Actuarial versus financial pricing of insurance[1]

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1 Introduction

This paper grew out of various discussions with academics and practitioners around the theme of the interplay between insurance and finance. Some issues were:

- Deregulation and the increasing collaboration between insurance markets and capital markets.
- The emergence of finance related insurance products, as there are catastrophe futures and options, PCS options, index linked policies, catastrophe bonds ....
- The emergence of integrated risk management practices for financial institutions, see Doherty (2000).
- Asset–liability and risk–capital based modelling (think of DFA (Dynamic Financial Analysis), DST (Dynamic Solvency Testing) and EV (Embedded Value)) subsuming simple liability modelling as the industry standard.
- The emergence of financial engineering as a widely accepted discipline, and its interface with actuarial science.

Besides these more general issues, specific questions were recently discussed in papers like Gerber and Shiu (1994), Embrechts and Meister (1997) and the references therein. An interesting approach on the financial pricing of insurance together with material for further reading is to be found in Phillips and Cummins (1995). An excellent, historical discussion on the evolution of actuarial versus financial pricing and hedging is Hans Bühlmann's lecture "Mathematical paradigms in insurance and finance." A web–version of this lecture is to be found under http://www.afshapiro.com/Buhlmann/index_Buhlmann.htm.

In this article, rather than aiming at giving a complete overview of the issues at hand, I will concentrate on some recent (and some not so recent) developments which from a methodological point of view offer new insight into the comparison of pricing mechanisms between insurance and finance.

2 The basics of insurance pricing

Any standard textbook on the mathematics of insurance contains the definition of *a fair insurance premium* and goes on to explain the various ways in which premium principles can be derived, including the necessary *loading*. See for instance Bowers et al. (1986), Bühlmann (1970) and Gerber (1979). The former gives a most readable introduction to the key issues of insurance premium calculation in a utility–based framework. In Bowers et al. (1989). p. 1, the following broad definition of insurance is to be found.

> *An insurance system is a mechanism for reducing the adverse financial impact of random events that prevent the fulfillment of reasonable expectations.*

Utility theory enters as a natural (though perhaps somewhat academic) tool to provide insight into decision making in the face of uncertainty. In determining the *value* of an economic outcome, represented as a random variable on some probability space $(\Omega, \mathcal{F}, P)$, the *expected value principle* leads at the *fair* or so–called *actuarial value* EX where E stands for the expectation with respect to the (physical!) measure P. Clearly inadequate as a premium principle (one should be prepared to pay more than EX), a utility function u enters in the premium–defining equation

$$u(w - \Pi) = E(u(w - X))$$

where w stands for current wealth, Π for the premium charged to cover the loss X when u is our utility. That means, u is an increasing twice differentiable function on $\mathbb{R}$ satisfying $u' > 0$ (more is better) and $u'' < 0$ (decreasing marginal utility). Through Jensen's inequality, the concavity of u immediately leads to

$$\Pi \geq EX$$

for our *risk averse* decision maker. Note that the fair premium EX is obtained for a linear utility. Similar considerations apply to the insurer who has utility v say, initial capital k and collected premium Θ covering the random loss X, then

$$v(k) = E(v(k + \Theta - X)) \,.$$

Again one easily concludes that

$$\Theta \geq EX \,.$$

An insurance contract is now called *feasible* whenever

$$\Pi \geq \Theta \geq EX \,.$$

Bowers et al. (1989), p. 10 summarise:

> *A utility function is based on the decision maker's preferences for various distributions of outcomes. An insurer need not be an individual. It may be a partnership, corporation or government agency. In this situation the determination of v, the insurer's utility function, may be a rather*

complicated matter. For example, if the insurer is a corporation, one of the management's responsibilities is the formulation of a coherent set of preferences for various risky insurance ventures. These preferences may involve compromises between conflicting attitudes toward risk among the groups of stockholders.

