Meta densities and the shape of their sample clouds

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Abstract

This paper compares the shape of the level sets for two multivariate densities. The densities are positive and continuous, and have the same dependence structure. The density f is heavy-tailed. It decreases at the same rate – up to a positive constant – along all rays. The level sets $\{f > c\}$ for $c \downarrow 0$ have a limit shape, a bounded convex set. We transform each of the coordinates to obtain a new density g with Gaussian marginals. We shall also consider densities g with Laplace, or symmetric Weibull marginal densities. It will be shown that the level sets of the new light-tailed density g also have a limit shape, a bounded star-shaped set. The boundary of this set may be written down explicitly as the solution of a simple equation depending on two positive parameters. The limit shape is of interest in the study of extremes and in risk theory since it determines how the maximal observations in different directions relate. Although the densities f and g have the same copula – by construction –, the shapes of the level sets are not related. Knowledge about the limit shape of the level sets for one density does not give any information about the limit shape for the other density.

Key words: Meta distribution, sample clouds, level sets, limit shape, multivariate extremes, densities with cubic level sets, power norming, copula.

Introduction

0.1 Dependence and the shape of sample clouds

For bivariate distributions the dependence structure is a rather complex issue. In a Gaussian world dependence may be specified by a single number, the correlation. As one moves from independence to comonotonicity the elliptic level sets of the density change shape, the circle changes into an ellipse which clings more and more closely to the diagonal. The correlation moves from zero to one.

For an elliptic Gaussian density the components of the maximum of a large number of independent observations will be asymptotically independent, however close the correlation is to one. Properly normalized, the partial maxima converge in distribution to a vector with independent Gumbel marginals.

Under the assumption of joint normality joint occurrence of extreme events is highly unlikely, whereas reality may point to the contrary. The latter property is one of the major weaknesses of the Gaussian copula model (as championed by [16]) within the framework of CDO pricing. It may have contributed, though perhaps in a minor way, to the current credit crisis.

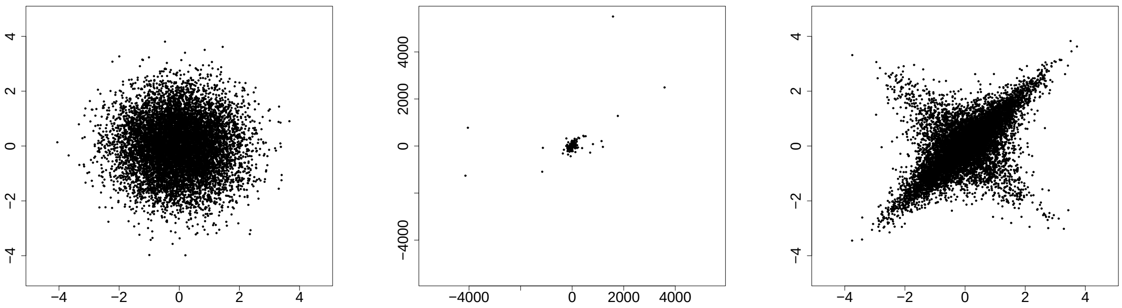
In the present paper, we consider meta distributions. These distributions allow us to model stronger forms of tail dependence while maintaining the desired Gaussian marginals. Let us illustrate this with an example.

Spherical Student t densities look somewhat like standard Gaussian densities, but the components of the coordinatewise maxima exhibit positive dependence. This dependence carries through to the max-stable limit vector. The marginal densities have heavier tails. A suitable increasing non-linear transformation will turn a random variable with a standard Gaussian density into a random variable with a standard Student t distribution with given parameter λ. The inverse transformation will map a sample from the Student t distribution into a standard Gaussian sample, moving in the far out sample points. If one applies this inverse transformation to each of the components of a vector from an elliptic Student t distribution with standard marginals one obtains a random vector with standard Gaussian components. The distribution of this new vector is not Gaussian. The marginals are Gaussian but the vector retains the dependence structure of the original heavy-tailed t distribution, also for the maxima. The new multivariate distribution is known as the meta distribution with standard Gaussian marginals based on the original elliptic t distribution. In more technical terms the meta distribution and the original distribution have the same copula.

As a parametric stationary model, meta distributions have been used in a wide range of applications, especially in the financial and actuarial literature (see [10]), but also in reliability theory (see [2]) and medical applications (see [13], [5]). The copula-based construction of multivariate distributions allows one to model marginal components and the dependence structure separately. This two-stage approach is perceived as an advantage in situations when only limited information on the interdependence of the marginal components is available. For a view on this, see [6]. The latter paper contains what is referred to as the "must-reads" on copulas, together with some references to papers more critical of this two-stage modelling approach. For more examples of meta distributions as well as references to areas of application of these models the reader is referred to [7].

The present paper addresses an important aspect of multivariate distributions - the limit shape of the sample clouds. Formally, a *sample cloud* is a random sample from a given distribution, a point process with a fixed number of points. If the scaled sample clouds converge onto a set, the boundary of this limit set will link the behaviour of extremal observations in different directions. Convergence of random samples and characterization of the shape of the limit set have been considered in [4] and [15].

In order to highlight the main notions of the paper, let us compare the behaviour of sample clouds



(a) (b) (c)

Figure 1: Bivariate sample clouds of $10,000$ points from (a) the standard normal distribution, (b) centered Cauchy distribution whose density has level sets shaped like the ellipse $5x^2+6xy+5y^2=1$, (c) the meta-Cauchy distribution with standard normal marginals and the original distribution as in (b).

from a multivariate Student t distribution and a Gaussian distribution. In both cases the sample clouds may be scaled to converge. Their asymptotic behaviour is different. The scaled sample clouds from the t distribution converge in law to a Poisson point process with a simple continuous intensity:

$$h(\mathbf{w}) = \eta(\omega)/r^{\lambda+d} \qquad r = \|\mathbf{w}\|_2 > 0,\ \omega = \mathbf{w}/r. \tag{0.1}$$

The function η on the unit sphere is continuous and positive. Here it ensures that h has elliptic level sets of the same shape as the level sets of the t density. The parameter λ denotes the degrees of freedom of the t distribution; d is the dimension of the underlying space. The scaled sample clouds from a standard Gaussian distribution on $\mathbb{R}^d$ have a fairly sharp boundary, because of the thin tails. They converge to a black ball. For the meta distribution with Gaussian marginals based on the elliptic Student t distribution the scaled sample clouds will also converge, but the limit shape is different. Fig. 1 shows bivariate sample clouds of ten thousand points for these three situations.

If one wants to step out of the Gaussian world, and use distributions with Gaussian marginals but a non-Gaussian dependence structure, the procedure above may be applied. One hunts around for a multivariate distribution whose dependence has the desired structure, and then transforms the marginals so as to obtain a meta distribution with standard Gaussian marginals and the dependence structure (copula) of the original distribution.

In this paper we assume that the marginals of the meta distribution all are equal to a given continuous positive symmetric light-tailed density g_d, standard Gaussian or Laplace. More generally one may assume

$$g_d(s) \sim as^b e^{-ps^\theta} \qquad s\to\infty, \quad a,p,\theta>0. \tag{0.2}$$

The original distribution has a continuous heavy-tailed density. This may be a multivariate Student t density with elliptic or cubic level sets (for information on l^p-norm spherical distributions an interested reader may consult [12]), or more generally a continuous positive density f whose tail behaviour is described by a continuous positive function h as in (0.1). Such tail behaviour implies that the shape of the level sets of f converges to the shape of the level sets of h.

The meta distribution has a continuous positive density g. The shape of the level sets of g depends on the level. We shall prove that the shape converges as the level goes to zero. Because of the light tails of the marginal density g_d in (0.2), the sample clouds from the meta distribution will also have this limit shape. The limit shape is non-convex, star-shaped, with continuous boundary, and highly symmetric. Fig. 3 and 4 show some examples in dimension $d = 2$ and 3. We shall derive a simple explicit expression for the boundary; see Theorems 2.4, 2.9 and equation (2.12).

Let us say a few words on the relation to multivariate extreme value theory. Our conditions ensure that sample clouds from the heavy-tailed density f, properly scaled, converge to a Poisson point process with intensity h. It follows that the coordinatewise maxima converge. Since the light-tailed density g_d lies in the domain of the Gumbel distribution for maxima, the coordinatewise maxima from the meta density also converge, by Galambos' theorem (see [9]). The limit distribution has the same dependence structure as the heavy-tailed max-stable limit distribution for f. Not only the coordinatewise maxima from the density g converge, but also the sample clouds from this density (with the same normalization). The limit is a Poisson point process on $\mathbb{R}^d$ with a continuous strictly positive intensity. The intensity is related to the intensity h in (0.1). The Poisson point process describes the edge of the sample cloud when zooming in on the positive vertex of the black limit set associated with the meta density. The structure of the edge of this limit set will be the subject of another paper. It is a second order phenomenon.

0.2 Structure and results

The body of the paper consists of three sections and an appendix. The first section introduces the meta transformation, contains definitions, formulates precise conditions on the heavy-tailed density f and the light-tailed marginal g_d, and investigates the behaviour of the meta transformation, and the effect on the meta density when changing the original density into a density which is asymptotic to it. The second section contains our main results. Here we determine the asymptotic form of the level sets of the meta density, and the asymptotic shape of the sample clouds from the meta distribution. The third section discusses domains of the limit shape, and shows how sensitive this shape is to perturbations of the original distribution. Section 4 presents our conclusions. The appendix contains technical results on regular variation, on von Mises functions, and on densities with cubic level sets.

The meta transformation is a continuous coordinatewise transformation K linking the original distribution function (df) F to the meta df $G = F \circ K$. We assume that F has a density f and give conditions under which the meta df G has a density. If f is positive and continuous and vanishes in infinity, and if the marginal densities g_i are continuous and positive on $\mathbb{R}$ then the meta density g will be continuous and positive. If the density $\tilde{f}$ is asymptotic to f and has the same marginals as f the meta transformations K and $\tilde{K}$ coincide and the meta densities g and $\tilde{g}$ will be asymptotic. After these general remarks we formulate the *standard assumptions* on the original density f and the marginal density g_d which will hold in the remainder of the paper.

The sample clouds from the meta density, properly scaled, converge almost surely. The limit set is the level set of a continuous function which is obtained from the meta density by scaling and power norming. Under the standard assumptions the limit set exists. It is a compact set. It is highly symmetric. It is invariant under permutations of the coordinates and under sign changes. The limit shape does not depend on the shape of the convex level sets of the density f; it is determined by two positive parameters. These are λ, the parameter which governs the rate of decrease of the density f along rays, and θ, the exponent in (0.2). The limit set is star-shaped. Its convex hull is a centered coordinate cube. Fig. 3a shows that in dimension $d = 2$, for certain values of the parameters, the limit set has the form of a flower with four symmetric petals.

The limit set does not change if we replace the density f by a density which is weakly asymptotic to f. However, the shape may change radically if one deletes the density on thin sectors along the axes. One may construct continuous densities $\tilde{f}$ with the same marginals as f such that the scaled sample clouds from f and $\tilde{f}$ converge to the same limiting Poisson point process. But the limit set for sample clouds from the meta density $\tilde{g}$ is a cube, or, alternatively, a cross consisting of the 2^d intervals linking the origin to the vertices of the cube. These results are surprising since the meta densities g and $\tilde{g}$ have the same multivariate extreme value limits. The limit shape of the sample clouds is of interest to risk analysis. However it is not clear how the shape relates to the asymptotic dependence in the underlying distribution. In Section 3.3 we shall discuss these issues in more detail.

1 The meta transformation

Altering the marginals of a multivariate df does not change the dependence structure of the underlying random vector. Starting with a random vector $\mathbf{Z}$ with continuous df F on $\mathbb{R}^d$ we alter the marginals to obtain a new df G with marginals G_i. We assume that the marginals G_i are continuous on $\mathbb{R}$ and strictly increasing on the interval $I_i = \{0 < G_i < 1\}$. Typically the marginals of G are equal and Gaussian with $I_i = \mathbb{R}$, exponential with $I_i = (0, \infty)$, or uniform with $I_i = (0, 1)$. These examples are motivated by models used in finance; see for instance [8] for the first and [16] for the second.

One may think of the theory developed here as an alternative to copulas. Gaussian marginals have the advantage that there exists a standard finite-dimensional class of multivariate Gaussian densities with standard normal marginals. Meta densities may be compared to these multivariate Gaussian densities. For sample clouds, it is more intuitive to assume that the distributions have unbounded support, and to look at points far out, if one is interested in extremes. In the chapter on copulas in Joe [14] the figures depict bivariate meta densities with Gaussian (rather than uniform) marginals.

