mathrm{Diff}^+(F)$. The isotopy class of h (the *monodromy* of the bundle) determines and is determined by the isomorphism class of $E \to S^1$.

In the special case of surface bundles the monodromy lies in the *mapping class group*, M_Σ, of isotopy classes of orientation-preserving diffeomorphisms of Σ fixing $\partial\Sigma$.

Lemma 5.1 (Dehn-Lickorish[20])**.** *When* $\partial\Sigma = \emptyset$*, the mapping class group* M_Σ *is generated by* right-handed Dehn twists *on non-separating simple closed curves in* Σ*.*

A Dehn twist along a simple closed curve C in an oriented surface is a localised diffeomorphism supported near a tubular neighbourhood $\nu C \cong S^1 \times [0,1]$. On $S^1 \times [0,1]$ it takes the form

$$(e^{i2\pi\theta}, t) \mapsto (e^{i2\pi(\theta+\eta t)}, t)$$

where $\eta = \pm 1$. The twist is *right-handed* if $\eta = +1$ and *left-handed* if $\eta = -1$.

5.2. **The Thurston-Winkelnkemper construction.**

Definition 5.2. *An open book decomposition* $\mathcal{B}(\mathbb{K}, F, f)$ *of a 3-manifold,* M*, consists of a link* $\mathbb{K}$ *(the* binding*) and a fibre bundle* $\mathbb{K} \to M \setminus \mathbb{K} \xrightarrow{f} S^1$ *where each fibre (or* page*) is the interior of a Seifert surface for* $\mathbb{K}$*.*

If a link $L \subset S^3$ is the binding for an open book decomposition then it is called a *fibred link*. I will specialise to the case of a connected binding. As a fibre bundle over S^1, this is isomorphic to $F(h) = F \times [0,1]/\sim$ where $(s,0) \sim (h(s),1)$ for some monodromy $h \in M_F$. Note that we may consider $M = F(h) \cup \partial F \times D^2$ by excising a tubular neighbourhood of the binding.

Example 5.3. *Consider* $S^3 \setminus \{\infty\}$ *as* $\mathbb{R}^3$ *via stereographic projection and let* K *be the y-axis. There is an open book decomposition as shown in figure 5.2 whose fibres are Seifert discs for* K *(half-planes).*

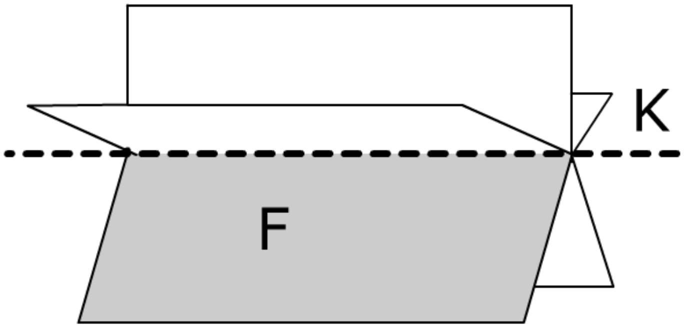


FIGURE 5. The simplest open book decomposition of S^3.

Definition 5.4. *A contact structure ξ on a 3-manifold, M, is compatible with an open book decomposition $\mathcal{B}(\mathbb{K}, F, f)$ of M if there is a contact 1-form α for which the Reeb vector field of ξ is transverse to the pages and tangent to the binding.*

Equivalently there must be a contact form, α, for ξ, such that $d\alpha$ restricts to a positive volume form on the pages and $\alpha|_{T\mathbb{K}} > 0$.

Theorem 5.5 (Thurston-Winkelnkemper, [26]). *Let $\mathcal{B}(\mathbb{K}, F, f)$ be an open book decomposition of a 3-manifold M. Then M admits a contact structure which is compatible with $\mathcal{B}$.*

Proof. Let h be the monodromy of $\mathcal{B}$. We assume that h fixes a collar neighbourhood $C = \partial F \times [0, 1)$ of F and we consider coordinates (θ, t) in this neighbourhood.

Let $\mathcal{S}$ be the set of 1-forms $\alpha \in \Gamma(T^*F)$ such that $d\alpha$ is a volume form on F and $\alpha|_U = (1 + t)d\theta$ for some neighbourhood $\partial F \subset U \subset C$. This set is clearly convex.

To see that $\mathcal{S}$ is non-empty: extend the form $dt \wedge d\theta$ on C to a volume form on the whole of F with total volume 1 and extend the 1-form $td\theta$ on C to a 1-form ψ on F. Since $\int_F \Omega - d\psi = 1 - \int_{\partial F} d\theta = 0$ (by Stokes's theorem) then $[\Omega - d\psi] = 0 \in H^2_{dR}(F)$. Hence there is a 1-form γ with $d\gamma = \Omega - d\theta$ and since both Ω and $d\psi$ vanish on C, we can assume that γ does also. Now $\psi + \gamma \in \mathcal{S}$.

Since the monodromy fixes C, $h^*\alpha \in \mathcal{S}$ for $\alpha \in \mathcal{S}$. Thus for every $\tau \in [0, 1]$ and $\alpha \in \mathcal{S}$ the 1-form $\tau\alpha + (1 - \tau)h^*\alpha$ is in $\mathcal{S}$ and clearly descends to a 1-form η on the bundle $F(h) = F \times [0, 1]/((x, 0) \sim (h(x), 1))$. In fact, $\eta = (1 + t)d\theta$ in a neighbourhood of $\partial F \times S^1$.