By specific choices of v (and/or u), various well–known premium principles can be derived. See for instance Goovaerts, de Vylder and Haezendonck (1984) for a detailed discussion, where also other approaches towards premium calculation principles are given.

a) The *net–premium principle* and its refinements are based on the equivalence principle yielding $\Theta = EX$. Resulting principles are:

   – the *expectation principle*

$$\Theta = EX + \delta EX\,;$$

   – the *variance principle*

$$\Theta = EX + \delta \mathrm{Var(X)}\,;$$

   – the *standard deviation principle*

$$\Theta = EX + \delta (\mathrm{Var(X)})^{1/2}\,;$$

   – the *semi–variance principle*

$$\Theta = EX + \delta E\left((X - EX)^+\right)^2\,.$$

   The above principles can also be linked to ruin–bounds over a given time period and indeed, often the loading factor is determined by setting sufficiently protective *solvency margins* which may be derived from *ruin estimates* of the underlying risk process over a given (finite) period of time.

b) Premium principles implicitly defined via *utility theory*. Besides the net–premium principle (linear utility) the following example is crucial:

   – the *exponential principle*

$$\Theta = \frac{1}{\delta}\,\log E\left(e^{\delta X}\right)$$

   for an appropriate $\delta > 0$. The utility function used in this case has the form

$$u(x) = -e^{-\delta x}$$

   referred to as the *exponential utility function* describing a model with *constant risk aversion* δ or *constant risk tolerance unit* δ^{-1}.

c) A further interesting class of examples, akin to Value–at–Risk measures in finance, are the so–called *quantile principles*. Suppose our loss variable X has distribution function F. Define the (generalised) inverse of F by

$$F^{\leftarrow}(y) = \inf\{x \in \mathbb{R} : F(x) \geq y\}, \quad 0 < y < 1 .$$

Then the $(1-\varepsilon)$–*quantile principle* corresponds to

$$\Theta = F^{\leftarrow}(1-\varepsilon) .$$

For $\varepsilon \downarrow 0$ we obtain the *probable maximal loss* (supposing that F has finite support). Though this risk measure is crucial in most Value–at–Risk (VaR) based risk management systems in finance (see for instance Basle Committee (1996)), Artzner et al. (1999) show that as a risk measure, $\Theta = F^{\leftarrow}(1-\varepsilon)$ fails to possess in general the crucial sub–additivity property defended in the latter paper as a key property for a viable risk measure. In Embrechts, McNeil and Straumann (1999), this failure of VaR is taken one step further and put into the context of the so–called Fundamental Theorems of Integrated Risk Management. The latter results summarise for which financial and insurance markets the VaR approach is unproblematic. At the same time, problems are highlighted where VaR is definitely the wrong measure to use.

d) A principle gaining increasingly in importance from a methodological point of view is the time–honoured *Esscher principle*

$$\Theta = \frac{E\left(Xe^{\delta X}\right)}{E\left(e^{\delta X}\right)}$$

for an appropriate $\delta > 0$. The latter can be obtained in various ways, for instance using a minimisation argument on a specific loss function (see for instance Heilmann (1987)). An economic foundation for the Esscher principle, using risk exchange, equilibrium pricing and Borch's theorem (Borch (1960)) has been given by Bühlmann (1980). An interesting generalisation of the latter paper is Bühlmann (1983). We shall come back to the Esscher principle later.

Besides the above justification of the various premium principles, an alternative approach would be to view Θ as a real function on a space of random variables (or even on a space of probability distributions) and then specify properties of Θ which we want a premium principle to posess. Typically some form of homogeneity and additivity is called for, further properties relate to convexity, iteration, order preservation and robustness. Without entering into details, see for instance Heilmann (1987), p. 136–137, mainly the exponential, standard deviation and variance principles get strong support across numerous publications. Hans Bühlmann proposes as pragmatic solution, *to use the standard deviation principle on the total portfolio and redistribute the resulting premium to the individual risks by using either the exponential or variance principle,* see Bühlmann (1984). The paper by Artzner et al. (1999) referred to above proposes a similar axiomatic approach towards financial Value–at–Risk. More recently, actuaries have been proposing so–called tail–distortion measures (see for instance Wang (1996), Wang, Young and Panjer (1997)) based on the economic work of Yaari (1987).

So far it seems that we are far away from the pricing mechanisms now standardly used in finance. It pays however to read the following remark in Bowers et al. (1989), p. 16 (indeed their Section 1.4 is well worth looking at in detail):

> *In a competitive economy, market forces will encourage insurers to price short–term policies so that deviations of experience from expected value will behave as independent random variables. Deviations should exhibit no pattern that might be exploited by the insured or insurer to produce consistent gains. Such consistent deviations would indicate inefficiencies in the insurance market.*

To someone working in finance, this sounds familiar! Before embarking on this *familiar theme* let me stress that the premium calculation discussion from an insurance point of view would now have to address *credibility theory* as a means towards differentiating premiums within a non–homogeneous portfolio. We shall not pursue this route here; see for instance Goovaerts et al. (1990) and the references therein for a first presentation.