Let $\mathbf{X}$ denote the vector with df G. The vector $\mathbf{X}$ lives on the open block $I = I_1 \times \cdots \times I_d$. There is a coordinatewise map $K : I \to \mathbb{R}^d$ of the form

$$\mathbf{z} = (z_1, \ldots, z_d) = (K_1(x_1), \ldots, K_d(x_d)) = K(\mathbf{x}) \qquad \mathbf{x} = (x_1, \ldots, x_d) \in I,$$

which allows us to write the original vector in terms of the new vector: $\mathbf{Z} \stackrel{\mathrm{d}}{=} K(\mathbf{X})$, where $\stackrel{\mathrm{d}}{=}$ denotes equality in distribution. This equality yields the basic relation:

$$G = F \circ K \qquad K_i = F_i^{-1} \circ G_i \qquad i = 1, \ldots, d. \tag{1.1}$$

The df F is assumed to be continuous. That is equivalent to continuity of the d marginals F_i. It does not ensure continuity of the meta transformation. We choose the marginals K_i to be left continuous so as to agree with the convention that inverse dfs F_i^{-1} are left continuous, see [18], page 3. For continuity of K, one needs the extra condition that the d marginal dfs F_i are strictly increasing on the interval $\{0 < F_i < 1\}$, see (1.1) above. This extra condition will be fulfilled if F has a density which is positive on $\mathbb{R}^d$ except perhaps on a set of finite Lebesgue measure. The inverse transformation K^{-1} is continuous without this extra condition. Because of formula (1.1) we prefer to work with K. Distributions with discontinuous marginals occur in practice, but the theory of the associated copulas is more complicated; see [11].

1.1 Definitions, Assumptions and Notation

A meta distribution is constructed by imposing the given marginals $G_1, \ldots, G_d$ onto the original df.

Definition 1. *Let $G_1, \ldots, G_d$ be continuous dfs on $\mathbb{R}$ which are strictly increasing on the intervals $I_i = \{0 < G_i < 1\}$. Consider a random vector $\mathbf{Z}$ in $\mathbb{R}^d$ with df F and continuous marginals F_i, $i = 1, \ldots, d$. Define the transformation*

$$K(x_1, \ldots, x_d) = (K_1(x_1), \ldots, K_d(x_d)), \qquad K_i(s) = F_i^{-1}(G_i(s)) \qquad i = 1, \ldots, d. \tag{1.2}$$

The df G in (1.1) *is the* meta distribution *(with* marginals G_i*) based on the* original *df F. The random vector $\mathbf{X}$ is said to be a* meta vector *for $\mathbf{Z}$ (with* marginals G_i*) if*

$$\mathbf{Z} \stackrel{\mathrm{d}}{=} K(\mathbf{X}). \tag{1.3}$$

The coordinatewise map $K = K_1 \otimes \cdots \otimes K_d$ which maps $\mathbf{x} = (x_1, \ldots, x_d) \in I = I_1 \times \cdots \times I_d$ into the vector $\mathbf{z} = (K_1(x_1), \ldots, K_d(x_d))$ is called the meta transformation. ◊

A meta transformation is basically a simple object. It is a vector of univariate increasing functions, each determined by two dfs on $\mathbb{R}$. The relations (1.1) and (1.3) are equivalent.

If $\mathbf{Z}$ has a density and we choose the meta distribution to have marginal densities, then $\mathbf{X}$ will have a density.

Proposition 1.1. *If the original vector has a density f, and if the marginals of the meta distribution have densities g_i, then the meta distribution has a density g. This density has the form:*

$$g(\mathbf{x}) = f(K(\mathbf{x})) \prod_i \frac{g_i(x_i)}{f_i(z_i)} \qquad z_i = K_i(x_i),\ x_i \in I_i = \{0 < G_i < 1\}. \tag{1.4}$$

The density g vanishes outside the block $I = I_1 \times \cdots \times I_d$.

Proof The formula (1.4) holds trivially in the univariate case. Let $\varphi \geq 0$ be a Borel function on $\mathbb{R}^d$, and set $\psi = \varphi \circ K$. Then $\mathbb{E}\psi(\mathbf{Z}) = \mathbb{E}\varphi(\mathbf{X})$ by (1.3). Set $P(\mathbf{x}) = \prod_i g_i(x_i)/f_i(K_i(x_i))$. This product is finite almost everywhere on I since f_i is positive almost everywhere on $K_i(I_i)$. The relation

$$\int h(\mathbf{z})d\mathbf{z} = \int h(K(\mathbf{x}))P(\mathbf{x})d\mathbf{x}$$

holds for all Borel functions $h \geq 0$ since it holds for functions of the form $h(\mathbf{x}) = h_1(x_1)\cdots h_d(x_d)$ by Fubini. Let $g(\mathbf{x}) = f(K(\mathbf{x}))P(\mathbf{x})$. With $h = \psi f$ we find:

$$\mathbb{E}\varphi(\mathbf{X}) = \mathbb{E}\psi(\mathbf{Z}) = \int \psi(\mathbf{z})f(\mathbf{z})d\mathbf{z} = \int \psi(K(\mathbf{x}))f(K(\mathbf{x}))P(\mathbf{x})d\mathbf{x} = \int \varphi(\mathbf{x})g(\mathbf{x})d\mathbf{x}$$

by the identity $\varphi = \psi \circ K$. It follows that $g = (f \circ K) \cdot P$ is the density of $\mathbf{X}$. ¶

One may write the equation (1.4) more symmetrically as an equality between two quotients:

$$q_g(\mathbf{x}) = \frac{g(\mathbf{x})}{g_1(x_1)\cdots g_d(x_d)} = \frac{f(\mathbf{z})}{f_1(z_1)\cdots f_d(z_d)} = q_f(\mathbf{z}) \qquad \mathbf{z} = K(\mathbf{x}). \tag{1.5}$$

These quotients describe the dependence structure of the dfs F and G. Their transformation is simple. If h denotes the density of the copula, then $q_h = h$ since the marginals are uniform on $(0,1)$. Hence

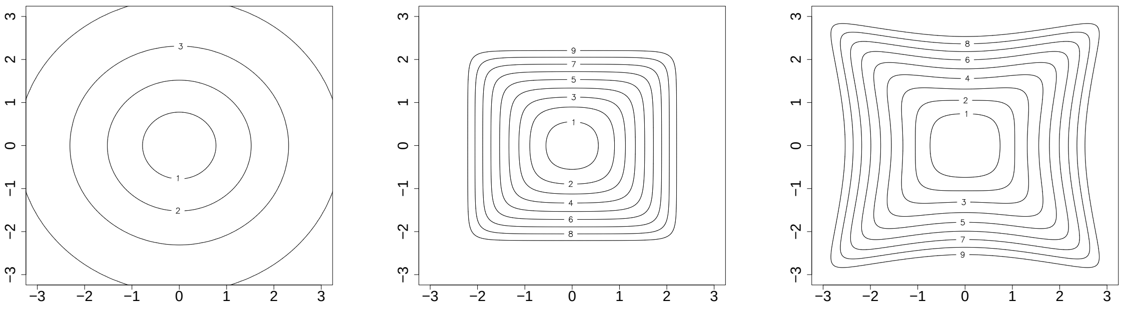
$$q_f(\mathbf{z}) = h(\mathbf{u}) \qquad z_i = F_i^{-1}(u_i),\ 0 < u_i < 1,\ i = 1, \ldots, d.$$

Corollary 1.2. *Suppose the marginals g_i of the meta density are continuous on I_i. Continuity of the density g in* (1.4) *on the block with edges I_i holds if the quotient*

$$q_f(\mathbf{z}) = \frac{f(\mathbf{z})}{f_1(z_1)\cdots f_d(z_d)} \tag{1.6}$$

is continuous on the block with edges $J_i = \{0 < F_i < 1\}$. Continuity of this quotient holds if the marginals f_i are continuous and positive on J_i, and f is continuous on the block. (But also if the components Z_i are independent, and $q_f \equiv 1$.)

The meta density g in (1.4) is the product of two factors. The first factor $f \circ K$ is the function f in the new coordinates $\mathbf{x}$, obtained by substituting $z_i = K_i(x_i)$. The second factor is the Jacobian determinant of the meta transformation. It is a product of univariate functions.



(a) f (b) $f \circ K$ (c) g

Figure 2: Level sets of (a) the density f of the bivariate spherical Student t distribution with $\lambda = 1$ degree of freedom, (b) the function $f \circ K$ where the meta transformation K transforms the t marginals to standard normal marginals, and (c) the meta density g. The levels are given as powers of 10.

Since K is defined coordinatewise, it transforms coordinate rectangles into coordinate rectangles. In this paper the dfs F_i will have heavy tails, and the dfs G_i will have light tails. If successive rectangles in $\mathbf{z}$-space increase by a factor two, then in $\mathbf{x}$-space the increase is much slower, each new rectangle adding a relatively thin border to the previous one. So large balls in $\mathbf{z}$-space (Fig. 2a) will be transformed into cubes with rounded edges in $\mathbf{x}$-space (Fig. 2b). For a spherically symmetric unimodal density f, the level sets of $f \circ K$ will be these rounded cubes. However, for the density g we also have to take the Jacobian into account. What does the function $g_i(x_i)/f_i(z_i)$ look like? The coordinates x_i and z_i are linked: $1 - G_i(x_i) = 1 - F_i(z_i)$. So x_i and z_i are quantiles for the same probability. Let us express the density in terms of the distribution tail. Suppose the marginal G_i is standard normal; then the density is heavier than the tail: $g_i(s) \sim s \cdot (1 - G_i(s))$, $s \to \infty$. Suppose the density f_i varies regularly with exponent $-\lambda - 1$ (see Definition 2); the density is lighter than the tail: $f_i(t) \sim (\lambda/t) \cdot (1 - F_i(t))$, $t \to \infty$. So $g_i(x_i)/f_i(z_i)$ is asymptotic to $x_i z_i/\lambda$ with x_i (and z_i) tending to $+\infty$. It will grow without bound. On the boundary of a cube, the Jacobian will be maximal in the vertices. The contribution of this product far outweighs the variation in the function $f \circ K$ on the boundary of large cubes. For elliptically symmetric Student densities f, the variation over the surface of a cube is bounded because of asymptotic scale invariance, and is negligible compared to the contribution of the Jacobian. For a bivariate spherically symmetric Student density f, the meta density g on the boundary of a large square $[-t, t]^2$ will be larger in the vertices than in the midpoints of the edges. See Fig. 2c. We conclude that for $c > 0$ sufficiently small the level sets $\{g > c\}$ will not be convex. The ridges along the 2^d diagonal halflines, due to the product of the quotients $g_i(x_i)/f_i(z_i)$, have a non-negligible influence on the density g. One of the aims of our paper is to make these qualitative remarks more precise. Fig. 6 shows how the shape of the bivariate meta density on horizontal lines depends on the vertical coordinate. Fig. 3 in the next section shows what shapes the level sets $\{g > c\}$ may assume as $c \to 0$.

1.2 Asymptotic properties of the marginals

For our exposition on the meta transformation it is useful to distinguish between the univariate behaviour, to be treated in this subsection, and the multivariate behaviour, in the next subsection. For simplicity, for the univariate behaviour we shall consider continuous dfs F_0 and G_0 on the halfline $[0,\infty)$ which vanish in the origin. We assume that G_0 is strictly increasing. Given explicit expressions for the asymptotic tail behaviour of these two dfs one can write down equally explicit expressions for the asymptotic behaviour of K_0.

First suppose $1-F_0(t)\sim c_0/t^\lambda$ for some $\lambda>0$ and $c_0>0$, and

$$1-G_0(s)\sim As^Be^{-ps^\theta} \qquad s\to\infty,\ A,p,\theta>0.$$

Then K_0 has a simple asymptotic form. The variables s and $t=K_0(s)$ satisfy $1-F_0(t)=1-G_0(s)$. So

$$c_0/t^\lambda\sim As^Be^{-ps^\theta}\Rightarrow t^\lambda\sim(c_0/A)s^{-B}e^{ps^\theta},$$

which gives, with $\tau=1/\lambda$, the explicit asymptotic equality

$$K_0(s)=t\sim(c_0/A)^\tau s^{-\tau B}e^{\tau ps^\theta} \qquad s\to\infty. \tag{1.7}$$

In general, the tail of G_0 is asymptotic to a *von Mises function*:

$$1-G_0(s)\sim e^{-\psi(s)} \qquad s\to\infty, \tag{1.8}$$

where ψ is a C^2 function with a positive derivative such that

$$a'(s)\to 0 \qquad s\to\infty, \qquad a(s)=1/\psi'(s). \tag{1.9}$$

The function $a(s)$ is the *scale function* of $1-G_0$, and

$$\frac{1-G_0(s+va(s))}{1-G_0(s)}\to e^{-v} \qquad s\to\infty,\ v\in\mathbb{R} \tag{1.10}$$

weakly on $\mathbb{R}$ and hence uniformly on $[c,\infty)$ for all $c\in\mathbb{R}$ (see e.g. Section 1.1 in [18]).

We assume that the marginal tails of the original df vary regularly.