Consider the 1-form $f^*d\tau$ i.e. the pullback of the form $d\tau$ on S^1 via the bundle projection f. Since $d\eta$ is a volume form on each fibre, $f^*d\tau \wedge d\eta$ is a non-degenerate 3-form on $F(h)$. Now

$$\alpha = \eta + Kf^*d\tau$$

will be a contact 1-form on $F(h)$ for sufficiently large K by compactness of $F(h)$ and the equation

$$\alpha \wedge d\alpha = \eta \wedge d\eta + Kf^*d\tau \wedge d\eta$$

Finally, we extend α to a contact 1-form $\tilde{\alpha}$ on $\partial F \times D^2(1)$. If $S^1 \times C$ is the collar neighbourhood of $\partial F(h)$ in $F(h)$ then $S^1 \times C \cup \partial F \times D^2(1) \cong \partial F \times D^2(2)$. In coordinates (r, ϕ) on $D^2(2)$ and θ on $\partial F = S^1$, the form $\alpha = rd\theta + Kd\phi$ on $S^1 \times C$. Near $r = 0$ there is a contact form $d\theta + r^2 d\phi$. Interpolate between these

$$\tilde{\alpha} = f_1(r)d\theta + f_2(r)d\phi$$

using smooth functions f_1, f_2 where

$$(f_1(r), f_2(r)) = \begin{cases} (-1, r^2) \text{ for } r \leq 1/2 \\ (r, K) \text{ on } r \geq 1 \end{cases}$$

and the functions are chosen so that $f_1'(r) < 0, f_2'(r) > 0$ on $r \in [1/2, 1]$. This ensures that

$$\alpha \wedge d\alpha = (f_1 f_2' - f_2 \wedge f_1')d\theta \wedge dr \wedge d\phi > 0$$

so that α is a contact form. It is easy to check compatibility. □

In fact, the contact structure given by the above construction is unique up to isotopy.

Theorem 5.6 (Giroux, [14]). *Let M be a 3-manifold and (for $i = 1, 2$) let $\xi_i = \ker \alpha_i$ be contact structures on M compatible with some open book decomposition $\mathcal{B}(\mathbb{K}, F, f)$. Then ξ_1 and ξ_2 are contact-isotopic.*

Proof. Let $\nu\mathbb{K} \cong D^2 \times S^1$ be a tubular neighbourhood of $\mathbb{K}$ with coordinates $(r, \phi) \times (\theta)$. Consider $f^* d\tau$ on $M \setminus \nu\mathbb{K}$ as in the previous proof and extend it to a form β on the rest of M as $f(r)d\phi$ for some smooth $f(r)$ with $f(0) = 0$, $f(1) = 1$. For large enough t,

$$\beta_{s,t} = (1 - s)(\alpha_0 + t\beta) + s(\alpha_1 + t\beta)$$

is a contact form and provides the required contact isotopy. □

5.2.1. *Example: Plumbing Hopf bands.* Recall that a torus knot is a knot in S^3 that can be isotoped to lie in the standard torus $S^1 \times S^1 \subset S^3$.

Proposition 5.7. *Any torus knot is the binding for some open book decomposition of S^3 whose monodromy is a product of right-handed Dehn twists.*

Proof. Consider the surface Σ_1 in the diagram. In fact it is a minimal Seifert surface for the Hopf link H. There is an open book decomposition for S^3 with binding H and Σ_1 as its fibre. The monodromy of this open book is a right-handed Dehn twist ϕ_{Σ_1} on the core circle.

Given two Hopf bands Σ_1 and Σ_2, the plumbing operation identifies a neighbourhood U_α of the arc $\alpha \subset \Sigma_1$ with a neighbourhood U_β of the arc $\beta \subset \sigma_2$. After smoothing corners, this produces a Seifert surface for a $(2, 3)$-torus knot K.

Thanks to Gabai's results [11] on the naturality of the Murasugi sum (of which this plumbing is a special case) we know that the resulting surface is the fibre of an open book $\mathcal{B}$ with binding K and monodromy $\phi_{\Sigma_2} \circ \phi_{\Sigma_1}$. We may inductively form all $(2, q)$-torus knots by further plumbings and these observations carry over: such knots are fibred by an open book $\mathcal{B}$ whose fibre is the plumbed Seifert surface, and the monodromy of $\mathcal{B}$ is a product of right-handed Dehn twists.

By plumbing in a different way we obtain all $(p, 2)$-torus knots, and plumbing the resulting surfaces will give the result. □

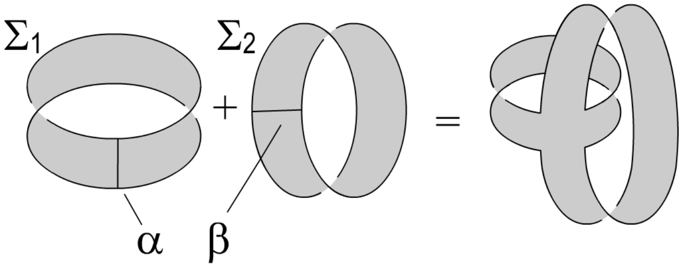


Figure 6. Plumbing two Hopf bands to construct an open book decomposition of S^3 with binding a $(2,3)$-torus knot.

6. Contact Surgery

6.1. **Legendrian neighbourhoods.** 2-handle surgery on 3-manifolds involves cutting out solid tori and gluing them back in using a diffeomorphism of the boundary $S^1 \times S^1$. In order for contact structures to extend over surgered manifolds we need to have a standard model for the contact structure on these tori. If we consider surgery along tori whose cores are Legendrian knots, the following proposition provides such a local model.

Proposition 6.1 (Legendrian neighbourhood theorem)**.** *Let* (M,ξ) *be a contact manifold with contact form* α *and let* $K \subset M$ *be a Legendrian knot. If* $\phi : K \to S^1$ *is a diffeomorphism then there are neighbourhoods* $K \subset U \subset M$ *and* $S^1 \times \{0\} \subset V \subset S^1 \times \mathbb{R}^2$ *and a contactomorphism* $\tilde{\phi} : U \to V$ *with* $\tilde{\phi}|_K = \phi$.