3 Pricing in finance

Stepping now from the insurance textbooks to the (mathematical) finance ones, one is immediately struck by the methodological change from an underlying probability space

$$(\Omega, \mathcal{F}, P)$$

to a so–called *filtered probability space*

$$\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P\right)$$

where $(\mathcal{F}_t)$ is an *increasing family of $\mathcal{F}$–sub–σ–algebras* representing the history of (or information contained in) the past and present of some underlying finance process. The point I want to make is not the abstract mathematics present in the above, much more I would like to stress the word

$$\textit{Information!}$$

Especially when it comes to differences between pricing in finance and insurance, it is exactly the description of the information available in the underlying market which becomes crucial. Before we look at some examples, first consider the typical Ansatz of *no–arbitrage pricing.* Our risk X in the previous paragraph typically becomes a *contingent claim* which, in the finite horizon case $[0, T]$, T being *maturity*, corresponds to X is $\mathcal{F}_T$–measurable. Hence X is determined by the underlying process for time values up–to and including, but *not* beyond, T. If we denote the underlying process by $(S_t)_{0 \le t \le T}$ and consider as an example the European call with strike K and maturity T, then

$$X = (S_T - K)^+ \; .$$

This random variable is akin to an excess–of–loss reinsurance treaty with priority K. Another example, especially relevant in the interplay between insurance and finance is the Asian option with strike K, i.e.

$$X = \left(\frac{1}{T} \int_0^T S_u \, du - K \right)^+ \; ,$$

this time similar to the familiar stop–loss treaty in reinsurance. As in the insurance case, we could start pricing these claims using the actuarial premium principle EX, i.e. the expectation under the physical measure P. In the insurance case, one could use ruin arguments to justify a loading factor as briefly explained in the previous section. In the finance context, the whole argument against using EX as a premium is based on the notion of

$$\textit{no–arbitrage!}$$

By now so much has been written on the subject that it is hardly possible even to begin a discussion having a reader in mind who wants to learn new things! Look at Cox and Rubinstein (1985) if you start from zero, move on to Hull (1993) to become more of an expert. If the lack of mathematics bothers you, look at Lamberton and Lapeyre (1996) or Duffie (1992). Other, useful texts are Baxter and Rennie (1996), Karatzas and Shreve (1997) and Musiela and Rutkowski (1997). An excellent text, stressing more the economic aspects of finance, but at the same time keeping a high mathematical standard is Björk (1998). This list is of course very incomplete. The above texts contain a lot of references for further reading. Finally if your hunger for mathematical precision concerning sentences like “certain statements about the (non–) existence of free lunches are *basically* equivalent to ...” is not yet stilled you must look again at your favourite textbook on functional analysis and read the fundamental paper by Delbaen and Schachermayer (1994). After all of this we know how to *correctly* price the above contingent claims in a no–arbitrage framework, namely the *correct* value at time t of X with risk–free interest rate r is

$$v_t = E^Q\left(e^{-r(T-t)} X \mid \mathcal{F}_t\right)$$

and the premium (to be charged at time $t = 0$) becomes

$$v_0 = E^Q\left(e^{-rT} X\right) . \tag{1}$$

For notational convenience we have used a constant risk free rate r. We could have used a more general (stochastic) model for r resulting in a value at time t

$$E^Q\left(\exp\left\{-\int_t^T r(s)\,ds\right\} X \mid \mathcal{F}_t\right) .$$

The main point is not the r, which is also there in the actuarial case though I did not make it explicit. The main point is the Q: we calculate the fair (no–arbitrage) premium also as an expectation but with respect to a *new* probability measure Q! This point should of course not disappear in a cloud of mathematical notation and sophistication. Indeed, this "change of measure" idea becomes very easy in a standard binomial tree model. The risk neutral probability measure Q changes the original measure P in order to give more weight to unfavourable events in a risk averse environment. In financal economics this leads to the concept of *price of risk* and in insurance mathematics it should explain the *safety loading.* The theory now tells us that in *nice cases* Q is the unique P–equivalent probability measure which turns (S_t) into a *martingale.* That martingales enter is not surprising, I could also have brought them to bear in the previous paragraph. What *is* surprising however is that they appear in a canonical way intimately linked to the economic notion of no–arbitrage. The latter is by now folklore and does not need further discussion here: besides all the references above you may find Varian (1987) entertaining.