Definition 2. *A measurable function h on* $(0,\infty)$ varies regularly *with* exponent ρ *(written $h\in RV_\rho$) if for all $x>0$,*

$$h(tx)/h(t)\to x^\rho \qquad t\to\infty.$$

The df F_0 has a tail which varies regularly with exponent $-\lambda<0$. Hence

$$1-F_0(t)\sim c_0e^{-\lambda r(\log t)} \qquad t\to\infty, \tag{1.11}$$

where r is a C^2 function (see Sections 11.2 and 18.1 in [1]) such that

$$r'(t)\to 1 \qquad r''(t)\to 0 \qquad t\to\infty. \tag{1.12}$$

The inverse function $q = r^{-1}$ satisfies the same asymptotic relations as r. Hence

$$K_0(s) = t \sim c_0^\tau e^{\varphi(s)} \qquad s \to \infty, \qquad \varphi(s) = \tau q(\psi(s)) \sim \tau\psi(s). \tag{1.13}$$

Differentiation gives:

$$\varphi'(s) = \tau q'(\psi(s))\psi'(s) \sim \tau/a(s) \qquad (1/\varphi')'(s) \to 0 \qquad s \to \infty. \tag{1.14}$$

Proposition 1.3. *Suppose F_0 and G_0 are continuous dfs on $[0,\infty)$ which vanish in the origin. Assume G_0 is strictly increasing and the tail is asymptotic to a von Mises function as in* (1.8) *with scale function $a(s)$. Assume $1 - F_0$ varies regularly with exponent $-\lambda < 0$. Set $K_0 = F_0^{-1}(G_0)$. Then $1/K_0(s)$ is asymptotic to a von Mises function with scale function $\lambda a(s)$, and*

$$K_0(s + v\lambda a(s))/K_0(s) \to e^v \qquad v \in \mathbb{R} \qquad s \to \infty. \tag{1.15}$$

Proof The first statement follows from (1.13) and (1.14). The limit relation in the display holds as in (1.10) since $1/K_0$ is asymptotic to a von Mises function with scale function $\lambda a(s)$. ¶

Corollary 1.4. *Let $\tilde{F}_0$ be a continuous df on $[0,\infty)$ which vanishes in the origin, and suppose $1 - \tilde{F}_0$ is asymptotic to $1 - F_0$ in ∞. Let $\tilde{G}_0 = G_0 = \tilde{F}_0(\tilde{K}_0)$. Then the functions $\tilde{K}_0$ and K_0 are asymptotic in ∞. Write $K_0(s) = t = \tilde{K}_0(\tilde{s})$. Then $\tilde{s} - s = o(a(s))$ for $s \to \infty$.*

Proof Asymptotic equality follows because $\tilde{F}_0^{-1}$ and F_0^{-1} are asymptotic in one by regular variation. The last relation follows from (1.15) by monotonicity of the functions K_0 and $\tilde{K}_0$. ¶

What do these results say about our multivariate dfs?

Suppose the df F_0 on $\mathbb{R}$ is continuous, $F_0(-t)/(1 - F_0(t)) \to C \in (0,\infty)$, and $1 - F_0 \in RV_{-\lambda}$, $\lambda > 0$. Also suppose the df G_0 has a continuous positive symmetric density on $\mathbb{R}$ and $1 - G_0$ (or the density) is asymptotic to a von Mises function with scale function $a(s)$ for $s \to \infty$. Then the results above hold both for $s \to \infty$ and for $-s \to \infty$ since $K_0(s) = F_0^{-1}(G_0(s)) = (1 - F_0)^{-1}(1 - G_0)(s)$. (If $F_i(t) = G_i(s)$ then $1 - F_i(t) = 1 - G_i(s)$.)

Now assume that F is a multivariate df with heavy-tailed marginals F_i which satisfy

$$F_i(-t) \sim c_i^- e^{-\lambda r(\log t)} \qquad 1 - F_i(t) \sim c_i^+ e^{-\lambda r(\log t)} \qquad t \to \infty, \tag{1.16}$$

where r is a C^2 function which satisfies (1.12). The $2d$ constants $c_i^\pm$ are positive. So the $2d$ marginal tails all vary regularly with the same exponent $-\lambda < 0$, and they are also balanced in the sense that they decrease at the same rate. The tails are asymptotic to constant multiples of each other. The condition will hold if the sample clouds from the distribution can be scaled by positive scalars to converge to a Poisson point process on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ with intensity h in (0.1). See Sections 16 and 17 in [1]. Assume the marginals of the meta df G are equal to the univariate df G_0 above. Under these conditions the $2d$ functions $-K_i(-s)$ and $K_i(s)$, $i = 1, \ldots, d$, are asymptotic to $(c_i^\pm)^\tau e^{\varphi(s)}$ for $s \to \infty$ as in (1.13).

1.3 Asymptotic behaviour of the multivariate functions

We assume that F is a multivariate df with continuous marginals $F_1, \dots, F_d$, and that the univariate dfs $G_1, \dots, G_d$ are continuous and strictly increasing on $\mathbb{R}$. We assume that the tails of the marginals of F vary regularly with negative exponents.

Proposition 1.5. *Suppose the assumptions above hold. Let $\tilde{F}$ have continuous marginals whose tails are asymptotic to those of the marginals of F. Then the meta transformations satisfy*

$$\frac{\|\tilde{K}(\mathbf{x}) - K(\mathbf{x})\|}{1 + \|K(\mathbf{x})\|} \to 0 \qquad \|\mathbf{x}\| \to \infty. \tag{1.17}$$

If the marginals F_i and $\tilde{F}_i$ are strictly increasing then the transformations K and $\tilde{K}$ are homeomorphisms of $\mathbb{R}^d$ onto itself, and the quotient above is continuous and bounded.

Proof The functions F_i^{-1} vary regularly in zero and in one. By regular variation $\tilde{F}_i^{-1}$ is asymptotic to F_i^{-1} in zero and one, and $\tilde{K}_i$ and K_i are asymptotic in $\pm\infty$ by (1.2). Hence $|\tilde{K}_i(x_i) - K_i(x_i)|/\|K(\mathbf{x})\| \to 0$ for $\|\mathbf{x}\| \to \infty$, whether x_i is bounded or not. This establishes (1.17). Adding one in the denominator in (1.17) ensures continuity of the quotient. ¶

Now assume F and $\tilde{F}$ have continuous densities f and $\tilde{f}$ on $\mathbb{R}^d$. Consider the corresponding meta densities g and $\tilde{g}$ with all marginals equal to a given continuous positive symmetric density g_d. We want to formulate conditions which ensure that:

- $\tilde{g}(\mathbf{x}) \sim g(\mathbf{x})$ for $\|\mathbf{x}\| \to \infty$;
- $\tilde{g}(\mathbf{x}) \sim g(\mathbf{x})$ for $\|\mathbf{x}\| \to \infty$ and $\min_i |x_i| \to \infty$;
- $\tilde{g}(\mathbf{x}) \asymp g(\mathbf{x})$ for $\|\mathbf{x}\| \to \infty$.

Recall that one writes $\tilde{h}(\mathbf{x}) \asymp h(\mathbf{x})$ for $\|\mathbf{x}\| \to \infty$ if $\tilde{h}$ and h are positive eventually, and both $\tilde{h}(\mathbf{x})/h(\mathbf{x})$ and $h(\mathbf{x})/\tilde{h}(\mathbf{x})$ are bounded outside a compact set; we refer to this type of asymptotic equivalence as *weak*. If the densities f and $\tilde{f}$ are positive and continuous, and agree outside a bounded set, and if the marginal densities agree, $\tilde{f}_i \equiv f_i$, for $i = 1, \dots, d$, then $\tilde{K}_i \equiv K_i$, and the meta densities $\tilde{g}$ and g agree outside a bounded set. If the marginals do not agree, then, even if the density f vanishes in infinity, the quotient $\tilde{g}/g$ need not be bounded, unless f is uniformly continuous. We now first look at the case where the densities $\tilde{f}$ and f are asymptotically equal: $\tilde{f}(\mathbf{z})/f(\mathbf{z}) \to 1$ for $\|\mathbf{z}\| \to \infty$.

Proposition 1.6. *Suppose the densities f and $\tilde{f}$ are continuous and positive outside a bounded set in $\mathbb{R}^d$, and asymptotic. Suppose the density f satisfies*

$$f(\mathbf{z}_n + \mathbf{p}_n)/f(\mathbf{z}_n) \to 1 \qquad \|\mathbf{z}_n\| \to \infty,\ \|\mathbf{p}_n\|/\|\mathbf{z}_n\| \to 0. \tag{1.18}$$

Let the marginal tails $F_i(-t)$ and $1-F_i(t)$ vary regularly with negative exponent for $t \to \infty$. Then this also holds for the densities. The marginal densities f_i and $\tilde{f}_i$ are continuous. The multivariate meta densities $\tilde{g}(\mathbf{x})$ and $g(\mathbf{x})$ are continuous and satisfy

$$g(\mathbf{x})/\tilde{g}(\mathbf{x}) \to 1 \qquad \min_i |x_i| \to \infty. \tag{1.19}$$

There exists a constant $C > 1$ such that $e^{-C} < \tilde{g}(\mathbf{x})/g(\mathbf{x}) < e^C$ for $\|\mathbf{x}\| > C$.

Proof Asymptotic equality of the densities f and $\tilde{f}$ implies asymptotic equality of their marginals by integration. The extra condition (1.18) on f also holds for $\tilde{f}$ and ensures that the marginal densities are continuous. Continuity of the meta densities g and $\tilde{g}$ follows by Proposition 1.1, and its corollary. The marginal densities also satisfy the condition $f_i(t_n + r_n)/f_i(t_n) \to 1$ for $|t_n| \to \infty$ and $r_n/|t_n| \to 0$. By Lemma A.1 the tails of the marginal densities vary regularly. The asymptotic equality $\tilde{K}_i \sim K_i$ in $\pm\infty$ established in the proof of Proposition 1.5 implies that the functions $g_i(s)/f_i(K_i(s))$ and $g_i(s)/\tilde{f}_i(\tilde{K}_i(s))$ are asymptotic for $s \to \pm\infty$. Condition (1.18) ensures that $\tilde{f}(\tilde{K}(\mathbf{x})) \sim f(K(\mathbf{x}))$ for $\|\mathbf{x}\| \to \infty$ by (1.17). Relation (1.19) follows. The last line follows from the next result. ¶

Proposition 1.7. *Suppose the densities f and $\tilde{f}$ are continuous on $\mathbb{R}^d$ and positive outside a bounded set, and $\tilde{f}(\mathbf{z}) \asymp f(\mathbf{z})$ for $\|\mathbf{z}\| \to \infty$. Also assume that $f(\mathbf{z}_n + \mathbf{p}_n) \asymp f(\mathbf{z}_n)$ if $\|\mathbf{z}_n\| \to \infty$ and $\|\mathbf{p}_n\|/\|\mathbf{z}_n\| \to 0$. Let the marginal densities f_i and $\tilde{f}_i$ be continuous. If the marginal tails $F_i(-t)$ and $1-F_i(t)$ vary regularly with negative exponent, and are asymptotic to the corresponding tails of the marginals of $\tilde{F}$, then the meta densities $\tilde{g}(\mathbf{x})$ and $g(\mathbf{x})$ satisfy*

$$\tilde{g}(\mathbf{x}) \asymp g(\mathbf{x}) \qquad \|\mathbf{x}\| \to \infty.$$

Proof Regular variation and asymptotic equality of the tails of the distribution imply that the functions $\tilde{K}_i$ and K_i are asymptotic in $\pm\infty$. Hence $\tilde{f}(\tilde{K}(\mathbf{x})) \asymp f(K(\mathbf{x}))$ by the arguments of the previous proposition, and similarly for the univariate functions $g_i(s)/f_i(K_i(s))$ and $g_i(s)/\tilde{f}_i(\tilde{K}_i(s))$ since $\tilde{f}_i(\tilde{K}_i(x_i)) \asymp f_i(K_i(x_i))$ in $\pm\infty$, and these functions are continuous and positive. ¶

2 The limit set

2.1 The standard set-up

Let us first introduce the multivariate heavy-tailed densities f.

Densities with level sets all of the same shape are easy to work with. Let D be a bounded convex open set which contains the origin. There is a unique function n_D, the *gauge function* of D, with the properties

$$\{n_D < 1\} = D \qquad n_D(r\mathbf{z}) = r n_D(\mathbf{z}) \qquad r > 0,\ \mathbf{z} \in \mathbb{R}^d. \tag{2.1}$$

For any continuous strictly decreasing positive function f_0 on $[0,\infty)$, the function $f:\mathbf{z}\mapsto f_0(n_D(\mathbf{z}))$ is unimodal with convex level sets all of the same shape. Assume f is a probability density. If the set D is symmetric, $-D=D$, then n_D is a norm, and the marginals $f_1,\dots,f_d$ are symmetric.

For densities $f=f_0\circ n_D$ there is a nice partial integration result:

$$\mathbb{P}\{\mathbf{Z}\in tD\}=f_0(t)|tD|+\int_0^t|sD||df_0(s)|=\int_0^t f_0(s)d|sD|.$$

The middle term is a limit of sums for horizontal slices; the right hand term is a limit for rings. Since $|sD|=s^d|D|$ one finds

$$\mathbb{P}\{\mathbf{Z}\notin tD\}=1-\mathbb{P}\{\mathbf{Z}\in tD\}=d|D|\int_t^\infty s^{d-1}f_0(s)ds. \tag{2.2}$$

If f_0 varies regularly with exponent $-(\lambda+d)<-d$ then f is integrable, the marginal densities vary regularly with exponent $-(\lambda+1)$ and the slowly varying functions $t^{\lambda+d}f_0(t)$ and $t^{\lambda+1}f_i(t)$, $i=1,\dots,d$ are asymptotic up to a constant factor. The remarks above remain valid if we assume asymptotic equality, $f(\mathbf{z})\sim f_0(n_D(\mathbf{z}))$ for $\|\mathbf{z}\|\to\infty$, and if D is a bounded open star-shaped set with continuous boundary: there exists a continuous positive function r_D on the unit sphere ∂B such that

$$D=\{r\zeta\mid\zeta\in\partial B,\ 0\le r<r_D(\zeta)\}. \tag{2.3}$$

As a matter of convenience we shall assume that f is positive.