Proof. The proof, as with most neighbourhood theorems, is an application of Moser's argument and can be found in [27], [13] or [23]. ☐

6.2. **Symplectic gluing.**

Definition 6.2. *Let* (X,ω) *be a symplectic manifold. A vector field* $V \in \Gamma(TX)$ *is a* symplectic dilation *if* $\mathfrak{L}_V\omega = \omega$.

A hypersurface, H, in a symplectic 4-manifold X is *of contact type* if there is a neighbourhood $U \supset H$ and a symplectic dilation defined on U which is transverse to H. By Cartan's formula for the Lie derivative on forms, $\mathfrak{L}_V\omega = d(\omega(V,-))$. Hence $\omega|_{TU}$ is exact and $\alpha := \omega(V,-)|_{TH}$ is a contact form.

Lemma 6.3. *Let* V *be a globally defined symplectic dilation on a symplectic manifold* (X,ω). *Then* $\mathfrak{L}_V\alpha = \alpha$.

Proof. By Cartan's formula

$$\mathfrak{L}_V\alpha = d(\alpha(V)) + d\alpha(V,-)$$

and $\alpha(V) = \omega(V,V) = 0$. By definition $d\alpha = \omega$ and hence $\mathfrak{L}_V\alpha = \omega(V,-) = \alpha$. ☐

If ϕ_t is the flow of a symplectic dilation then this tells us that $\phi_t^*\alpha$ is a conformal rescaling of α, which leaves the contact distribution unchanged.

Corollary 6.4. *Let* H *be a hypersurface of contact type for a globally defined symplectic dilation* V *on* (X,ω). *Let* ϕ_t *be the flow of* V. *Then* $\phi_t|_H : H \to H_t$ *is a contactomorphism between hypersurfaces of contact type.*

If $H = \partial U$ for some codimension-0 submanifold $U \subset X$ and V points out of U, we say $(H, \ker \alpha)$ is strongly compatible with ω.

Definition 6.5. *A symplectic cobordism between* (M_1, ξ_1) *and* (M_2, ξ_2) *is a symplectic manifold with boundary* $-M_1 \cup M_2$ *such that the contact structure* $\xi_1 \cup \xi_2$ *is strongly compatible with* ω*. We call* M_1 *the* concave end *and* M_2 *the* convex end *of the cobordism.*

Here is an example of a symplectic cobordism from $\emptyset$ to S^3 with its tight contact structure. Such cobordisms are called symplectic fillings and their convex ends are called symplectically fillable contact manifolds.

Example 6.6. *The 3-sphere* $S^3 = \{(x_1, ..., x_4) : \sum_{i=1}^4 x_i^2 = 1\} \subset \mathbb{R}^4$ *has a contact form*

$$\alpha(x_1, x_2, x_3, x_4) = -x_3 dx_1 - x_4 dx_2 + x_1 dx_3 + x_2 dx_4$$

It bounds the region $B^4 = \{(x_1, ..., x_4) : \sum_{i=1}^4 x_i^2 \leq 1\} \subset \mathbb{R}^4$*, which supports a symplectic 2-form*

$$\omega = 2dx_1 \wedge dx_3 + 2dx_2 \wedge dx_4 = d\alpha$$

The symplectic dilation is

$$\frac{1}{2} \sum_{i=1}^{4} x_i \frac{\partial}{\partial x_i}$$

The symplectic cobordism category of contact 3-manifolds is the category whose objects are contactomorphism classes of positively oriented contact 3-manifolds (M, ξ) and whose morphisms $(M_1, \xi_1) \prec (M_2, \xi_2)$ are symplectomorphism classes of symplectic cobordisms from representatives of M_1 to representatives of M_2.

Proposition 6.7. *The relation of symplectic cobordism is reflexive and transitive.*

Proof.

- *Identity:* Any contact 3-manifold (M, ξ) admits a *symplectisation*, namely the symplectic manifold $(M \times \{t \in (0, \infty)\}, d(t\alpha))$ for some contact form α on (M, ξ). The sections $M_t = M \times \{t\}$ are now hypersurfaces of contact type and since the flow of the symplectic dilation d/dt provides contactomorphisms $M_t \cong M_{t+\epsilon}$ we get a symplectic cobordism from the contactomorphism class of (M, ξ) to itself.
- *Composition:* See the gluing lemma below.

□

Lemma 6.8 (Gluing lemma)**.** *Let*

- (X_1, ω_1), (X_2, ω_2) *be closed symplectic 4-manifolds,*
- $U_1 \subset X_1$, $U_2 \subset X_2$ *be compact 4-dimensional submanifolds with* ω_1*-,* ω_2*-compatible contact boundaries* $(\partial U_1, \xi_1)$*,* $(\partial U_2, \xi_2)$ *respectively,*
- $\phi : (\partial U_1, \xi_1) \to (\partial U_2, \xi_2)$ *be a contactomorphism.*

Then the union $U_1 \cup_\phi X_2 \setminus U_2$ *is a symplectic 4-manifold.*

Proof. We essentially follow [Et1]. Let V_i be the symplectic dilation and $\alpha_i = \omega_i(V_i, -)$ the corresponding contact form for $(\partial U_i, \xi_i)$. Define S to be the symplectisation of $(\partial U_2, \xi_2)$. Since $\phi^*(\alpha_2) = f\alpha_1$ for some $f > 0$, we can embed $(\partial U_1, \xi_1)$ in S as a graph:

$$\psi_1 : x \mapsto (\phi(x), f(x))$$

By compactness, f is bounded. Without loss of generality we take f to be everywhere less than 1. We embed ∂U_2 in S via $\psi_2 : x \mapsto (x, 1)$.