In order to work out the price quoted under (1), we need to get hold of Q. At this point we have to look more carefully at the meaning of *nice cases.* Two such cases are the *binomial tree model* and *geometric Brownian motion*, in finance often referred to as the Cox–Ross–Rubinstein, respectively Black–Scholes model. The thing that makes them nice is that they are *complete models.* The latter means that any contingent claim X can be attained through a self–financing trading strategy, mathematicians would say that X has to satisfy an Itô representation with respect to (S_t). A very readable account on this is Jensen and Nielsen (1995). From the introduction to the latter paper I have borrowed the comment below.

> *Theories and models dealing with price formation in financal markets are divided into (at least) two markedly different types. One type of models*

> *is attempting to explain levels of asset prices, risk premium etc. in an* absolute *manner in terms of the so–called* fundamentals. *A crucial model of this type includes the well known rational expectation model equating stock prices to the discounted value of expected future dividends. Another type of models has a more modest scope, namely to explain in a* relative manner *some asset prices in terms of other, given and observable prices.*

It should be clear from the discussion so far that the present section adheres more to the latter approach, whereas the former leans more closely to the actuarial approach of the previous section, though the difference between both is not so sharp as I make it sound. Let us now return to the needed notion of *nice cases*. In summary, the theory of no–arbitrage leads to linear pricing functionals. If our market is such that

a) we have sufficiently many basic building blocks in the market so that new assets can be represented as linear combinations of these building blocks, and

b) these building blocks have a unique price,

then the market is termed *complete*. If not, the market is *incomplete*. In the former cases (completeness) prices are unique (Q is unique) whereas in the second case (typical in insurance) without further information on investor specific preferences, only bounds on prices can be given (Q is not unique). This brings us to the main observation:

Within the no–arbitrage framework
nice cases = complete markets!

See Jensen and Nielsen (1995) for some elementary examples of a complete market and no arbitrage, a complete market and an arbitrage opportunity and finally an incomplete market with no arbitrage opportunity. Besides the Cox–Ross–Rubinstein (binomial) and Black–Scholes (geometric Brownian motion) models, further *nice cases* (complete models) for instance include

- multi–dimensional Brownian motion and some special types of diffusions,
- $(N_t - \lambda t)_{t\geq 0}$ with (N_t) a homogeneous Poisson process with intensity λ, and
- square integrable point process martingales $\left(N_t - \int_0^t \lambda_s \, ds\right)_{t\geq 0}$.

For remarks on this and further references, see Embrechts and Meister (1997). The *not so nice* (incomplete) *cases* typically occur as

- stochastic volatility models,

– processes with jumps of random size (e.g. stable processes, compound Poisson processes, jump diffusions, non–Brownian Lévy processes),

– so–called models with friction, i.e. including transaction costs and investment constraints.

For the latter models, in general no unique martingale price exists and holding an option is a genuinely risky business. If pricing is for instance embedded in a utility maximisation framework, then a unique measure emerges in a very natural way. See for instance Davis and Robeau (1994), Embrechts and Meister (1997) and references therein. By now a *must* for all interested in incomplete markets is the so–called Föllmer–Schweizer–Sondermann approach based on the minimisation of expected squared hedge error. See Föllmer and Sondermann (1986), Föllmer and Schweizer (1989) and the interesting discussion by Dybvig (1992). This pioneering work has by now been expanded to include superhedging (i.e. pricing so that portfolio reaches at least the terminal value) or quantile hedging (as before, but now only with a sufficiently high probability). Various other risk measures for the hedge–shortfall are currently looked at in the literature. See for instance Föllmer and Leukert (2000) and Cvitanic and Karatzas (1999).

4 Back to insurance

At the Bowles Symposium on Securitization of Risk, Georgia State University, Atlanta (1995), John Finn and Morton Lane (see Finn and Lane (1997)) brought the methodologists back with their feet on the ground by saying:

> *There is no right price of insurance; there is simply the transacted market price which is high enough to bring forth sellers, and low enough to induce buyers.*

This market price is not necessarily an equilibrium price. Uniqueness and no–arbitrage are not guaranteed. Nevertheless, from a methodological point of view, the following problem is of interest:

Find a martingale approach to premium calculation principles in an arbitrage–free market!