Definition 3. *In the* standard set-up, *f is a positive continuous density, asymptotic to $f_0(n_D(\mathbf{z}))$ for $\|\mathbf{z}\|\to\infty$, with f_0 continuous, strictly decreasing and regularly varying with exponent $-(\lambda+d)$ for some positive λ, and with D a bounded open set containing the origin, star-shaped and with a continuous boundary. The meta density g has equal marginals g_d, where the density g_d is continuous, positive, symmetric, and asymptotic to a von Mises function $e^{-\psi}$ with scale function $a=1/\psi'$ whose derivative vanishes in infinity.* ♢

The von Mises condition (1.9) ensures that

$$g_d(s+va(s))/g_d(s)\to e^{-v}\quad\text{uniformly in } v\in[c,\infty)\qquad s\to\infty,\ c\in\mathbb{R}.$$

The distribution tail $1-G_d$ satisfies the same limit relation for the same scale function. It is known that a df H_0 lies in the domain of attraction of the Gumbel distribution for maxima if and only if it is asymptotic to a df G_0 with a continuous density g_0 which is asymptotic to a von Mises function $e^{-\psi}$. See e.g. Proposition 1.4 in [18].

In order to have a limit shape we need to impose an extra condition on the marginal density g_d:

$$\text{The function } \psi \text{ above varies regularly in infinity with exponent } \theta>0. \tag{2.4}$$

The distribution tail then satisfies a similar condition since $1 - G_d(s) \sim a(s)g_d(s)$ for $s \to \infty$ and $|\log a(s)| = o(\psi(s))$, see Proposition A.2. The weaker assumption is a basic condition on the marginal distributions in [4], but it is dropped in [15]; see also Cor. 9.16 in [1].

This condition is satisfied by the normal density, the Laplace density, and by densities g_d of the form (0.2). The distribution tail $1 - G_d$ then also has this form. It is asymptotic to $As^B e^{-ps^\theta}$ with $a = p\theta A$ and $b = B + \theta - 1$.

If we assume (2.4) then the meta density g in the standard set-up has level sets which may be scaled to converge to a limit set, as will be shown below. The shape of the limit set E depends only on the exponents λ and θ. The scaled densities will diverge on the interior of E and tend to zero off the closure of E. Let $\mathbf{1} = (1, \ldots, 1)$ denote the vertex of the standard cube $C = [-1, 1]^d$. There exists a compact set E such that

$$\frac{g(s\mathbf{u})}{g(s\mathbf{1})} \to \begin{cases} \infty & \mathbf{u} \in \mathrm{int}(E) \\ 0 & \mathbf{u} \in E^c \end{cases} \qquad s \to \infty.$$

In order to obtain a proper limit function for the quotient, one has to use power norming. Construct functions $(g(s\mathbf{u})/g(s\mathbf{1}))^{\epsilon(s)}$ where the exponent $\epsilon(s)$ vanishes for $s \to \infty$. This dampens the exponential decrease. We shall see that the exponent $\epsilon(s) > 0$ may be chosen so that the quotient converges to a continuous function uniformly on compact sets in $\mathbb{R}^d$. The limit function has a zero in the origin and it equals one precisely on the boundary of the set E. It is simpler to work with logarithms. Write $g = e^{-\gamma}$. Below we shall prove that

$$\frac{\gamma(s\mathbf{1}) - \gamma(s\mathbf{u})}{\psi(s)/\lambda} \to \chi(\mathbf{u}) \qquad s \to \infty, \ \mathbf{u} \neq 0.$$

The limit function χ has a simple structure. It is symmetric with respect to permutations of the coordinates, and sign changes. It depends only on the exponents λ and θ. The boundary of the limit set E is $\{\chi = 0\}$.

2.2 Limit sets for sample clouds

Sample clouds from light-tailed distributions tend to have clearly defined boundaries. For sample clouds from the meta density g above there is a limit shape. If $\mathbf{X}_1, \mathbf{X}_2, \ldots$ are independent observations from the meta density g and we choose $r_n > 0$ such that $ng(r_n\mathbf{1}) \to 1$, then the n points from the scaled sample cloud $N_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$ will roughly fill out the limit set E, as we will see.

Definition 4. *Let E be a compact set in $\mathbb{R}^d$ and μ_n finite measures. We say that μ_n* converge onto E *if $\mu_n(\mathbf{p} + \epsilon B) \to \infty$ for any ϵ-ball centered in a point $\mathbf{p} \in E$, and if $\mu_n(U^c) \to 0$ for all open sets U containing E. The finite point processes N_n* converge onto E *if $\mathbb{P}\{N_n(U^c) > 0\} \to 0$ for open sets U containing E, and if*

$$\mathbb{P}\{N_n(\mathbf{p} + \epsilon B) > m\} \to 1 \qquad m > 1, \ \epsilon > 0, \ \mathbf{p} \in E.$$

Proposition 2.1. *If N_n is an n-point sample cloud from a probability distribution π_n on $\mathbb{R}^d$, then N_n converges onto E if the mean measures $\mu_n = n\pi_n$ converge onto E.*

Proof For any Borel set A the number $N_n(A)$ has a binomial-$(n, \pi_n A)$ distribution. Hence $\mathbb{P}\{N_n(A) > m\} \to 1$ for all $m > 1$ if and only if $\mu_n A \to \infty$ and $\mathbb{P}\{N_n(A) > 0\} \to 0$ if and only if $\mu_n A \to 0$. ¶

Example 1. The sample clouds from a standard normal distribution on the plane, scaled by $\sqrt{2 \log n}$ will converge onto the closed unit disk; the sample clouds from a meta distribution with standard Gaussian marginals based on a Student distribution will converge with the same scaling onto a compact set E, but E has a different shape. Compare Figures 1a and 1c. The scaling constants may be determined by the marginals, see Proposition 2.5. ♢

A detailed analysis of the almost sure convergence of scaled sample clouds from multivariate distributions with rapidly varying tails in terms of random sets is given in [4] and [15].

2.3 The limit function χ for densities with cubic level sets

In this subsection we assume that $\mathbf{Z}$ has density $f(\mathbf{z}) = f_0(\|\mathbf{z}\|_\infty)$ for a continuous strictly decreasing function f_0 on $[0, \infty)$ which varies regularly with exponent $-(\lambda + d)$. Some results on the construction and properties of probability densities with cubic level sets are given in the Appendix, Section A.2. The marginal densities are equal and symmetric, continuous and strictly decreasing on $[0, \infty)$. The marginal density f_d varies regularly with exponent $-(\lambda+1)$. The slowly varying functions for f_0, f_d and $1-F_d$ are asymptotically equal up to a positive constant. In a log-log plot, a regularly varying function becomes a function whose slope tends to the exponent. Recall that one may write $1 - F_d(t) \sim e^{-\lambda r(\log t)}$, where r is C^2 and satisfies (1.12).

Let g be the meta density with marginals equal to g_d, where g_d is assumed continuous, positive and symmetric, and asymptotic to a von Mises function $e^{-\psi}$, see (1.8). We also assume (2.4), that ψ varies regularly with exponent $\theta > 0$. The meta transformation is $K : \mathbf{x} \mapsto \mathbf{z} = (K_0(x_1), \dots, K_0(x_d))$. Recall that s and $t = K_0(s)$ are linked by $1 - G_d(s) = 1 - F_d(t)$. By (1.13)

$$\kappa_0(s) := \log K_0(s) \sim \tau\psi(s) \qquad s \to \infty, \qquad \tau = 1/\lambda. \tag{2.5}$$

For the derivative $K_0'(s)$ the equalities $1 - F_d(t) = 1 - G_d(s)$ and $K_0(s) = t$ give:

$$K_0'(s) = \frac{g_d(s)}{f_d(t)} = \frac{1 - F_d(t)}{f_d(t)} \frac{g_d(s)}{1 - G_d(s)} \sim \tau t/a(s) = \tau K_0(s)/a(s).$$

Hence by (2.5) and Proposition A.2

$$\kappa_1(s) := \log K_0'(s) \sim \log(\tau) - \log a(s) + \kappa_0(s) \sim \tau\psi(s) \qquad s \to \infty. \tag{2.6}$$

We are now ready to determine the shape of the level sets of the meta density

$$g(\mathbf{x}) = f(K(\mathbf{x}))K_0'(x_1)\cdots K_0'(x_d).$$

The first factor again is unimodal with cubic level sets. It is constant on the upper face of the cube $[-s,s]^d$. It suffices to look at the density g on the cone generated by this face. Let Π_s be the upside down pyramid which is the convex hull of this face and the origin. It consists of all points $\mathbf{x}$ of the form $|x_i| \le x_d \le s$. We have argued above that on a cube the density g is maximal in the vertices. Consider the quotient $g(s\mathbf{u})/g(s\mathbf{1})$. Write $g = e^{-\gamma}$, and

$$\chi_s : \mathbf{u} \mapsto \frac{\gamma(s\mathbf{1}) - \gamma(s\mathbf{u})}{\tau\psi(s)} = A_s(\mathbf{u}) + B_s(\mathbf{u}) \tag{2.7}$$

where $\mathbf{u} = (u_1, \ldots, u_d)$ with $|u_i| \le u_d = v > 0$ and $A_s(\mathbf{u})$ is the contribution due to the first factor $f(K(\mathbf{x}))$ in the expression for g. Observe that

$$\varphi_s(\mathbf{u}) := -\log f(K(s\mathbf{u}))) = -\log f_0(K_0(sv)) \sim (\lambda + d)\log(K_0(sv)) \sim (\lambda + d)\tau\psi(sv) \qquad s \to \infty \tag{2.8}$$

by (2.5). Hence, by (2.4)

$$A_s(\mathbf{u}) = \frac{\log f(K(s\mathbf{u}) - \log f(K(s\mathbf{1}))}{\tau\psi(s)} \sim \frac{\varphi_s(\mathbf{1}) - \varphi_s(\mathbf{u})}{\varphi_s(\mathbf{1})/(\lambda + d)} \to (\lambda + d)(1 - v^\theta) \qquad s \to \infty, \tag{2.9}$$

and by (2.6)

$$B_s(\mathbf{u}) = \frac{\kappa_1(su_1) - \kappa_1(s)}{\tau\psi(s)} + \cdots + \frac{\kappa_1(su_d) - \kappa_1(s)}{\tau\psi(s)} \to (u_1^\theta - 1) + \cdots + (u_d^\theta - 1). \tag{2.10}$$

Theorem 2.2. *Let g be the meta density introduced above. Then for $v = \|\mathbf{u}\|_\infty > 0$,*

$$\chi_s(\mathbf{u}) := \frac{\log(g(s\mathbf{u})/g(s\mathbf{1}))}{\psi(s)/\lambda} \to \chi(\mathbf{u}) = |u_1|^\theta + \cdots + |u_d|^\theta + \lambda - (\lambda + d)v^\theta \qquad s \to \infty. \tag{2.11}$$

Convergence is uniform on compact subsets of $\mathbb{R}^d$.

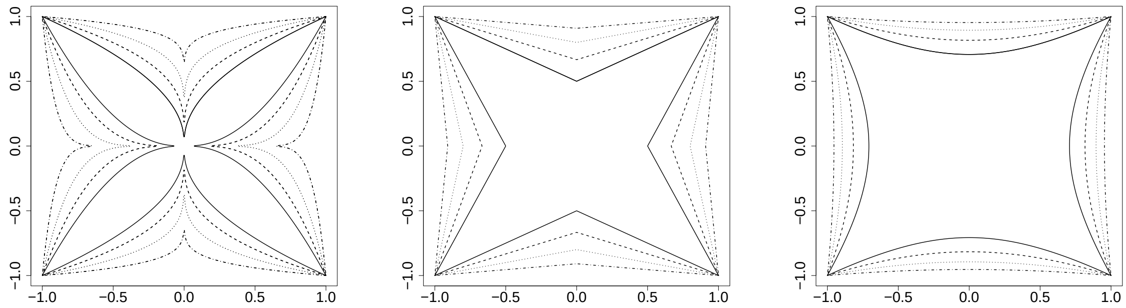
Hence, the *limit set* is given by

$$E := E_{\lambda,\theta} := \{\mathbf{u} \in \mathbb{R}^d \mid |u_1|^\theta + \cdots + |u_d|^\theta + \lambda \ge (\lambda + d)\|\mathbf{u}\|_\infty^\theta\}. \tag{2.12}$$

2.4 The shape of the limit set

For dimension $d \ge 2$ the shape of the limit set is determined by two positive parameters, the exponents λ and θ.

For $d = 2$ the set E consists of four symmetric petals with vertices in $(\pm 1, \pm 1)$, as shown in Fig. 3. The symmetry of the limit shape is due to the symmetry and equality of the marginals of the meta density. These symmetry conditions were imposed to keep the presentation simple. The sharp point of the petal at the vertex $(1, 1)$ follows from our basic assumption that large values of the two components of the meta



(a) $\theta = 0.1$ (b) $\theta = 1$ (c) $\theta = 2$

Figure 3: Possible shapes of the limit set $E_{\lambda,\theta}$ for $d = 2$ and different values of parameters λ and θ. Each plot corresponds to a given value of θ. The line legend specifies the value of λ: $\lambda = 1$ (solid), $\lambda = 2$ (dashed), $\lambda = 4$ (dotted), $\lambda = 10$ (dotdash).

vector should be dependent. Given these boundary conditions, the petals may still be convex, concave, or have linear edges. All three cases are present in Fig. 3.

