

# The Eells-Salamon twistor correspondence

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The Eells-Salamon<sup>1</sup> twistor correspondence is a dictionary for translating certain problems in the Riemannian geometry of an oriented Riemannian 4-manifold  $(M, g)$  into the almost complex geometry of a related 6-manifold  $(Z, J)$  (the twistor space of  $M$ ). More precisely it is a correspondence between  $g$ -minimal surfaces in  $M$  and  $J$ -holomorphic curves in  $Z$ . We will begin by reviewing the construction and geometry of the twistor space  $Z$  before explaining the twistor correspondence.

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<sup>1</sup>As this seminar is being given at ETH, I feel it necessary to point out that this is not Dietmar.

# Twistor space

The space  $Z$  is a bundle over  $M$ :

$$\begin{array}{c} Z \\ \downarrow \tau \\ M \end{array}$$

The fibre  $F_p = \tau^{-1}(p)$  is the space of complex structures  $\psi$  on  $T_p M$  which agree with the orientation on  $M$  and are  $g$ -orthogonal, i.e.

$$g(\psi v, \psi w) = g(v, w)$$

This space admits a transitive action of  $SO(4)$  and the stabiliser of any given  $\psi$  is  $U(2) = SO(4) \cap GL(2, \mathbb{C})$ , i.e.

$$F_p \cong SO(4)/U(2)$$

We claim that  $SO(4)/U(2)$  is diffeomorphic to  $S^2$ . This is not so hard to see: fix an orthonormal basis  $e_1, \dots, e_4$  of  $T_p M$ . For any  $\psi \in F_p$ ,  $\psi(e_1)$  is a unit vector in  $S^2 \subset \langle e_2, e_3, e_4 \rangle$ . Once that has been fixed we need only specify how  $\psi$  acts on the orthogonal complement of  $\langle e_1, \psi(e_1) \rangle$  but this is 2-dimensional and therefore there's a unique orthogonal complex structure compatible with the orientation (rotation by  $\pi/2$  anticlockwise). The only choice we had was of  $\psi(e_1) \in S^2$ . Therefore we see that  $Z$  is an  $S^2$ -bundle over  $M$ .

The twistor fibre admits a natural  $SO(4)$ -invariant Kähler structure:

- The complex structure  $j$  is given by considering  $T_\psi F_p \subset \text{End}(T_p M)$  as a subspace passing through the origin (by translating) and then allowing  $\psi$  to act. Clearly  $\psi^2 = -1$ . Moreover,  $\psi$  preserves  $T_\psi F_p$ . To see this suppose  $\alpha \in T_\psi F_p$  so that to first order in  $\alpha$

$$g((\psi + \alpha)v, (\psi + \alpha)w) = g(v, w) \text{ and } (\psi + \alpha)^2 = -1$$

i.e.

$$g(\psi v, \alpha w) + g(\alpha v, \psi w) = 0 \text{ and } \psi \alpha = -\alpha \psi$$

now it's easy to see that after replacing  $\alpha$  by  $\psi \alpha$  these equations still hold.

- The space  $\text{End}(T_p M)$  inherits a metric from  $g$  at  $p$  and this restricts to an  $SO(4)$ -invariant metric  $g_F$  on  $F_p \subset \text{End}(T_p M)$ .

$Z$  is a subset of the endomorphism bundle  $\text{End}(TM)$ . The Levi-Civita connection induces a connection on  $\text{End}(TM)$  which descends to a connection on the sphere bundle  $Z$ . We write  $\mathcal{H} \subset T_\psi Z$  and  $\mathcal{V} = T_\psi F_p \subset T_\psi Z$  for the horizontal and vertical spaces of the connection and the twistor projection  $\tau$  respectively. With respect to the splitting  $T_\psi Z = \mathcal{H} \oplus \mathcal{V}$  we can define various geometric structures, including:

- a metric

$$g \oplus g_F$$

- almost complex structures

$$J_\pm = \psi \oplus \pm j$$

$J_+$  is called the Atiyah-Hitchin-Singer almost complex structure. It sometimes turns out to be integrable: this holds iff  $(M, g)$  is a *self-dual* 4-manifold, which illustrates yet another beautiful translation between the (almost) complex geometry of the twistor space and the Riemannian geometry of  $M$ . By contrast  $J_-$  is never integrable. It is called the Eells-Salamon almost complex structure and it's what we're interested in today.