This is exactly the title of a paper by Delbaen and Haezendonck (1983) which both finance experts and actuaries (of the third kind, dixit Bühlmann (1987)) are strongly advised to read. Further important papers in this context are the classic Borch (1962), Doherty and Schlesinger (1983), Sondermann (1991) and Venter (1991).

The Delbaen–Haezendonck paper starts from the basic underlying risk process (over the finite horizon $[0,T]$)

$$X\left(N_t\right)=\sum_{k=1}^{N_t} X_k\,,\quad 0\le t\le T\,,$$

where (X_k) are iid claims with common distribution function F, (N_t) a homogeneous Poisson process with intensity $\lambda>0$ so that

$$N_t=\sup\left\{n\in\mathbb{N}: T_1+\cdots+T_n\le t\right\}$$

where the (T_k) are iid with $\operatorname{Exp}(\lambda)$ distribution. The random variable T_k denotes the (random) occurrence time of the kth claim X_k. We neglect for the moment all IBNR effects. The processes (X_k) and (T_k) are assumed to be independent.

With the above definition $X\left(N_t\right)$ becomes a compound Poisson process with distribution

$$P\left(X\left(N_t\right)\le x\right)=\sum_{k=0}^{\infty} e^{-\lambda t}\,\frac{(\lambda t)^k}{k!}\,F^{k*}(x)\,,\quad x\ge 0\,,$$

where F^{k*} denotes the kth convolution of F, i.e.

$$F^{k*}(x)=P\left(X_1+\cdots+X_k\le x\right)\,.$$

Suppose now that at each time t the company can sell the remaining risk of the period $]t,T]$ for a given (predictable) premium p_t, hence the underlying price process (S_t) has the form

$$S_t=\Theta_t+X\left(N_t\right)\,,\quad 0\le t\le T\,.$$

Hence the company's liabilities S_t at time t consist of two parts: *first* the part $X\left(N_t\right)$ of claims up to the time t, and *second* the premium Θ_t for the remaining risk $X\left(N_T\right)-X\left(N_t\right)$ (unknown at time t). Delbaen and Haezendonck (1989) at this point conclude:

> *The possibility of buying and selling at time t represents the possibility of "take–over" of this policy. This liquidity of the market should imply that there are no arbitrage opportunities and hence by the Harrison–Kreps theory* (Harrison and Kreps (1979)) *there should be a risk neutral probability distribution Q such that $\{S_t: 0\le t\le T\}$ is a Q–martingale.*

If one further imposes that

$$\Theta_t=\theta(T-t)\,,\quad 0\le t\le T\,,$$

where θ is a *premium density*, then one can show that in sufficiently many reinsurance markets, the linearity of the premiums implies that under Q the risk process

$\{X(N_t) : 0 \leq t \leq T\}$ remains a compound Poisson process. The basic solution then reduces exactly to those equivalent measures Q which preserve the compound Poisson property of $\{X(N_t) : 0 \leq t \leq T\}$. Within this no–arbitrage–insurance context, a viable premium density then takes on the form

$$\theta_Q = E^Q(X(N_1)) = E^Q(N_1)\; E^Q(X_1)\ , \tag{2}$$

resulting both in a change of claim–size as well as claim–intensity of the underlying risk process. Under certain measurability conditions, those Q–measures which give rise to such viable premium principles take on the following form (formulated in terms of distribution functions)

$$F_Q^{(\beta)}(x) = \frac{1}{E(\exp\{\beta(X_1)\})} \int_0^x e^{\beta(y)} dF(y)\,, \quad x \geq 0\,,$$

where $\beta : \mathbb{R}^+ \to \mathbb{R}$ is increasing so that

$$E(\exp\{\beta(X_1)\}) < \infty \quad \text{and} \quad E(X_1 \exp\{\beta(X_1)\}) < \infty\,.$$

The resulting premium density $\theta_{Q(\beta)}$ then satisfies

$$\theta_p = E(N_1)\; E(X_1) < \theta_{Q(\beta)} < \infty\,,$$

hence taking satefy loading into account.