On the cone $\{|u| < v\}$ the level set ∂E is the graph of the function

$$u \mapsto v = c(\lambda + |u|^\theta)^{1/\theta} \qquad c = (1+\lambda)^{-1/\theta}.$$

The function is symmetric on $[-1, 1]$. It is convex on $[0, 1]$ for $0 < \theta \leq 1$ and concave for $\theta \geq 1$. The basic constants are the minimum v_{00} in $u = 0$ and the slope s_{00} in $u = 1$:

$$v_{00} = (1 + 1/\lambda)^{-1/\theta}, \qquad s_{00} = 1/(1+\lambda). \tag{2.13}$$

For $d \geq 2$ on the inverted pyramid Π_1 the level set $\Gamma = \{\chi = 0\}$ may be described as the graph of a function: $v = v(u_1, \ldots, v_{d-1})$ by solving

$$(\lambda + d - 1)v^\theta = \lambda + |u_1|^\theta + \cdots + |u_{d-1}|^\theta. \tag{2.14}$$

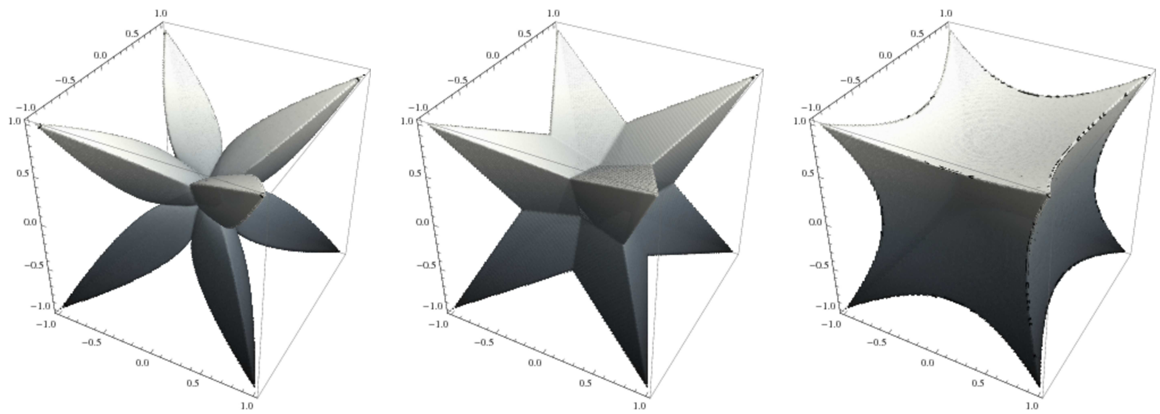
Let us first consider this function $(u_1, \ldots, u_{d-1}) \mapsto v$ on the whole space $\mathbb{R}^{d-1}$. Let H be the set above the graph. It intersects horizontal hyperplanes $v = v_0$ in the sets

$$|u_1|^\theta + \cdots + |u_{d-1}|^\theta \leq C(v_0). \tag{2.15}$$

The constant $C(v_0) = (\lambda + d - 1)v_0^\theta - \lambda$ is positive for

$$v_0 > v_{00} = 1/(1 + (d-1)/\lambda)^{1/\theta}. \tag{2.16}$$

The quantity v_{00} is the minimum of the function v. For $\theta \geq 1$ the set H is convex and the level sets of the function v are disks in the l^θ norm.



(a) $\theta = 0.5$ (b) $\theta = 1$ (c) $\theta = 2$

Figure 4: Possible shapes of the limit set $E_{\lambda,\theta}$ for $d = 3$, $\lambda = 1$, and different values of θ.

In particular, for $\theta = 2$ the graph of v is a cylinder symmetric hyperbola with asymptotic cone

$$v = c\|(u_1, \ldots, u_{d-1})\| \qquad c = 1/\sqrt{\lambda + d - 1}.$$

The point $\mathbf{1}$ lies on the hyperbola. On the inverted pyramid Π_1 the limit set E is the complement of the convex set H above the hyperbola: $E \cap \Pi_1 = \Pi_1 \setminus H$. By symmetry the same holds for the remaining $2d-1$ pyramids into which the cube $C = [-1, 1]^d$ may be split. Let S be the boundary of a smaller cube v_0C with $0 < v_0 < 1$. On each of the faces, the set E is the complement of the disk of radius $r_0 = \sqrt{C(v_0)}$ in (2.15). For $\lambda = 1$ and $d = 3$ we find $r_0^2 = 3v_0^2 - 1$, and hence for values $v_0 > 1/\sqrt{2}$ the intersection of E with the boundary S of the cube $[-v_0, v_0]^3$ will consist of eight disjoint components around the eight vertices of the cube. This phenomenon, in dimension $d = 2$, is already visible in Fig. 1c.

For $\theta = 1$ and $d = 3$ the set above the graph of v is the convex cone C with top in $(0, 0, 1/(\lambda + 2))$ which intersects the horizontal plane $v = 1$ in the rotated square $|u_1| + |u_2| \leq 2$ (since $\mathbf{1} \in \partial C$). See Fig. 4 for 3-dimensional visualizations of the limit set $E_{\lambda,\theta}$.

Proposition 2.3. *The limit set E is star-shaped with continuous boundary. It is symmetric for permutations and sign changes of the coordinates. It converges to the standard cube $C = [-1, 1]^d$ for $\lambda \to \infty$.*

Proof Let Π_+ denote the cone $\|\mathbf{u}\|_\infty \leq u_d$. The boundary ∂E contains the intersection of Π_+ with the graph of v. It suffices to observe that this intersection is closed, and that each ray in Π_+ intersects the graph of v in a unique point for $\lambda > 0$ by homogeneity. Symmetry follows from the symmetry of f and g_d. The limit relation holds by (2.16) and the inclusion $E \subset C$. ¶

The limit shape of a sample cloud from a distribution on $[0, \infty)^d$ with equal marginals G_d which satisfy (2.4) has the form

$$\{\mathbf{u} \in [0, \infty)^d \mid u_1^\theta + \cdots + u_d^\theta \leq 1\}$$

if the vector has independent components, see [4]. There is a superficial resemblance with our limit shape (2.12) with the role of the diagonals taken over by the axes.

2.5 Results

We can now formulate and prove the basic result of this paper, dropping the condition of cubic level sets for the original heavy-tailed density f. This result will then be refined. We prove almost sure convergence of the scaled sample clouds, and give a number of simple procedures for defining scaling constants. It will be shown that the level sets $\{g = c\}$ may be enclosed between the boundaries of scaled copies of the limit set E, and we give a simple upper bound for the tail of the intensities $g_n(\mathbf{u}) = nr_n^d g(r_n\mathbf{u})$ of the scaled sample clouds $N_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$.

Theorem 2.4. *Let f and g_d satisfy the assumptions of the standard set-up. Let condition* (2.4) *hold. Let g denote the meta density with marginals g_d associated with the density f. Let $r_n > 0$ satisfy $g_d(r_n) \sim 1/n$. Let $E = E_{\lambda,\theta}$ be the closed subset of the standard cube introduced in* (2.12). *Here λ is the parameter associated with f, and θ the exponent of regular variation of the function ψ in* (2.4). *Then the level sets $\{g \geq 1/n\}$ scaled by r_n converge to E. For the sequence of independent observations $\mathbf{X}_n$ from the meta density g, the scaled sample clouds $N_n = \{\mathbf{X}_1/r_n, \ldots, \mathbf{X}_n/r_n\}$ and their mean measures converge onto E.*

Proof First assume $f(\mathbf{z}) = c_0 f_0(\|\mathbf{z}\|_\infty)$. The denominator $\psi(s)$ in (2.7) increases without bound as $s \to \infty$. Hence the quotient $g(s\mathbf{1})/g(s\mathbf{u})$ goes to zero uniformly on any compact set disjoint from the closure of E, and to infinity on compact sets in $\text{int}(E)$ for $s \to \infty$. The theorem holds for sample clouds from g by the arguments above. See Proposition 2.5 below for the choice of the scaling constants r_n. Now observe that in the standard set-up $f(\mathbf{z}) \asymp c_0 f_0(\|\mathbf{z}\|_\infty)$ because ∂D fits in between two scaled cubes, and hence also $\tilde{f}(\mathbf{z}) \asymp c_0 f_0(\|\mathbf{z}\|_\infty)$. By Proposition 1.7 the densities g and $\tilde{g}$ corresponding to f and $\tilde{f}$ satisfy $\tilde{g} \asymp g$. Hence they have the same limit set. ¶

There are many ways in which the scaling constants r_n may be defined. Assume g_0 is a symmetric density on $\mathbb{R}$ with df G_0, and g_0 is asymptotic to the von Mises function $c_0 e^{-\psi}$ in infinity. One may define r_n by $\psi(r_n) = \log n$, $g_0(r_n) = 1/n$, $1 - G_0(r_n) = 1/n$ or $r_n g_0(r_n) = 1/n$.

Proposition 2.5. *These four sequences are asymptotically equal.*

Proof Let $\psi(b_n) = \log n$. Let $\epsilon > 0$. We have to show that the interval $[e^{-\epsilon} b_n, e^{\epsilon} b_n]$ eventually contains r_n for the other three definitions. This is so if the difference $\Delta(r) = \psi(e^\epsilon r) - \psi(r)$ is sufficiently large. We need the three relations: $\Delta(r) >> 1$, $\Delta(r) >> |\log a(r)|$ and $\Delta(r) >> \log r$. The last follows from $1/\psi'(r) = a(r) = o(r)$, and implies the first. The second is proved in the Appendix, Proposition A.2. ¶

Any sequence r_n which is asymptotic to one of the four sequences above for the marginal g_d may be used to scale the sample clouds from the meta density g. (Because the projections of the scaled sample clouds onto the vertical axis will then converge onto $[-1, 1]$.)

Let us now take a closer look at the meta density g and the scaled densities $g_n(\mathbf{u}) = r_n^d g(r_n\mathbf{u})$. We want to give bounds on the tails of g, and on the size of g_n inside E and outside. The constants r_0 below may differ from line to line.

The limit (2.11) yields the elegant relation

$$\frac{\gamma(tr\mathbf{p}) - \gamma(sr\mathbf{q})}{\psi(r)} \to t^\theta - s^\theta \qquad r \to \infty, \tag{2.17}$$

uniformly in $\mathbf{p}, \mathbf{q} \in \partial E$ and $0 \leq s, t \leq c$ for any $c > 1$. (Since $(\chi(s\mathbf{q}) - \chi(t\mathbf{p})) \cdot \tau = t^\theta - s^\theta$.)

For any $\epsilon > 0$ there exist constants $\delta > 0$ and $r_0 > 1$ such that

$$\gamma(e^\epsilon r\mathbf{p}) - \gamma(r\mathbf{q}) \geq e^\delta \psi(r)\theta\epsilon \qquad \mathbf{p}, \mathbf{q} \in \partial E,\ r \geq r_0. \tag{2.18}$$

(Choose $\delta > 0$ so that $e^{\theta\epsilon} - 1 \geq e^{2\delta}\theta\epsilon$.)

Let $m(r)$ and $M(r)$ denote the minimum and the maximum of the function g on $r\partial E$, the boundary of the set rE. The minimum may be much smaller than the maximum. However for any $\epsilon > 0$ eventually $M(e^\epsilon r) < m(r)$. By the inequality above for any $C > 1$ and $\epsilon > 0$

$$M(e^\epsilon r) << e^{-\theta\epsilon\psi(r)} m(r)/r^C \qquad r \to \infty \tag{2.19}$$

since $\psi(r) >> \log r$. The relation also holds if we define $m(r)$ to be the minimum of g over rE, and $M(r)$ the maximum over rR where R is the closure of the ring $2E \setminus E$. Instead of (2.18) use

$$\gamma(tr\mathbf{p}) - \gamma(sr\mathbf{q}) \leq e^\delta \psi(r)\theta\epsilon \qquad \mathbf{p}, \mathbf{q} \in \partial E,\ 0 \leq s \leq 1,\ e^\epsilon \leq t \leq 2e^\epsilon,\ r \geq r_0.$$

The inequality (2.19) now shows that $M(e^\epsilon r) > M(r)$ eventually, hence $g(\mathbf{x}) \leq M(r)$ for $\mathbf{x} \in rE^c$ and $r \geq r_0$, which shows that $M(r)$ is the maximum of g over the closed complement of rE for $r \geq r_0$. Hence (2.19) also holds if we redefine:

$$m(r) = \min\{g(\mathbf{x}) \mid \mathbf{x} \in rE\} \qquad M(r) = \max\{g(\mathbf{x}) \mid \mathbf{x} \in \mathrm{cl}(rE^c)\} \qquad r > 0.$$

In the standard set-up the level sets of g may be quite complicated, like a shore line with many small islands. The "shape" is expressed in the following proposition, which is an immediate consequence of the inequality (2.19) with the new interpretation of $m(r)$ and $M(r)$.