## Gauss lifts

Given a 2-plane  $\pi$  in  $T_p M$  there is a unique  $\psi_\pi \in F_p$  making it holomorphic (this is the same argument that showed us  $F_p \cong S^2$ ). We use this to construct the *Gauss lift* of an immersion  $\iota : \Sigma \looparrowright M$  of a 2-manifold  $\Sigma$ .

- This is an immersion  $\mathfrak{Gauss}(\iota) : \Sigma \rightarrow Z$  and it is defined by sending  $z \in \Sigma$  to  $\psi_{\iota_* T_z \Sigma} \in F_{\iota(z)}$ .
- Clearly  $\tau \circ \mathfrak{Gauss}(\iota) = \iota$  (hence the name “lift”).
- The tangent spaces  $\mathfrak{Gauss}(\iota)_* T_z \Sigma$  are  $\mathfrak{Gauss}(\iota)(z)$ -invariant by construction and hence one can pullback the complex structures  $\mathfrak{Gauss}(\iota)(z)$  along  $\mathfrak{Gauss}(\iota)$  to get a complex structure  $j_\iota$  on  $\Sigma$ .

Suppose that  $(\Sigma, g)$  is a Riemann surface and  $\iota : \Sigma \looparrowright M$  an immersion. Then it's not hard to see that the complex structure determined by the conformal structure of the metric  $\iota^*g$  is precisely  $j_\iota$ . In the case when the immersion is conformal, i.e.  $g = j_\iota^*g$  we can obtain a local coordinate expression for the Gauss lift as follows. Let  $z$  be a local complex coordinate on  $\Sigma$  such that,  $\iota_*\partial_z = \iota_*\partial_x - i\iota_*\partial_y = ce_1 - ice_2$  where  $e_1, e_2$  are orthogonal unit vectors and  $\mathbb{R} \ni c(z) > 0$  and extend  $e_1, e_2$  to a local orthonormal frame  $e_1, \dots, e_4$ . We can write the Gauss lift as

$$\mathcal{G}_{\text{auss}}(\iota)(s) = e_1 \wedge e_2 + e_3 \wedge e_4 = \frac{1}{2ic^2}(1 + \star)\iota_*\partial_z \wedge \iota_*\partial_{\bar{z}}$$

where  $e_1 \wedge e_2$  is understood to mean the endomorphism sending  $e_1$  to  $e_2$  and  $e_2$  to  $-e_1$  (we're being pretty fuzzy here and identifying  $\Lambda^2 T_p M$  with  $\text{End}(T_p M)$  using the metric),  $\star$  is the Hodge star (so  $e_1 \wedge e_2 \wedge \star(e_1 \wedge e_2) = e_1 \wedge \dots \wedge e_4$ , i.e.  $\star(e_1 \wedge e_2) = e_3 \wedge e_4$ ). You should also be careful because my sign conventions are screwy relative to those of Eells and Salamon.

# The twistor correspondence

Now we come to the main theorem.

## Theorem (Eells-Salamon twistor correspondence)

*There is a one-to-one correspondence (given by Gauss lifting) between a) conformal harmonic immersions of Riemann surfaces  $(\Sigma, j)$  in  $M$  and b) nonvertical  $J_-$ -holomorphic maps  $\Sigma \rightarrow Z$ .*

Non-vertical means “not contained in a twistor fibre” (a constant map from  $\Sigma$  to  $M$  is not an immersion!). In the direction a) to b) the correspondence goes via Gauss lifting. In the other direction the correspondence goes by composition with  $\tau$ .

A story for another day is why conformal harmonic maps have anything to do with minimal immersions.