Special choices of β now lead to special premium principles, all consistent within the no–arbitrage set–up. Examples are (see (2) for the notation used):

a) $\beta \equiv \alpha > 0$, then

$$\begin{aligned} E^{Q(\beta)}(N_1) &= e^\alpha E(N_1) = e^\alpha \lambda\,, \\ E^{Q(\beta)}(X_1) &= E(X_1)\ \ (\textit{expected value principle})\,; \end{aligned}$$

b) $\beta(x) = \log(a + bx)$, $b > 0$ and $a = 1 - bE(X_1) > 0$, then

$$\begin{aligned} E^{Q(\beta)}(N_1) &= \lambda\,, \\ E^{Q(\beta)}(X_1) &= E(X_1) + b\,\mathrm{Var}(X_1)\ \ (\textit{variance principle})\,; \end{aligned}$$

c) $\beta(x) = \alpha x - \log E\left(e^{\alpha X_1}\right)$, $\alpha > 0$, then

$$\begin{aligned} E^{Q(\beta)}(N_1) &= \lambda\,, \\ E^{Q(\beta)}(X_1) &= \frac{E(X_1 \exp\{\alpha X_1\})}{E(\exp\{\alpha X_1\})}\ \ (\textit{Esscher principle})\,. \end{aligned}$$

For further details on the above approach see Delbaen and Hazendonck (1989); Meister (1995) completes the proof of the main result in the latter paper, generalises the approach to mixed Poisson and doubly stochastic Poisson processes and applies the results obtained to the pricing of CAT–futures. A summary on the latter is to be found in Embrechts and Meister (1997). See again Venter (1991) for a critical discussion on no–arbitrage pricing in reinsurance. The basics of mathematical modelling in insurance can for instance be found in Rolski et al. (1998).

5 Final discussion

As we have seen above, in a sufficiently liquid (re–)insurance market, classical insurance–premium principles can be reinterpreted in a standard no–arbitrage pricing set–up. The variety of premium principles used is explained through the inherent incompleteness of the underlying risk process in so far that a whole family of equivalent martingale measures exist. First of all, the necessary liquidity assumptions imposed may at present be rather unrealistic, definitely so in the case where single catastrophe risks for instance are to be priced. The introduction of CAT–futures by the Chicago Board of Trade in 1992, together with its new generation of PCS–options aims at offering such liquidity. Though being just a first step, I am convinced that the resulting *securitisation* attempts for insurance risk will eventually yield markets where the methods briefly discussed above will become applicable. Various important issues I have not addressed, as there are the secondary market problem, the resulting issue of risk–adjusted capital, the overall problem of securitisation of insurance risk, the emergence of all–finance products etc. No doubt many more papers will competently discuss some of the above problems. Especially the paper by Phillips and Cummins (1995) and the references therein should be consulted for a lot of interesting ideas. I would like to end with a brief summary of some further ongoing research which I deem to be relevant.

First, a key question concerns the choice between working under *the physical measure P or under the (in the incomplete case, an) equivalent martingale measure.* When it comes down to pricing simple risks or products in clearly iliquid insurance markets, the physical measure P gives us an objective description of underlying randomness. This quickly gives rise to interesting methodological questions. As an example, take the pricing of CBOT CAT–futures. Suppose $X(N(t))$ now represents a homeowners pool's losses over a period $[0, t]$. A key component in the pricing of a CAT–future amounts to estimating distributional properties of quantities like

$$L_t = \left(\frac{X(N(t))}{cE(X(N(t)))} - K \right)^+ \tag{3}$$

for some strike (loss–ratio) K and loading factor c. Various approaches exist, including *no–arbitrage pricing* using a mixture of a geometric Brownian motion and a homogeneous Poisson model (Cummins and Geman (1995)), *equilibrium pricing* based on a counting process model (Aase (1994)), *utility and risk minimisation pricing* based on the general class of doubly–stochastic Poisson processes (Meister (1995)), pricing using so–called *implied loss distributions* (Finn and Lane (1997)) and finally actuarial pricing using *moment bounds* on the underlying loss ratios (Brockett, Cox and Smith (1997)). In non–life insurance, the risk process $(X(N(t)))$ typically exhibits heavy–tailed behaviour for the claim–size distribution F (see Embrechts, Klüppelberg and Mikosch (1997) for a comprehensive discussion on this), moreover, the pool construction makes the claim intensity large so that in order to price L_t in (3) under P, one needs estimates for

$$E\left(L_t\right) = \frac{1}{cE\left(X\left(N_t\right)\right)} \int_{\gamma E(X(N_t))}^{\infty} P\left(X\left(N_t\right) - E\left(X\left(N_t\right)\right) > x\right)\, dx\,. \tag{4}$$

Here $\gamma = Kc - 1 > 0$. Letting $t \to \infty$ in (4) one is eventually faced with estimates of the type

$$P\left(Z_1 + \cdots + Z_{n(t)} > x(t)\right)$$

for $t \to \infty$ where the Z_is are heavy–tailed iid random variables and both $n(t) \to \infty$, $x(t) \to \infty$. This is precisely the set–up encountered in large deviation theory but under the non–standard assumption of heavy–tailedness (e.g. Pareto or lognormal distributions). For further details, see Embrechts, Klüppelberg and Mikosch (1997).