Proposition 2.6. *For any $\epsilon > 0$ there exists $c_0 > 0$ such that for $c \in (0, c_0]$ the level set $\{g = c\}$ lies between the boundary of r_1E and r_2E, where $r_1 = e^{-\epsilon}r$ and $r_2 = e^\epsilon r$, and the boundary of rE contains a point of the level set $\{g = c\}$.*

Let $g_n(\mathbf{u}) = na_n^d g(a_n\mathbf{u})$, where we choose a_n so that $g_n(a_n\mathbf{1}) = 1$. Consider the behaviour of g_n on the boundaries Γ_k of the sets $e^{k\epsilon}E$ for $k = 0, 1, 2, \ldots$. Observe that $g_n \leq 1$ holds on Γ_1, and even $g_n \leq e^{-\theta\epsilon\psi(a_n)}$. The function g_n decreases by a factor at least $e^{\theta\epsilon\psi(a_n)}$ as one moves from Γ_k to the next boundary curve Γ_{k+1}. If we define $M_n(r)$ as the maximum of g_n over the closed complement of rE, then $M_n(e^\epsilon r) \leq r^{\theta\psi(a_n)}$ for $r \geq 1$, and hence in terms of the gauge function n_E

$$g_n(e^\epsilon \mathbf{u}) \leq 1/n_E(\mathbf{u})^{\theta\psi(a_n)} \qquad \mathbf{u} \in E^c. \tag{2.20}$$

The function g_n is the density of the mean measure μ_n of the scaled sample cloud $\{\mathbf{X}_1/a_n, \ldots, \mathbf{X}_n/a_n\}$. For large n

$$\mu_n(e^\epsilon E^c) = e^{d\epsilon} \int_{E^c} g_n(e^\epsilon \mathbf{u}) d\mathbf{u} \leq e^{d\epsilon} \int_{E^c} \frac{d\mathbf{u}}{n_E(\mathbf{u})^{\theta\psi(a_n)}}. \tag{2.21}$$

The integral on the right may be computed explicitly using (2.2). It is finite and asymptotic to $|E|/\theta\psi(a_n)$. By symmetry it has the scaling property

$$\int_{rE^c} \frac{d\mathbf{u}}{n_E(\mathbf{u})^{\theta\psi(a_n}} = \frac{r^d}{r^{\theta\psi(a_n)}} \int_{E^c} \frac{d\mathbf{u}}{n_E(\mathbf{u})^{\theta\psi(a_n)}} \qquad r > 0. \tag{2.22}$$

We see that the projection $\bar{\mu}_n$ of μ_n on the vertical axis satisfies $\bar{\mu}_n[e^\epsilon, \infty) \to 0$.

One may also look at the curve Γ_{-1}. Assume $e^{-\epsilon} \geq 1/2$. Then

$$g_n(e^{-\epsilon}\mathbf{u}) \geq e^{\epsilon\theta\psi(a_n/2)} \qquad \mathbf{u} \in E,\ n \geq n_0. \tag{2.23}$$

Hence $\mu_n(U) \to \infty$ for any non empty open set $U \subset e^{-\epsilon}E$, and $\bar{\mu}_n[e^{-2\epsilon}, \infty) \to \infty$. Since $\epsilon > 0$ is arbitrary we see that the measures $\bar{\mu}_n$, with density $na_n g_d(a_n s)ds$, converge onto $[-1, 1]$. This gives:

Theorem 2.7. *The sequences a_n and b_n, defined by $na_n^d g(a_n \mathbf{1}) = 1$ and $\psi(b_n) = \log n$, are asymptotic.*

Instead of the vertex $\mathbf{1}$ one may take any point $\mathbf{q} \in \partial E$ to define the scaling sequence.

Proposition 2.8. *Let $\mathbf{q}_n \in \partial E$ and $k \in \mathbb{Z}$. Define r_n by $nr_n^k g(r_n \mathbf{q}_n) = 1$. Then $r_n \sim b_n$.*

Proof For $k = d$ this follows from Proposition 2.6; else use (2.19) and $\psi(a_n) \sim \psi(b_n) >> \log b_n$ as noted in the proof of Proposition 2.5. ¶

Theorem 2.9. *The scaled sample clouds from the meta density converge almost surely onto E.*

Proof By the inequalities (2.21) and (2.22), and $\psi(r_n) \sim \psi(b_n) = \log n$ (by regular variation of ψ),

$$\mathbb{P}\{\mathbf{X}/r_n \in e^{2\epsilon}E^c\} \leq \mu_n(e^{2\epsilon}E^c)/n \sim e^{2d\epsilon - \theta\psi(a_n)\epsilon}/n << 1/n^{1+\epsilon\theta/2} \qquad n \to \infty.$$

Hence almost surely $\mathbf{X}_n/r_n \in e^{2\epsilon}E$ eventually. Similarly the probability p_n that there is no scaled sample point in an open set $U \subset e^{-\epsilon}E$ is small, $\sum p_n < \infty$. Hence U a.s. eventually contains a point of each scaled sample. By cutting up U into m horizontal open slices of positive volume we see that U a.s. eventually contains m points. We may conclude that $N_n(\mathbf{p} + \epsilon B) \to \infty$ holds almost surely for any open ϵ-ball centered in a point $\mathbf{p} \in E$. Here we give the asymptotics on $p_n \sim e^{-\mu_n(U)}$: The upper bound on g_n in (2.23) together with regular variation of ψ gives $\epsilon\theta\psi(r_n/2) \sim c \log n$ with $c = \epsilon\theta/2^\theta$. So $\mu_n(U) \geq |U|n^{c/2}$ eventually for all open sets $U \subset e^{-\epsilon}E$. Then $n^{c/2} >> \log n$ implies $p_n \leq 1/n^2$ eventually. ¶

3 The domain of the limit shape

The scaled sample clouds from the density f in Theorem 2.4 converge in distribution to a Poisson point process N with intensity h in (0.1) weakly on the complement of any centered ball. If one replaces the density f by a density $\tilde{f}$ which is asymptotic to f, the sample clouds from $\tilde{f}$ will converge with the same scaling. Convergence need not hold if $\tilde{f}$ is weakly asymptotic to f, $\tilde{f} \asymp f$. By Proposition 1.7 the meta densities g and $\tilde{g}$ are weakly asymptotic if f and $\tilde{f}$ are. The sample clouds from g and $\tilde{g}$ will converge onto the set E with the same scaling.

That raises the question in how far one can alter the original heavy-tailed distribution (with density f) and still retain the same limit set E with the same scaling constants for the meta distribution with marginals g_d. We shall consider probability distributions with the marginal densities of f. This condition on the marginals sometimes obscures the argument. Hence we shall also look at discrete distributions, and then assume that the vectors $\mathbf{X}$ and $\mathbf{Z}$ are related by $\mathbf{Z} = K(\mathbf{X})$ where K is the meta transformation associated with the multivariate density f and the marginal density g_d. In the section on sensitivity (Section 3.2), it is shown that small perturbations of the distribution with density f, perturbations that do not affect the marginals or the convergence of the scaled sample clouds from the density f, may drastically alter the limit shape of the scaled sample clouds from the meta distribution.

3.1 Domains

Suppose f satisfies the standard assumptions from Section 2.1, Definition 3. There exist scaling constants r_n such that the mean measures $d\rho_n(\mathbf{w}) = nr_n^d f(r_n\mathbf{w})d\mathbf{w}$ of the scaled sample cloud $N_n = \{\mathbf{Z}_1/r_n, \ldots, \mathbf{Z}_n/r_n\}$ converge to the limit measure $d\rho(\mathbf{w}) = h(\mathbf{w})d\mathbf{w}$ weakly on the complement of any centered ball, where h is defined in (0.1). What probability measures $\tilde{\pi}$ will yield the same asymptotic behaviour? In particular one may ask: If one replaces the density f by a discrete probability measure, how large and how far apart are the atoms allowed to be? There is a simple criterium in terms of partitions.

Definition 5. *A* regular partition *of* $\mathbb{R}^d$ *is a countable collection of disjoint bounded Borel sets* A_n *of positive volume,* $|A_n| > 0$*, for which the following conditions hold:*

1) every bounded set in $\mathbb{R}^d$ *is covered by a finite number of sets* A_n*;*

2) each set A_n *may be enclosed in a ball of radius* r_n *centered in* $\mathbf{z}_n \in A_n$*,*

$$A_n \subset \mathbf{z}_n + r_n B, \tag{3.24}$$

such that $r_n = o(\|\mathbf{z}_n\|_2)$*.*

If (A_n) is a regular partition, if $\mathbf{z}_n \in A_n$, and if $\tilde{\pi}$ is a probability measure on $\mathbb{R}^d$ with mass $p_n = \int_{A_n} f(\mathbf{z})d\mathbf{z}$ in $\mathbf{z}_n$, then a sequence of independent observations $\tilde{\mathbf{Z}}_1, \tilde{\mathbf{Z}}_2, \ldots$ from the discrete distribution

$\tilde{\pi}$ has the same asymptotic behaviour as observations $\mathbf{Z}_n$ from the density f. Convergence $nr_n^d f(r_n\mathbf{w}) \to h(\mathbf{w})$ for $\mathbf{w} \neq 0$ implies weak convergence of $\rho_n = n\epsilon_{r_n^{-1}}(\tilde{\pi})$ to $d\rho(\mathbf{w}) = h(\mathbf{w})d\mathbf{w}$ on the complement of centered balls; the sample clouds $\tilde{N}_n = \{\tilde{\mathbf{Z}}_1/r_n, \dots, \tilde{\mathbf{Z}}_n/r_n\}$ converge in distribution to the Poisson point process N with intensity h weakly on the complement of centered balls. Here ϵ_r is the scalar expansion $\epsilon_r : \mathbf{x} \mapsto r\mathbf{x}$. Similar results hold for all probability measures $\tilde{\pi}$ for which $\tilde{\pi}A_n \sim \int_{A_n} f(\mathbf{z})d\mathbf{z}$. There also is a converse, which shows that densities f in the standard set-up play an important role in the characterization of domains of attraction.

Theorem 3.1. *Suppose f satisfies assumptions of the standard set-up in Definition 3. Let r_n be the associated scaling constants, and $d\rho(\mathbf{w}) = h(\mathbf{w})d\mathbf{w}$ the limit measure. If $\tilde{\pi}$ is a probability measure on $\mathbb{R}^d$ such that the mean measures $\mu_n = n\epsilon_{r_n}^{-1}(\tilde{\pi})$ of the scaled sample clouds converge weakly to ρ on the complement of any centered ball, then there exists a regular partition (A_n) such that $\tilde{\pi}(A_n) \sim \int_{A_n} f(\mathbf{z})d\mathbf{z}$.*

Proof See [1], Theorem 16.27.

Now turn to the meta distribution with density g. Assume the standard conditions. Also assume regular variation of the function ψ in the exponent of the von Mises function $e^{-\psi}$ as in (2.4). The measures with densities $nr_n^d g(r_n\mathbf{u})$ converge onto the set $E = E_{\lambda,\theta}$ for a suitable choice of scaling constants r_n by Theorem 2.4.

Proposition 3.2. *Let (A_n) be a regular partition, and $\tilde{\pi}$ a probability measure such that $\tilde{\pi}(A_n) \sim \int_{A_n} g(\mathbf{x})d\mathbf{x}$. Then the measures $\tilde{\rho}_n = n\epsilon_{r_n}^{-1}(\tilde{\pi})$ converge onto the set E.*

Proof For any $\delta > 0$ there exists a constant $r_0 > 1$ such that for $r \geq r_0$ all sets A_n/r which intersect the cube $[-2,2]^d$ have diameter less than δ. We first show that $\rho_n(S) \to \infty$ for any ball $S = \mathbf{x} + \epsilon B$ in the interior of E. The ball $S' = \mathbf{x} + (\epsilon - \delta)B$ is covered by the union U_n of the sets A_k/r_n which are contained in S for $n \geq n_0$, and hence

$$\tilde{\rho}_n(S) \geq \tilde{\rho}_n(U_n) \sim \int_{U_n} h_n(\mathbf{w})d\mathbf{w} \geq \int_{\mathbf{x}+(\epsilon-\delta)B} h_n(\mathbf{w})d\mathbf{w} \to \infty.$$

Similarly the integral of h_n over the complement of the cube $[-3/2, 3/2]^d$ goes to zero, and this implies that the $\tilde{\rho}_n$ measure of the union V_n of all atoms A_k/r_n which intersect the complement of $[-2,2]^d$ goes to zero. By the same argument the $\tilde{\rho}_n$ measure of a compact ball at distance $> \delta$ from E will vanish for $n \to \infty$. ¶

Now replace the original measure with density f by the probability distribution $\tilde{\mu} = K(\tilde{\pi})$. Then

$$\tilde{\mu}(B_n) \sim \int_{B_n} f(\mathbf{z})d\mathbf{z} \qquad B_n = K(A_n). \tag{3.25}$$

If we choose $\tilde{\pi}$ to have marginals g_d, like g, then the probability measure $\tilde{\mu}$ will have the same univariate marginals as the density f. For any probability measure $\tilde{\mu}$ on $\mathbb{R}^d$ which satisfies (3.25) for the partition (B_n), and which has the same univariate marginals as the density f, the scaled sample clouds from the meta distribution with marginals g_d converge onto the set E with the scaling constants $r_n \sim b_n$ where $\psi(b_n) = \log n$ as in Theorem 2.7.