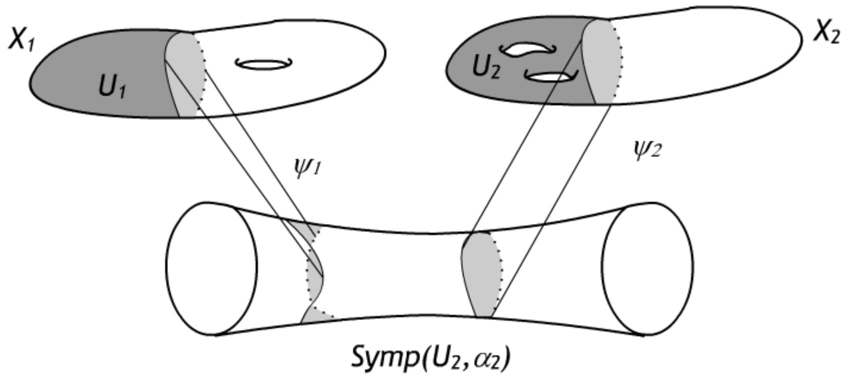


FIGURE 7. Gluing symplectic manifolds along hypersurfaces of contact type.

The idea now is to find neighbourhoods $N_i \subset X_i$ of ∂U_i which we can identify symplectomorphically with neighbourhoods $N_i' \subset S$ of $\psi_i(\partial U_i)$. I sketch the construction of the neighbourhoods N_2 and N_2'. There is a similar construction for N_1 and N_1'.

There are diffeomorphic tubular neighbourhoods, N_2 and N_2', of ∂U_2 and $\psi_2(\partial U_2)$ respectively. Say these are both diffeomorphic to T, that is

$$N_2 \overset{g_0}{\overset{\sim}{\to}} T \overset{g_1}{\overset{\sim}{\leftarrow}} N_2'$$

with $g_1|_{\psi_2(\partial U_2)} \circ \psi_2 = g_0|_{\partial U_2}$. Set $Y = g_0(\partial U_2)$. Since V_2 and $\partial/\partial t$ are transverse to ∂U_2 and $\psi_2(\partial U_2)$ respectively we may assume that $g_{0*}V_2 = g_{1*}\partial/\partial t = V$.

The symplectic forms $\eta_0 = g_0^{-1*}\omega_2$ and $\eta_1 = g_1^{-1*}d(t\alpha_2)$ are cohomologous (since they are both exact) and they agree on Y. This allows us to interpolate linearly through symplectic forms between them on a neighbourhood B of Y and by Moser's stability theorem, $g_0 \circ g_1^{-1}|_{g_1(B)}$ is isotopic to a symplectomorphism.

If Q is the subset $\{(y,t) \in S : t \in [f(y), 1]\} \cup N_1' \cup N_2'$, we can glue $U_1 \cup N_1$, Q and $(X_2 \setminus U_2) \cup N_2$ via the corresponding symplectomorphisms to obtain a global symplectic structure.

$\square$

6.3. **Surgery.** Let $(M_1, \xi_1) \prec (M_2, \xi_2)$ be a symplectic cobordism (realised by a symplectic manifold Z) and L a Legendrian knot in M_2. If $\sigma : S^1 \times D^2 \to \nu K$ is a framing for K then we can attach a copy of $D^4 = D^2 \times D^2$ to Z by identifying one half of the boundary $\partial D^4 = S^1 \times D^2 \cup D^2 \times S^1$ with νK via the framing σ.

This process is called surgery and can be done on any knot in any 3-manifold to yield a cobordism between M_2 and the surgered $M_2' = M_2 \setminus \nu K \cup D^2 \times S^1$. Since we are working in the symplectic cobordism category we want to be able to extend our symplectic structure over the cobordism obtained by surgery and to do this we need constraints on the framing, resulting in *Weinstein surgery.*

Proposition 6.9 (Weinstein [27]). *If $\sigma - \sigma_{contact} = -1$ then the symplectic structure on Z extends naturally to the union $Z \cup_{\sigma} D^4$, and the contact structure on (M_2, ξ_2) yields a contact structure ξ_2' on $M_2' = M_2 \setminus \nu K \cup D^2 \times S^1$. This gives a symplectic cobordism $(M_1, \xi_1) \prec (M_2', \xi_2')$.*

Proof. Consider $\mathbb{R}^4$ with the standard symplectic structure and the global symplectic dilation $V = -\nabla f$ where $f = x_1^2 + x_3^2 - \frac{1}{2}(x_2^2 + x_4^2)$. The hypersurface $X_+ = f^{-1}(1)$ is transverse to V and hence of contact type. The circle $S = \{x_2 = x_4 = 0\} \cap X_+$ is Legendrian. Consider a neighbourhood $\nu S \subset X_+$ and let Σ be a hypersurface diffeomorphic to $S^1 \times D^2$ which is transverse to V and such that $\nu S \cup \Sigma \cong S^3$ is closed.

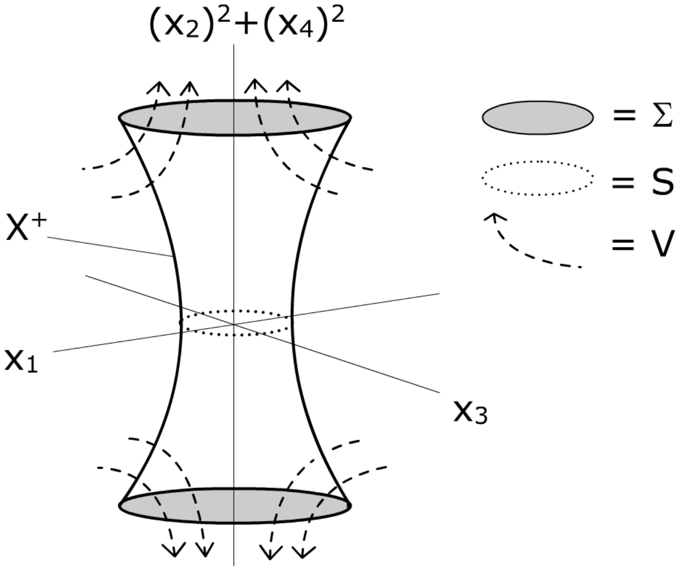


FIGURE 8. The symplectic 2-handle we are adding to X.