Define the pullback diagram

$$\begin{array}{ccc}
 \iota^* Z & \xrightarrow{\hat{\iota}} & Z \\
 \hat{\tau} \downarrow & & \downarrow \tau \\
 \Sigma & \xrightarrow{\iota} & M
 \end{array}$$

Notice first that  $\mathcal{G}\text{auss}(\iota)_* \partial_Z$  projects via  $\tau_*$  to  $\iota_* \partial_Z$  and hence

$$\mathcal{G}\text{auss}(\iota)_* \partial_Z = \widetilde{\iota_* \partial_Z} + \hat{\iota}(\iota^* \nabla)_{\partial_Z} \mathcal{G}\text{auss}(\iota)(z)$$

$J_-$ -holomorphicity means

$$J_- \mathcal{G}\text{auss}(\iota)_* \partial_Z = i \mathcal{G}\text{auss}(\iota)_* \partial_Z$$

Since  $J_-$  preserves the  $\mathcal{H} \oplus \mathcal{V}$  splitting, this equation must hold for the horizontal and vertical components individually. Horizontally, we have seen that  $\mathcal{G}\text{auss}(\iota)(z) \iota_* \partial_Z = i \iota_* \partial_Z$  precisely when  $\iota$  is conformal. Harmonicity will come from analysing the vertical component.

For ease on the eye, let's write  $\sigma(z) := \mathfrak{Gauss}(\iota)(z)$  and  $\iota^*\nabla = \nabla$  and omit  $\hat{\iota}$ . We want to understand when

$$-j\nabla_z\sigma = i\nabla_z\sigma$$

We may assume that  $\iota$  is conformal, which means that (as a complex structure on  $\iota^*Z$ )

$$\sigma(z) = (1 + \star)e_1 \wedge e_2$$

Note that the tangent space to  $F_{\iota(z)}$  at  $\sigma(z)$  is spanned by

$$\alpha = (1 + \star)e_1 \wedge e_3, \quad \beta = (1 + \star)e_1 \wedge e_4$$

and  $-j$  sends  $\beta$  to  $\alpha$ . Holomorphicity therefore means that the  $\alpha - i\beta$ -component of  $\nabla_z\sigma$  vanishes.

# Harmonicity

If we fix a metric on  $\Sigma$  compatible with the conformal structure  $g$  then we can talk about harmonic maps  $\iota : \Sigma \rightarrow M$ . We consider  $d\iota$  as a section of the bundle  $T^*\Sigma \otimes \iota^*TM$  equipped with the Levi-Civita connections coming from both manifolds. The tensor  $\nabla d\iota$  is then a section of  $T^*\Sigma \otimes T^*\Sigma \otimes \iota^*TM$ . The *tension* of  $\iota$  is defined to be the contraction

$$\text{tr}(\nabla d\iota)$$

where the trace contracts  $T^*\Sigma \otimes T^*\Sigma$  via the metric.

Now

$$d\iota = dz \otimes (e_1 - ie_2) + d\bar{z} \otimes (e_1 + ie_2)$$

and  $\nabla_z dz \propto dz \otimes dz$ ,  $\nabla_{\bar{z}} d\bar{z} \propto d\bar{z} \otimes d\bar{z}$  and  
 $\text{tr}(dz \otimes dz) = \text{tr}(d\bar{z} \otimes d\bar{z}) = 0$ . Thus

$$\text{tr}(\nabla d\iota) \propto dz \otimes d\bar{z} \otimes \nabla_{\bar{z}}(e_1 - ie_2) + d\bar{z} \otimes dz \otimes \nabla_z(e_1 + ie_2)$$

Since the Levi-Civita tensor is torsionfree,

$$\begin{aligned} \nabla_z(e_1 + ie_2) - \nabla_{\bar{z}}(e_1 - ie_2) &= [\iota_* \partial_z, \iota_* \partial_{\bar{z}}] \\ &= \iota_* [\partial_z, \partial_{\bar{z}}] \\ &= 0 \end{aligned}$$

so  $\iota$  is harmonic if and only if  $\nabla_z e_1 = -i\nabla_z e_2$ .