A further, always recurring theme in the realm of actuarial versus financial pricing of insurance is the *Esscher pricing principle.* Originally brought into insurance in order to approximate the total claim–size distribution

$$P(X(N(t)) \leq x)\,, \quad x > 0\,, \tag{5}$$

especially for large values of x, the so–called Esscher transform (also referred to as exponential tilting) now plays a fundamental role as an actuarial pricing mechanism in finance. Whereas in the estimation of formulas like in (5) only an exponential transformation of the underlying distribution function F of the claim–sizes is needed (see for instance the excellent Jensen (1995)), when it comes down to the pricing of derivatives in insurance and finance, the Esscher transform has to be defined on stochastic processes. A very readable account is Gerber and Shiu (1994) where the Esscher transform is defined for exponentials of Lévy processes (i.e. processes with stationary and independent increments). Under the Esscher transformed probability measure, discounted price processes are martingales, hence no–arbitrage prices can be calculated. Various discussants to the above paper stress that, in the incomplete case, the Esscher price is just one of many viable prices. A key question for further research is then: what makes the Esscher price special when it comes to pricing under infinitely many equivalent martingale measures? New results concerning partial answers to this question are appearing. Some examples are:

a) In Meister (1995), it is shown that both in an exponential utility maximisation framework as well as in a general market equilibrium set–up, the Esscher price occurs as the unique solution, i.e. the unique no–arbitrage price, in the incomplete market of a compound Poisson risk process however constrained by utility or equilibrium considerations.

b) In Delbaen, Bühlmann, Embrechts and Shiryaev (1996), the notion of Esscher transform is generalised to *conditional Esscher transforms* which allow to apply the exponential tilting technique to a general class of semimartingales. A for insurance relevant version of the above in discrete time is Delbaen, Bühlmann, Embrechts and Shiryaev (1998).

c) We already discussed that in the incomplete case, uniqueness of a pricing martingale measure can only be achieved through imposing certain optimality conditions. This leads to possible candidates like the minimal martingale measure of Föllmer and Schweizer (1989) and the variance–optimal measure of for instance Schweizer (1995). An obvious question is now: how does the Esscher pricing relate to the above constrained pricing mechanisms? A partial (mathematical) answer to this question is given in Grandits (1999). The main point of the latter paper can be summarised as follows. First, the L^2 theory traditionally used in order to construct risk minimising measures is generalised to L^p for $1 < p < \infty$ leading to a so–called p–optimal measure. If the latter converges to a martingale measure for the underlying process as $p \to 1$, then the limit point *must be the Esscher measure!*

The above points clearly show that concerning the Esscher transform *there is more available than first meets the eye!* Further results are to be expected.

In the introduction I promised that the paper would very much be on work in progress and not a complete overview. As a consequence, I have left out many relevant references and approaches; this should not be interpreted as an ordering of importance. As a matter of fact, I most strongly believe that financial as well as actuarial pricing of insurance products will increasingly involve more sophisticated statistical methodology. Much more than at present is encountered within the realm of finance. A discussion of some of these methods, essentially related to extremal events are for instance to be found in Embrechts, Klüppelberg and Mikosch (1997) and references therein. A lot of updated work including free software, additional examples, related research are to be found on http://www.math.ethz.ch/finance.

It has been common belief that actuaries have to learn from finance specialists when it comes to pricing and modelling the asset side of their books. Whereas this undoubtedly is true, at the same time I would strongly advice finance specialists to have a closer look at some of the recent developments within the actuarial world. As always, a bridge can be walked in two directions. I very much hope that, besides the existence of a financial bridge to actuarial pricing my summary will also have indicated that there is something like an actuarial bridge to financial pricing. As always, the truth will lie somewhere in the middle, the developments taking place right now make me believe that we are converging steadily to this unifying theory.

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