In order to get some insight in these partitions (B_n) in **z**-space, we consider standard partitions in **x**-space generated by centered coordinate cubes $s_nC = [-s_n, s_n]^d$ where $0 < s_1 < s_2 < \cdots$ and $s_{n+1} \sim s_n \to \infty$. We create a regular partition by dividing the square rings $R_n = s_{n+1}C \setminus s_nC$ into blocks whose edges are $o(s_n)$. This may be done by dividing each side of the cube $s_{n+1}C$ into subintervals by a symmetric partition:

$$s_{n,0} = 0 < s_{n,1} < \cdots < s_{n,m} = s_n < s_{n+1} \qquad s_{n,-k} = -s_{n,k}, \qquad m = m_n \to \infty.$$

The sequence m_n may go to infinity quite slowly, say $m_n = [1 + \log n]$. One may take all $2m$ subintervals of $[-s_n, s_n]$ of equal length, but it suffices that δ_n, the length of the maximal interval is $o(s_n)$ for $n \to \infty$. If we assume for simplicity that the marginals of f are equal and symmetric, then the image of a centered coordinate cube in **x**-space is a centered coordinate cube in **z**-space, and the $2^d((m_n + 1)^d - m_n^d)$ blocks in the square ring $[-s_{n+1}, s_{n+1}]^d \setminus [-s_n, s_n]^d$ are mapped by the meta transformation K into blocks in the square ring $R_n = [-t_{n+1}, t_{n+1}]^d \setminus [-t_n, t_n]^d$, where $t_n = K_d(s_n)$. These blocks are determined by the partition

$$t_{n,0} = 0 < t_{n,1} < \cdots < t_{n,m} = t_n < t_{n+1} \qquad t_{n,-k} = -t_{n,k} \qquad t_{n,k} = K_d(s_{n,k}).$$

What does the partition in **z**-space look like? The cubes $t_nC = [-t_n, t_n]^d$ with $t_n = K_d(s_n)$ are huge. For any $\delta \in (0, 1)$ one may choose $s_n \sim s_{n+1} \to \infty$ such that

$$t_n = n^{n^{n^{1-\delta}}}.$$

Since $t_n \ll \exp(e^{n^{1-\delta/2}})$ we shall assume $t_n = e^{r_n}$ where $r_n = \exp(n^{1-\delta})$. Then $r_{n+1}/r_n \to 1$, but

$$t_n^{1-1/n} \ll t_{n-1} \ll \frac{t_n}{e^n} \ll \frac{t_n}{\log t_n} \ll t_n.$$

The partition in **x**-space is regular if we subdivide the ring $R_n = t_{n+1}C \setminus t_nC$ into blocks $B_{n\mathbf{k}}$ according to an exponential subdivision of the sides given by $t_{nk} = e^{r_{nk}}$ with $r_{nk} = (k/m)r_n$ for $k = 1, \ldots, m = m_n$.

The probability mass $p_{n\mathbf{k}}$ of the block $B_{n\mathbf{k}}$ is the integral of f over this block. The masses of the blocks in a given ring R_n are very unequal. In the bivariate case the twelve rectangles, which contain the vertices of the square $[-t_n, t_n]^2$, contain almost all mass of the ring.

If one replaces the density f by a discrete measure with mass $p_{n\mathbf{k}}$ in a point $\mathbf{z}_{n\mathbf{k}}$ in the block $B_{n\mathbf{k}}$ then the image of the sample clouds associated with a sequence of independent observations $\tilde{\mathbf{Z}}_1, \tilde{\mathbf{Z}}_2, \ldots$ from this discrete probability distribution under the original coordinatewise map K^{-1} will converge onto the set E if one scales by r_n. Alternatively on each block one can define a measure $\mu_{n\mathbf{k}}$ of mass $p_{n\mathbf{k}}$ on a curve linking two diametric vertices of the block $B_{n\mathbf{k}}$ such that the measure $\mu_{n\mathbf{k}}$ has the same univariate marginals as the restriction of the density f to the block. One is free to choose a copula for each block.

The sum μ then has the same marginals as f. The meta distribution is $K^{-1}(\mu)$. Sample clouds from this distribution have the same asymptotic shape as the sample clouds from the original meta density g.

The probability distribution π above is an example of a perturbation of the original distribution with density f for which the meta transformation is the same, and for which the sample clouds from the meta distribution have the same asymptotic behaviour as those from the meta density g based on f. The original distribution given by the density f may be distorted to a considerable extent without affecting the meta transformation K, or the first order asymptotic behaviour of the sample clouds from the meta distribution, as described by the limit set $E = E_{\lambda,\theta}$.

3.2 Sensitivity

The limit shape turns out to be sensitive to slight perturbations of the original density. Proposition 3.2 shows that the tails of the meta distribution with density g may be mangled without destroying the convergence of the scaled sample clouds to the limit set $E_{\lambda,\theta}$. Below we show that small changes in the density f may, however, alter the limit set E drastically.

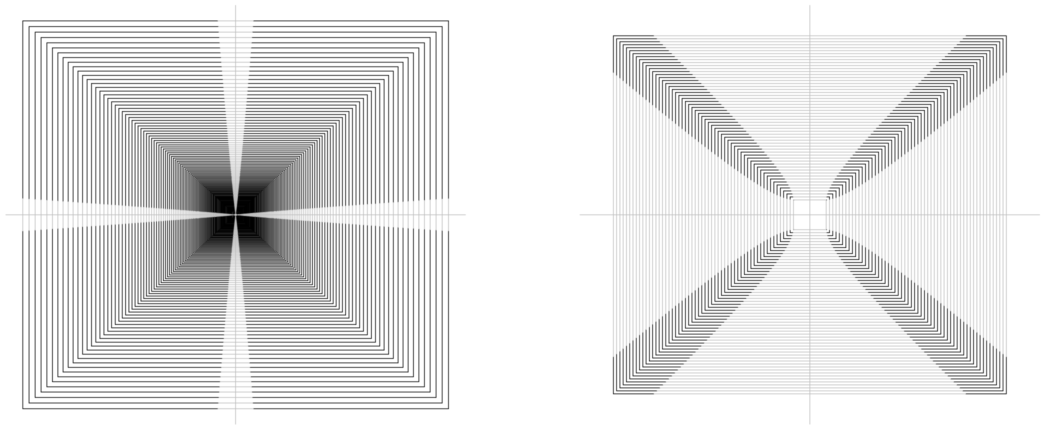
Example 2. Assume f is a density on the plane with square level sets and Student t marginal densities which decrease like $1/2t^2$. We delete the mass on a thin strip T along the positive vertical axis:

$$T = \{(x, y) \mid |x| \leq \frac{y}{\log y},\ y \geq e\}.$$

This strip is asymptotically negligible since $|x|/y \leq 1/\log y \to 0$. We like the marginals to be equal and symmetric, and hence also delete f on the three sets obtained by reflections in the diagonals. One may compensate for the lost mass by increasing the density in a compact neighbourhood of the origin. The new density $\tilde{f}$ may be assumed to have equal continuous positive marginals. These satisfy $\tilde{f}_2(t) \sim 1/2t^2$. Choose $K_i(s) = e^s$ for $s \geq s_0$. Then $g_d(s) \sim e^{-s}/2$. Let $C_n = e^n C$ be the square with edge length $2e^n$ in $\mathbf{z}$-space. The corresponding square in $\mathbf{x}$-space is nC, with edge length $2n$. The interval $[-e^n, e^n] \times \{e^n\}$ maps onto $[-n, n] \times \{n\}$. The subinterval $[-e^n/n, e^n/n] \times \{e^n\}$ is deleted. It maps onto $[-n + \log n, n - \log n] \times \{n\}$. See Fig. 5.

We now have a density $\tilde{f}$ close to f such that the meta density $\tilde{g}$ vanishes everywhere except on a thin strip around the diagonals. Hence the scaled sample clouds from $\tilde{g}$ converge onto the cross E_{00} consisting of the two diagonals in the standard square $C = [-1, 1]^2$. The functions $f_r(\mathbf{z}) = \tilde{f}(r\mathbf{z})/\tilde{f}(r\mathbf{1})$ still converge (almost everywhere, and in $\mathbf{L}^1$ on ϵB^c for all $\epsilon > 0$) to the function $1/\|\mathbf{w}\|_\infty^3$. The sample clouds from the density $\tilde{f}$ and from the density f converge to the same Poisson point process with intensity $h(\mathbf{w}) = 1/\|\mathbf{w}\|_\infty^3$. The coordinatewise maxima from the densities $\tilde{f}$ and f have the same limit; so do the coordinatewise extremes from the meta densities $\tilde{g}$ and g by Galambos's theorem. ◊

There is a converse. Restrict f to the union U of T and its three reflections, multiply by $\log \|\mathbf{z}\|_\infty$ to ensure that the marginal densities decrease asymptotically like $1/2t^2$, and make some changes on a



(a) $C_n = e^n C$ in $\mathbf{z}$-space (b) $C_n = nC$ in $\mathbf{x}$-space

Figure 5: The squares with edge lengths $2e^n$ in $\mathbf{z}$-space (a) with subintervals $[-e^n/n, e^n/n] \times \{e^n\}$ deleted and the images $C_n = nC$ in $\mathbf{x}$-space (b) with the corresponding subintervals $[-n + \log n, n - \log n] \times \{n\}$ deleted.

compact set to ensure that the two marginals are equal, symmetric and positive. The meta transformation K above with $K_i(s) = e^s$ for $s \geq s_0$ will give a meta density $\tilde{g}$ with marginals $\tilde{g}_d(s) \sim e^{-s}/2$ for $s \to \infty$. Now the sample clouds from $\tilde{g}$ converge onto E, even though the density $\tilde{f}$ lives on a thin set which clings to the axes. The scaled sample clouds from $\tilde{f}$ converge to a Poisson point process which lives on the axes. The max-stable limit vector for f (and hence for g) has independent components. $\diamondsuit$

With some extra effort one may create for any star-shaped compact set $E_0 \subset [-1, 1]^d$ with continuous boundary a continuous positive density $\tilde{f}$ with the same marginals as f such that the sample clouds from $\tilde{f}$ and f converge to the same Poisson point process with the same scaling, and such that the sample clouds from $\tilde{g}$ converge onto the set $E = E_0 \cup E_{00}$, where E_{00} is the diagonal cross, the union of the intervals linking the origin to the vertices of the bounding cube. For $E_0 = [-1, 1]^d$ one can construct a density $\overline{g}$ with cubic level sets and marginals $\overline{g}_d$ such that $\overline{g}_d(s) \ll g_d(s)$ but $\overline{g}_d(cs) \gg g_d(s)$ for any $c \in (0, 1)$ (see Lemma A.3 and Proposition A.4 for details). Then the marginals of $f + \overline{f}$ and of f are asymptotic and one may use a meta transformation to replace $f + \overline{f}$ by a density $\tilde{f}$ with marginals f_d.

3.3 Discussion

In situations where chance plays a role the asymptotic description often consists of two parts, a deterministic term, catching the main effect, and a stochastic term, describing the random fluctuations around the deterministic part. Thus the average of the first n observations converges to the expectation; under additional assumptions the difference between the average and the expectation, blown up by a factor $\sqrt{n}$, is asymptotically normal. Empirical dfs converge to the true df; the fluctuations are modeled by a time changed Brownian bridge. For a positive random variable the n point sample clouds N_n scaled by the $1-1/n$ quantile converge onto the interval $[0,1]$ if the tail of the df is rapidly varying; for convergence of the maxima one needs the extra condition that the tail is asymptotic to a von Mises function.

Example 3. Suppose F is the df of the absolute value of a standard Gaussian variable. The scaled sample clouds N_n/r_n converges onto $[0,1]$ if $1-F(r_n)=1/n$, but $(N_n-r_n)/a_n$ with $a_n=1/\sqrt{2\log n}$ converges (in distribution) to a Poisson point process with intensity e^{-s}. Convergence of the scaled sample clouds onto the interval $[0,1]$ with this scaling sequence r_n will hold for any df G which agrees with F in a sequence of points $t_n\to\infty$ provided $t_{n+1}/t_n\to 1$. The tails of F and G may differ considerably. With some effort one may construct a sequence t_n and a df G such that

$$G(t_n)=F(t_n)\qquad (1-G(t_n-))/(1-F(t_n))=n^{n^{\sqrt{n}}}.$$

Convergence to the first order deterministic term is a much more robust affair than convergence of the random fluctuations around this term.

It is surprising that in the theory developed in this paper perturbations of the original distribution which do not affect the second order fluctuations at the vertices may drastically alter the shape of the limit set, the first order term.

This peculiarity of the theory of limit shapes for meta distributions could be due to the nature of the meta transformation. The map K is highly nonlinear. It respects coordinatewise maxima, but destroys geometric objects: ellipsoids, convex sets, hyperplanes, cones and rays.

It should also be noted that a limit shape is less stable than a limit point. We have assumed the marginals of the meta density to be equal and symmetric. That is natural. We are free to choose the marginals and the dependence structure separately, hence we choose well behaved marginals. If the marginal densities are not symmetric even the convex hulls of the sample clouds will have a limit (a coordinate box) only under special balance conditions for the upper and lower quantiles of the marginal distributions.

A more technical explanation for the peculiar sensitivity of the limit shape is the incompatibility of the partition (A_n) and $(B_n=K(A_n))$ of Section 3.1. Both partitions may be seen as partitions of $\mathbf{z}$-space. The partition (A_n) is associated with the max-stable limit distribution; the partition (B_n) is associated with the limit shape of the sample clouds from the meta distribution. If we regard the atoms of the partition (B_n) as nerve cells, then the region around the axes is far more sensitive than the remainder of the space, and it is not surprising that cutting away these regions has drastic effects on the limit.