By the Legendrian neighbourhood theorem, K and S have contactomorphic neighbourhoods in M_2 and S^3 respectively. The gluing lemma then tells us that using the framing given by moving S off itself in (say) the x_2-direction we can glue in a symplectic manifold. It is clear from 3.1 that this a -1-framing relative to σ_{contact}, as the contact plane twists precisely once around $S^1 \times \{0\}$ relative to the constant (say horizontal) framing. □

If we want to perform surgery on Legendrian knots with a framing $+1$ relative to σ_{contact} then we will still obtain a contact manifold, even though the resulting cobordism does not necessarily support a compatible symplectic structure. The handle we use for this is that shown in figure 6.3.

Notice that in both these cases the boundary of Σ is of contact type and so the contact structure on $M_2 \setminus \nu K$ extends over $M_2 \setminus \nu K \cup \Sigma$.

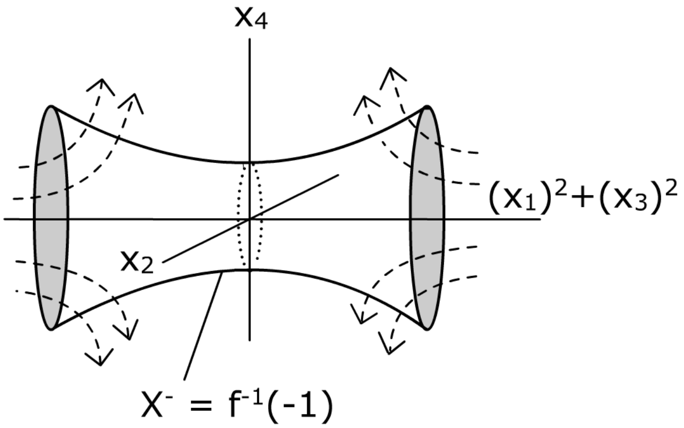


FIGURE 9. The 2-handle for +1-surgery.

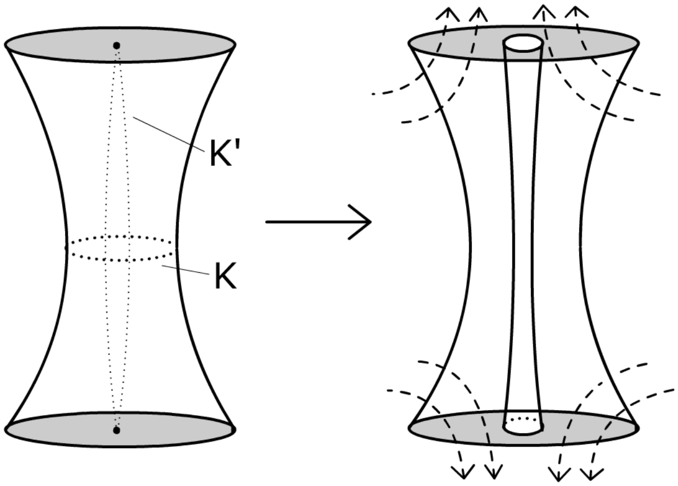


FIGURE 10. Cancellation of ±1 surgeries.

Lemma 6.10 (Cancellation)**.** *Let K be a Legendrian knot in (M, ξ) and K' a pushoff of K along the contact framing. Then -1-surgery on K followed by $+1$-surgery on K' yields a contact 3-manifold which is contactomorphic to (M, ξ).*

Proof. In the boundary S^3 of our symplectic 2-handle, K' links with K to form a Hopf link (recall figure 3.1). Figure 6.3 shows what happens when we now perform $+1$-contact surgery on K'. Now by flowing along the symplectic dilation we can contract the union of our 2-handles and obtain a contactomorphism between the original and surgered contact structures. □

Ding and Geiges [3] have developed a complete theory of this contact Dehn surgery. The main result is a contact analogue for Lickorish's surgery theorem for 3-manifolds [19].

Theorem 6.11. *Any contact 3-manifold is obtained by surgery along a link* $\mathbb{L}^+ \cup \mathbb{L}^-$ *in* (S^3, ζ_{std}) *where the framing is* $+1$ *on* $\mathbb{L}^+$ *and* -1 *on* $\mathbb{L}^-$.

7. Concave Filling

The aim of this section is to prove the concave filling theorem:

Theorem 7.1. *For any contact 3-manifold* (M, ξ) *there is a symplectic cobordism* $(M, \xi) \prec \emptyset$ *(a* symplectic cap*).*

This immediately tells us that the symplectically fillable objects form an equivalence class under symplectic cobordism. The most important corollary for us is

Corollary 7.2. *Symplectically fillings of contact 3-manifolds can be embedded in closed symplectic 4-manifolds.*

Proof. Simply glue the cobordisms $\emptyset \prec (M, \xi)$ and $(M, \xi) \prec \emptyset$ to obtain the required closed symplectic 4-manifold. $\square$

There are several techniques [1, 7, 8, 12] for proving theorem 7.1. I will follow [1], though I have managed to avoid any mention of Stein domains. We first review some basic definitions and terminology.