We'll now show that if  $\iota$  is conformal and harmonic then  $\mathfrak{Gauss}(\iota)$  is holomorphic. Since  $\nabla$  commutes with  $\star$

$$\begin{aligned} \sigma \cdot (\alpha - i\beta) &= (1 + \star)(\nabla_z e_1 \wedge e_2 - e_1 \wedge \nabla_z e_2) \cdot \\ &\quad (e_1 \wedge e_3 - e_2 \wedge e_4 - ie_1 \wedge e_4 - ie_2 \wedge e_3) \\ &= 2((\nabla_z e_1) \cdot e_4 - (\nabla_z e_2) \cdot e_3 + i(\nabla_z e_1) \cdot e_3 + i(\nabla_z e_2) \cdot e_4) \end{aligned}$$

Harmonicity implies  $\nabla_z e_1 = -i\nabla_z e_2$  so this whole expression vanishes. For the converse, observe that we had a (whole  $S^1$  of) choice of  $e_3$  and  $e_4$  (only the plane  $e_3 \wedge e_4$  was fixed) and the above equation holds for all such choices. We can therefore compare the  $e_3$  and  $e_4$  components of  $\nabla_z e_1$  and  $-i\nabla_z e_2$  separately and see that they agree. It remains to show that the  $e_1$  and  $e_2$  components agree.

Recall that by construction  $c(e_1 - ie_2) = \iota_* \partial_z$ . Therefore

$$\nabla_z(e_1 + ie_2) = c(\nabla_1 - i\nabla_2)e(e_1 + ie_2)$$

We've seen that this is real by torsionfreeness of the Levi-Civita connection so we know

$$\nabla_z(e_1 + ie_2) = c(\nabla_1 e_1 + \nabla_2 e_2)$$

Let's ignore  $c$ . The  $e_1$  and  $e_2$  components of this are

$$\Gamma_{11}^1 + \Gamma_{22}^1, \Gamma_{11}^2 + \Gamma_{22}^2$$

where  $\Gamma_{jk}^i$  are the Levi-Civita connection coefficients for the pullback metric on  $\Sigma$ . Since the coordinate directions spanned by  $e_1$  and  $e_2$  are conformal for the pullback metric, we know that  $ds^2 = c^2(dx_1^2 + dx_2^2)$ . But then

$$\Gamma_{11}^1 = c^{-2}\partial_1 c^2, \Gamma_{22}^1 = -c^{-2}\partial_1 c^2, \Gamma_{11}^2 = -c^{-2}\partial_2 c^2, \Gamma_{22}^2 = c^{-2}\partial_2 c^2$$

so all these components of the tension vanish from conformality.

## References

The paper which I have bastardised for the purposes of this talk is

- Eells, J. and Salamon, S. (1985) *Twistorial construction of harmonic maps of surfaces into four-manifolds* Annali della Scuola Normale Superiore di Pisa, Classe di Scienze 4<sup>e</sup> série, tome 12, no. 4, pp. 589–640.

Being a local coordinate fan I found their notation quite offputting and hence I decided to write up these notes. The paper

- Atiyah, M., Hitchin, N. and Singer, I. (1978) *Self-duality in four-dimensional Riemannian geometry* Proceedings of the Royal Society of London, A. 362, 425–461

is another excellent paper which uses a slightly different almost complex structure on twistor space whose geometry mirrors the Riemannian geometry of the 4-manifold. This is actually what most people mean when they talk about twistor spaces (e.g. the twistor space of  $S^4$  is the standard complex  $\mathbb{C}P^3$ , and the twistor fibration is the map  $\mathbb{C}P^3 \rightarrow S^4$  coming from the Hopf fibration  $S^3 \rightarrow S^7 \rightarrow S^4$  by dividing out each fibre by a Hopf fibration  $S^3 \rightarrow S^2$ ).