7.1. Lefschetz fibrations.

Let $f : X \to D^2$ be a smooth map from a compact, oriented 4-manifold X to the disc which is a submersion outside a finite critical set $p_1, ..., p_k$. This is called a Lefschetz fibration if there are neighbourhoods $U_1, ..., U_k$ of $p_1, ..., p_k$ and $V_1, ..., V_k$ of $f(p_1), ..., f(p_k)$ respectively, which are domains for complex coordinate charts in which f has the form $f(z_1, z_2) = z_1^2 + z_2^2$. We also require that the natural orientations of the complex charts agree with those of X and D^2.

We call a fibre regular if it contains no critical point and singular otherwise (use F to denote a regular fibre). By small deformations of the fibration we can assume that fibres contain at most one critical point and without loss of generality we can take the critical values p_i to lie on the circle of radius $1/2$ and to be ordered by increasing θ (where θ is the angular polar coordinate on D^2).

Let $D_r = \{x \in D^2 : |x| < r\}$ and consider $A_r = f^{-1}(D_r)$. Notice that $Re(f|_{U_i}) = Re(z_1)^2 + Re(z_2)^2 - Im(z_1)^2 - Im(z_2^2)$ is a Morse function on U_i with index 2 at p_i and hence $A_{1/2+\epsilon}$ is obtained from $A_{1/2-\epsilon} = F \times D^2$ by attaching 2-handles along loops K_i (*vanishing cycles*) in regular fibres F_i with framing -1 relative to the "surface framing" (taking the complementary direction to TK_i in F_i). For details, see [16].

Let E be a disc in D^2 containing exactly one critical value. The monodromy of the fibre bundle $f|_{\partial E}$ is a right-handed Dehn twist around K_i.

Conversely, given a Lefschetz fibration $f : X \to D^2$, it is clear that we can extend the Lefschetz fibration over a 2-handle attached to a knot $K \subset f^{-1}(b)$ for some regular value b. The resulting fibration has K as a vanishing cycle.

Suppose the *global monodromy* of a Lefschetz fibration $f : X \to D^2$ is trivial, that is the monodromy of the bundle $f|_{\partial D^2}$ is the identity in M_F. Then we can glue a copy of $D^2 \times F$ to X along ∂X and we obtain a Lefschetz fibration over

S^2 (defined in exactly the same way as Lefschetz fibrations over the disc). This is a closed manifold if the fibres are closed. This next powerful theorem of Gompf vindicates all of these definitions,

Theorem 7.3 (Gompf [15]). *The total space of a Lefschetz fibration $X \to S^2$ with homologically essential, closed fibres is a closed symplectic manifold and the regular fibres are symplectic submanifolds.*

The proof is a symplectic version of the argument used to prove theorem 5.5. We say that a Lefschetz fibration with homologically essential fibres is allowable, and we use the term *positive allowable Lefschetz fibration* (PALF) for an allowable Lefschetz fibration over D^2. (The term "positive" refers to the condition on orientations in the definition: had we not have required this, the monodromy could consist of left-handed Dehn twists and Gompf's theorem would no longer hold).

PALFs induce open book decompositions on their boundaries. Explicitly, the binding is the boundary of some fixed regular fibre F and the fibration of $\partial X \setminus \partial F$ is given by $\pi \circ f|_{\partial X \setminus F}$ (where $\pi : D^2 \setminus \{f(F)\} \to \partial D^2$ is the linear retraction).

7.2. **The existence of a PALF.**

Lemma 7.4 (Plamenevskaya [24], Appendix A). *Let $\mathbb{L}$ be a Legendrian knot in (S^3, ζ_{std}). There is an open book decomposition $\mathcal{B}$ of S^3 whose binding is a torus knot and such that*

- *$\mathbb{L}$ is contained in a page of $\mathcal{B}$,*
- *$\mathbb{L}$ is Legendrian in cont($\mathcal{B}$),*
- *cont($\mathcal{B}$) is isotopic to ζ_{std} via an isotopy which fixes $\mathbb{L}$,*
- *The Reeb vector field for cont($\mathcal{B}$) defines both the contact framing and the surface framing for $\mathbb{L}$.*

Proof. We begin by constructing the page. The technique is to isotope $\mathbb{L}$ through Legendrian knots to put it inside a surface on which we know $d\alpha$ is a volume form (here $\alpha = dz - ydx$ is the standard contact form on $\mathbb{R}^3 \subset S^3$). Namely, we consider the front projection of $\mathbb{L}$ and denote by t_i the points where $\mathbb{L}$ intersects the yz-plane Π. The segments of $\mathbb{L}$ between consecutive t_i can be isotoped through Legendrian knots so that (away from their endpoints) they have gradient ± 1 in the front projection. Consider the straight part of each segment and extend this to a straight line in $\mathbb{R}^3$. Thicken this to have width ϵ in the $\pm x$-directions to get a rectangular strip containing most of the segment.

Clearly on this strip, $d\alpha = dx \wedge dy$ is a volume form. We do this for each segment, and then attach "bands" b_i in neighbourhoods of the points t_i so that $\mathbb{L}$ is contained in the resulting union. These bands are copies of $I \times I$ attached to the boundaries of consecutive strips and b_i can be arranged to contain $\mathbb{L}$ in a neighbourhood of t_i. Denote the union of all strips and bands by F' and truncate this surface by slicing the ends off strips so that $\mathbb{L}$ is still contained in the surface and denote the result by F.

Finally, we notice that the boundary of F is a torus knot, K. We can isotope K around on $F \setminus \mathbb{L}$ to make it transverse to the characteristic foliation on F, so without loss of generality the boundary of F is transverse to the contact distribution.

Now we thicken F fixing K to obtain a handlebody U_1 and an open book decomposition of U_1 which induces the standard contact structure. Since K is a torus knot, we can fibre $U_2 = S^3 \setminus U_1$ with binding K. Notice that the fibre constructed

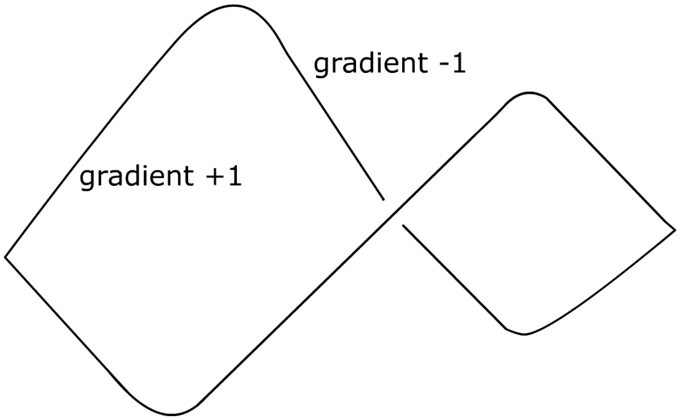


FIGURE 11. The knot after isotopy.

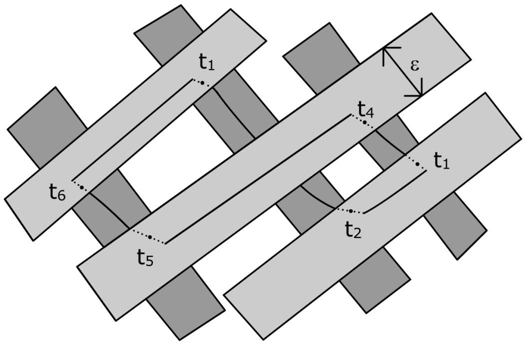


FIGURE 12. Constructing the fibre.

in the course of this lemma coincides with the fibre constructed in proposition 5.7. Thus we have an open book decomposition of $S^3 = U_1 \cup U_2$. The construction in the proof of the Thurston-Winkelnkemper theorem yields a contact form on U_2 which agrees with α on ∂U_2.

Since K is a torus knot, the monodromy, h, of the open book decomposition is a product of non-separating positive Dehn twists and hence the Lefschetz fibration over D^2 with monodromy h is a symplectic filling for the contact structure it induces. By Eliashberg's classification of contact structures on S^3 this must be isotopic rel ∂U_2 to the standard (tight) structure ζ_{std} and hence the restriction to U_2 is also tight. By corollary 4.5 this is isotopic to the restriction $\zeta_{\mathrm{std}}|_{U_2}$, so we can isotope the contact structure induced from the open book decomposition to the standard one via an isotopy which is the identity on U_1 and hence fixes $\mathbb{L}$.

The fact that the contact and surface framings agree is trivial by construction: the Reeb field gives a section of both. $\square$

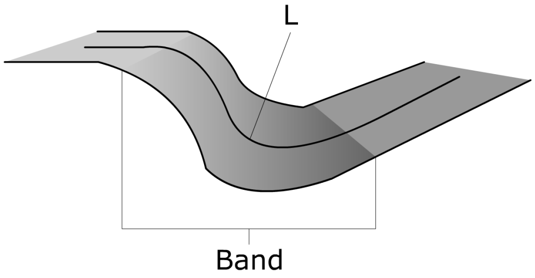


Figure 13. Attaching bands.

That is the technical part of the theorem over, the rest is just surgery with a careful choice of framings.

Theorem 7.5. *Let* (X, ω) *be a symplectic 4-manifold obtained by attaching Weinstein 2-handles to* (D^4, ω_0) *(that is, by contact* -1*-surgery along a Legendrian link* $\mathbb{L}$ *in* (S^3, ζ_{std})*). Then there is a positive allowable Lefschetz fibration* $X \to D^2$ *and furthermore this can be chosen so that the open-book decomposition induced on* ∂X *is compatible with the contact structure on* ∂X*.*

Proof. Let $\mathcal{B}$ be the open book decomposition given by lemma 7.4 for the link $\mathbb{L}$. Let F be a fibre for the open book decomposition and let h be the monodromy of the fibration. Let $Y \to D^2$ be a Lefschetz fibration with this regular fibre and monodromy and inducing $\mathcal{B}$ on $\partial Y = S^3$.

Now we perform surgery on the total space of the Lefschetz fibration by attaching Weinstein 2-handles along the components of $\mathbb{L}$. The last point in lemma 7.4 tells us that the surface and contact framings of $\mathbb{L} \subset S^3$ agree. Recall that Weinstein surgery required a framing of -1 relative to σ_{contact} and that attaching 2-handles to a Lefschetz fibration required a framing of -1 relative to the surface framing. This implies that the Lefschetz fibration will extend over a Weinstein surgery, hence over (X, ω). Clearly the contact structure on ∂X will agree with the open book decomposition induced by the Lefschetz fibration. The fibres are homologically essential by construction. □

7.3. **Proof of the theorem.**

Lemma 7.6. *Any PALF* $f : X \to D^2$ *embeds in a closed symplectic manifold.*

Proof. Let F be a regular fibre and $\mathcal{B}$ the induced open book decomposition with ∂F as binding. Performing a -1-surgery to attach a 2-handle along ∂F allows us to cap the regular fibres with discs and by the discussion in 7.1 the result, X' admits a Lefschetz fibration $f' : X \to D^2$. Let $D(\gamma_n) \cdots D(\gamma_1)$ be the global monodromy of this fibration and let $f'' : X'' \to D^2$ be the Lefschetz fibration with global monodromy $D(\gamma_1)^{-1} \cdots D(\gamma_n)^{-1}$ (notice that this monodromy can be expressed in terms of right-handed Dehn twists by lemma 5.1, so this does define a positive Lefschetz fibration). Since

$$D(\gamma_n)\cdots D(\gamma_1)\cdot D(\gamma_1)^{-1}\cdots D(\gamma_n)^{-1}=1$$

these glue to give a Lefschetz fibration $\tilde{X}\to S^2$ and $\tilde{X}$ is symplectic by theorem 7.3. □

Theorem 7.7. *A contact 3-manifold* (M,ξ) *admits a symplectic cap.*

Proof. By the Ding-Geiges surgery theorem (M,ξ) is obtained from (S^3,ζ_{std}) by contact $+1$-surgery along a link $\mathbb{L}^+$ and contact -1-surgery along a link $\mathbb{L}^-$. We can push $\mathbb{L}^+$ off itself along the contact framing to give a link in M and perform -1 surgery on it to obtain a contact manifold (M',ξ') (this surgery gives a cobordism $(M,\xi)\prec(M',\xi')$). By the cancellation lemma, (M',ξ') is the result of -1-surgery on a link in (S^3,ζ_{std}).

The -1-surgery presentation of (M',ξ') specifies a cobordism $U:\emptyset\prec(M',\xi')$ and by theorem 7.5 this cobordism supports a PALF which induces an open book decomposition on M' compatible with the contact structure ξ'. This embeds in a closed symplectic manifold (X,ω) by 7.6 and hence we can use the gluing lemma to glue the cobordism $(M,\xi)\prec(M',\xi')$ to $X\setminus U$.

The result is a symplectic cap on (M,ξ). □

8. The tightness criterion

The tightness criterion follows now from the observation that no closed symplectic manifold with $b_2^+>1$ can contain spheres of self-intersection $+1$. This may be proved using Seiberg-Witten theory (applying the methods of [17] and the non-vanishing result of [25]).

Theorem 8.1 (Tightness criterion)**.** *A symplectically fillable contact 3-manifold is necessarily tight.*

Proof. Assume there is a symplectically fillable, overtwisted contact 3-manifold, (M,ξ). If K is the Legendrian boundary of an overtwisted disc then (by example 3.2) we can form a Legendrian unknot, K', with $\mathrm{tb}(K')=2$.

Let (X,ω) be a symplectic filling for (M,ξ). Performing -1 surgery along $K'\subset M$ will yield a symplectic 4-manifold (X',ω') with a contact structure (M',ξ') on its boundary and further Weinstein surgeries allow us to increase b_2^+ so that it is strictly greater than 1.

The union of the Seifert disc, Σ', for K' and the core of the 2-handle is a smoothly embedded sphere in X', with self-intersection 1. See this as follows. We move K' off itself via the -1-framing in M to give a knot K''. This can be capped by another copy of the core which does not intersect the first, however the corresponding Seifert surfaces might intersect in X. Since we are performing surgery with coefficient -1 relative to the contact framing, the linking number of K' and K'' is -1 and hence a disc bounding K' will intersect K'' exactly once (with sign $+1$). By pushing the Seifert surface Σ'' for K'' down into X, we check that $\Sigma''\cap M=K''$ so $\Sigma''\cap\Sigma'$ contains exactly one point and, checking orientations, this gives intersection number 1 as required.

If we now glue X' to a symplectic cap for (M',ξ') then we get a closed symplectic 4-manifold with $b_2^+>1$ and an embedded sphere of self-intersection 1. □

9. Conclusion

We have now seen that all contact 3-manifolds admit a cobordism to the empty set (the concave filling theorem) but that, by contrast, there is a strong condition on the global geometry of those which are symplectically fillable (the tightness criterion). This is a striking illustration of the asymmetry of the symplectic cobordism relation between contact 3-manifolds.

One might ask if the property of tightness is sufficient to deduce that a contact 3-manifold must be symplectically fillable. This is not the case, the relation of symplectic cobordism is more subtle than that. Paolo Lisca in [21] demonstrated three examples of 3-manifolds which supported no symplectically fillable contact structure. It was subsequently shown in [22] that two of these did support tight contact structures (in the same paper, they construct infinitely many more tight, non-fillable examples). In the absence of the tightness criterion this was accomplished using subtle invariants of spin^c 3-manifolds coming from Heegaard-Floer theory. Lisca's third example of a space with no symplectically fillable contact structure, the Poincaré homology sphere with a particular choice of orientation, turned out not to support any (positive) tight structure ([9]).

There are still many open questions about symplectic cobordism between contact 3-manifolds. For instance, can there exist cobordisms from tight contact 3-manifolds to overtwisted ones? Certainly not if the tight manifold is symplectically fillable, but the existence of tight, non-fillable manifolds opens up new areas of subtlety to explore.

I finish with a sketch of the partially ordered set of contact 3-manifolds under symplectic cobordism. It makes sense to divide the set into three: the overtwisted, the symplectically fillable and the tight but non-fillable contact 3-manifolds (considering $\emptyset$ as a fillable manifold). Etnyre and Honda [10] showed that an overtwisted contact 3-manifold admits a symplectic cobordism to any other contact 3-manifold. The concave filling theorem gives cobordisms from any contact manifold to $\emptyset$ and hence to any fillable manifold. Together with the tightness criterion this allows us to draw figure 9, where different arrow-styles indicate that there are a) always, b) never, c) potentially cobordisms between manifolds at the endpoints.

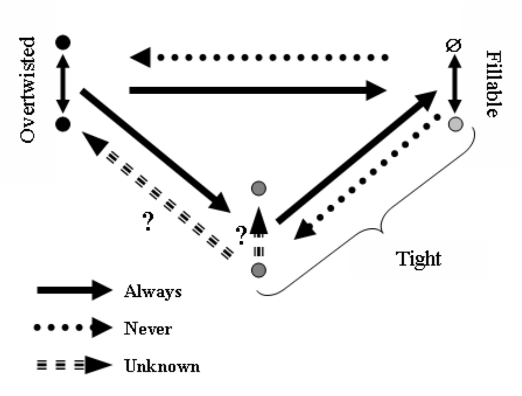


Figure 14. A heuristic sketch of the symplectic cobordism relation.

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