The limit shape describes the variation in the distribution of large observations as the direction changes. Insight in this variation is important for risk analysis. If one assumes that the loss function is known, and increases as one moves out in the state space on which the density lives, then, given the rate at which the tails decrease along rays, the limit shape of the level sets will determine the asymptotic distribution of high losses. Unfortunately the non-linear nature of the meta transformation destroys the sense of direction. Under the transformation K rays in $\mathbf{x}$-space turn into curves in $\mathbf{z}$-space which are attracted towards the $2d$ halfaxes; under the inverse transformation rays in $\mathbf{z}$-space turn into curves in $\mathbf{x}$-space which are attracted towards the 2^d semidiagonals. For densities f in the standard set-up the direction of large sample points is fairly uniformly distributed; the variation is determined by the function η in (0.1), a continuous positive function on a compact set. In the meta distribution the large observations cluster around the 2^d semidiagonal rays, the components are either asymptotically comonotonic or countermonotonic.

There is a dual result. For light-tailed densities with elliptic level sets the meta densities with Student t marginals concentrate around the axes. If the original density is Gaussian with spherical level sets, the meta vector has independent t distributed components, and so has the max-stable limit. Scaled sample clouds from this multivariate t distribution converge to a Poisson point process on $\mathbb{R}^d \setminus \{\mathbf{0}\}$ whose mean measure lives on the axes.

The shape of the level sets and sample clouds of the meta density reflect the structure of the density. In Section 1 we observed that the Jacobian in the expression for the meta density creates ridges along the semidiagonal rays. In order to obtain more insight into the structure of these ridges, we depict in Fig. 6 two sections at the levels $y = 2$ and $y = 6$ of the bivariate meta density of Fig. 1c. Fig. 6 suggests that the ridges are steep, with the mass concentrated along the centre, the points on the diagonal. This is due to rapid variation of the density. Let us see what happens as the level y goes to infinity.

For both the original vector $\mathbf{Z}$ and for the meta vector $\mathbf{X}$ the conditional density given the value of the vertical component Z_d or X_d may be written down without ado. Scale the conditional distribution by the value of the vertical component, and let this value go to infinity. The conditional distributions converge. For the vector $\mathbf{Z}$ with the heavy-tailed density the limit distribution is continuous on $\mathbb{R}^{d-1} \times \{1\}$ with density $\propto h(w_1, \ldots, w_{d-1}, 1)$ where h is the limit function in (0.1). For the meta vector the conditional distributions converge to a discrete probability distribution concentrated in the 2^{d-1} vertices δ of the standard cube in the positive halfspace $\{u_d \geq 0\}$. The probability distribution is given by

$$p(\delta) = \rho(Q(\delta) \cap \{w_d \geq 1\})/\rho\{w_d \geq 1\} \qquad \delta_i \in \{-1, 1\},$$

where $Q(\delta)$ denotes the orthant containing the point δ, and ρ is the infinite measure with density h. The numbers $p(\delta)$ reflect the asymmetry of the distribution of the tails of f in the upper halfspace.

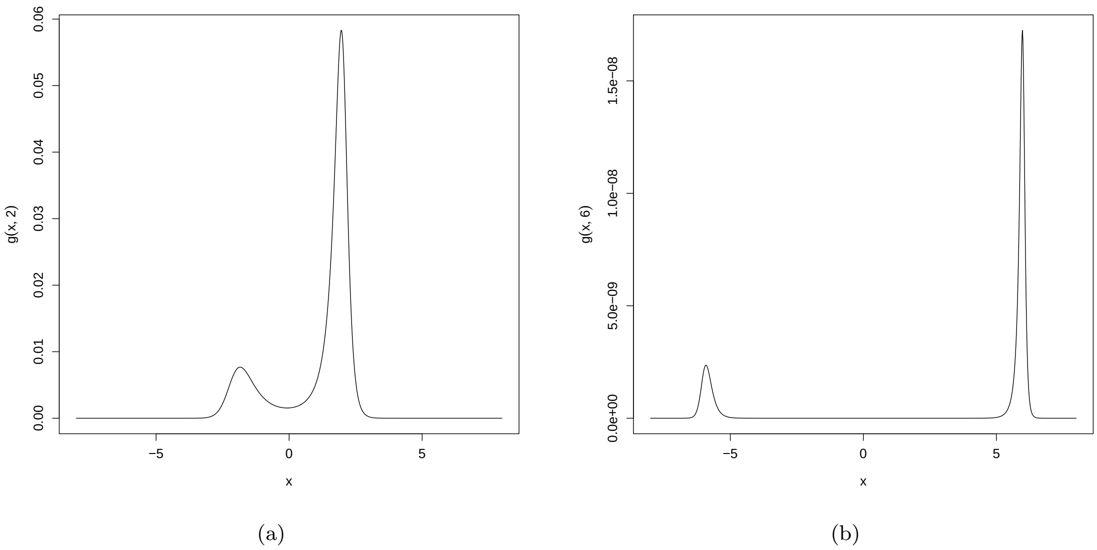


Figure 6: Sections of a bivariate meta-Cauchy density $g(x, y)$ with standard normal marginals at levels (a) $y = 2$ and (b) $y = 6$. The original density has Cauchy marginals with scale parameter $\sqrt{5/4}$ and level sets shaped like the ellipse $5x^2 + 6xy + 5y^2 = 1$.

4 Conclusions

Gaussian models may perform well for multivariate data but still fail in describing extremal situations. This failure may be due to the tail behaviour of the marginals. It may also be due to a non-Gaussian dependence structure. This paper addresses the second cause, but also touches on the first.

The setting in which we work is rather limited. The issue of importing asymptotic dependence for large observations in a Gaussian world is an important one. We focus on a class of dependence structures determined by a well circumscribed, well understood family of heavy-tailed densities. These densities have exemplary limit behaviour - under scalar normalization they converge to a continuous positive function, the intensity of the limit point process for the sample clouds. In the meta world we weaken the condition of Gaussian marginals, also allowing Weibull tails. But we retain our assumption of good behaviour for the marginals: the marginal densities are equal, continuous, positive, and symmetric.

In this limited setting we obtain precise and explicit results on the asymptotic behaviour of the meta density. These results are of interest in themselves. They should also help to clear up the relation between three fundamental concepts in multivariate asymptotics: dependence (copula); asymptotic dependence (shape of the sample clouds); and limit behaviour (of the coordinatewise extremes and high risk scenarios; refer to [1] for a definition of the latter).

For applications it is important to know the effect of small changes in the original distribution. (Weak) asymptotic equality of the densities has no effect on the limit shape, but if one replaces the original density by a probability measure with the same limit behaviour for extremes, even if one preserves the marginals, the shape of the sample clouds from the meta distributions may change. Here our results are of a rather sketchy nature, but they should cast some light on the relation between asymptotic dependence and asymptotic equivalence for multivariate probability distributions, and on the invariance of these concepts under transformations which preserve the copula.

We now list a number of concrete conclusions:

- If we import the dependence structure of an elliptic Student t distribution into a distribution with Gaussian marginals the components of the max-stable limit vector will no longer be independent. The probability that the coordinatewise maximum of the sample cloud is attained by one of the points of the sample is bounded away from zero as the size of the sample goes to infinity.
- The meta density g has a simple form. It may be described in terms of the Gaussian marginals and the level sets $\{g > c\}$. The level sets have a limit shape as c decreases to zero.
- A limit shape for the level sets of the meta densities exists whenever the original density f is continuous and may be scaled to converge to a function $h(\mathbf{w}) = \eta(\omega)/r^{\lambda+d}$ as in (0.1), and when the prescribed marginals are equal, continuous, positive, symmetric, and asymptotic to a von Mises function $e^{-\psi}$, where moreover $\psi \in RV_\theta$, $\theta > 0$. The shape is determined by the two positive exponents λ and θ, see Fig. 3.
- The limit shape of the level sets of the meta density is also the limit shape of the sample clouds. The scaled sample clouds converge almost surely.
- The scaling constants r_n for this convergence are only determined up to asymptotic equality. One may choose r_n to satisfy $g_d(r_n) = 1/n$, but there are various other explicit definitions in terms of the df G_d or the meta density g.
- It is not clear what is the relation between the copula and the asymptotic dependence of a distribution. The meta transformation preserves the copula, but in the first order asymptotics of the meta distribution, expressed in the shape of the sample cloud, all information about the asymptotic dependence of the original distribution is lost, at least in the standard set-up treated in this paper, and only the parameter λ is still visible. Given the marginal density g_d the limit shape of the sample clouds from the meta distribution is determined by one positive constant, the parameter λ in the asymptotic decay of the heavy-tailed density f.
- There is no relation between the limit shape of the level sets of the original densiy f (which is the common shape of the level sets of h), and the limit shape of the level sets of the meta density g.
- The limit shape describes the variation in the size of the extremes in terms of the direction. It helps the risk analyst to locate the directions in which the risk region is most likely to be entered.

- The limit shape is invariant under weak asymptotic equivalence of the original distributions. It is sensitive to excision of the original density around the axes. The exact relation between the original distribution and the limit shape of the meta density (with Gaussian marginals say), is still unclear.
- The relation between the limit laws for the coordinatewise maxima from the original distribution and from the meta distribution is simple. If there is a limit law for the coordinatewise maxima from the original distribution then the coordinatewise maxima from the meta distribution will converge provided the upper tails of the marginals of the meta distribution lie in the domain of attraction of a univariate extreme value distribution. This follows by Galambos's theorem. However, compared to the limit shape, the coordinatewise extrema are a second order phenomenon, which only comes to life when one zooms in on one of the vertices of the limit shape.
- The behaviour of the sample cloud at the edge will be investigated more fully in a future publication.

A Appendix

The appendix collects a number of technical results.

A.1 On regular variation

If a density varies regularly with exponent $-1-\lambda$ with $\lambda > 0$ then the distribution tail varies regularly with exponent $-\lambda$ (see Karamata's theorem 0.6 in [18]). The converse holds if the density is decreasing, see [3], Theorem 1.7.2, or if it satisfies a growth condition:

Lemma A.1. *Suppose the density f on $\mathbb{R}$ is positive and continuous on a neighbourhood of ∞ and satisfies $f(x_n) \sim f(y_n)$ for $x_n \sim y_n \to \infty$. If the distribution tail $R(x) = \int_x^\infty f(t)dt$ varies regularly with exponent $-\lambda < 0$, then the density varies regularly with exponent $-\lambda - 1$.*

Proof Let $x_n \to \infty$ and write $f_n(t) = f(x_n t)/c_n$ with $c_n = R(x_n)/x_n$. Then f_n is a density on $[1, \infty)$. Write $f_n(1) = a_n^2 \lambda$. Choose $s_n > 1$ minimal with $f_n(s_n) = a_n \lambda$. If $s_n \to 1$ then $a_n \to 1$ since $s_n x_n \sim x_n$ implies $f(s_n x_n) \sim f(x_n)$. So suppose $s_n \geq s > 1$. If $a_n > a > 1$ then $\int_1^s f_n(t)dt > a(s-1)\lambda > a(1-1/s^\lambda)$. Write

$$\int_1^s f_n(t)dt = \int_1^\infty f_n(t)dt - \int_s^\infty f_n(t)dt = 1 - \int_s^\infty \frac{f(x_n t)}{c_n} dt = 1 - \frac{1}{c_n x_n} \int_{s x_n}^\infty f(y)dy = 1 - \frac{R(s x_n)}{R(x_n)}.$$

We find $R(sx_n)/R(x_n) < 1 - a(1 - 1/s^\lambda) = 1/s^\lambda - (a-1)(1 - 1/s^\lambda)$, which contradicts the regular variation of R. Similarly for $a_n \leq a < 1$. Then $R(sx_n)/R(x_n) > 1 - a(1 - 1/s^\lambda)$. ¶

A von Mises function has the form $e^{-\psi}$, where ψ is a C^2 function on $[c, \infty)$ with a positive derivative, and where the scale function $a = 1/\psi'$ has a derivative which vanishes in infinity. Regular variation of the scale function implies regular variation of ψ. The converse need not hold.

Example 4. Let $\psi(t) = t + \sqrt{t}\cos\sqrt{t}$. Then $\psi(t) \sim t$ implies that ψ varies regularly. The derivative is $\psi'(t) = 1 - (\sin\sqrt{t})/2 + (\cos\sqrt{t})/2\sqrt{t}$. Hence $\psi''(t)$ vanishes, and so does $a'(t)$ since $\psi'(t) > 1/3$ eventually. Take $t_n = (2n\pi + \pi/2)^2$ and $s_n = (2n\pi - \pi/2)^2$. Then $s_n \sim t_n$, but $\psi'(t_n) \to 1/2$ and $\psi'(s_n) \to 3/2$. So ψ' does not vary regularly, and neither does the scale function $1/\psi'$. $\diamondsuit$

The function ψ is increasing and unbounded. The scale function a satisfies $a(t) = o(\psi(t))$ for $t \to \infty$. This implies that $\log(1 + a(t)) = o(\psi(t))$ for $t \to \infty$. However $|\log a(t)|$ will also be large if $a(t)$ becomes very small.

Proposition A.2. *Let ψ be a C^2 function on $[c, \infty)$ with a positive derivative. Set $a(t) = 1/\psi'(t)$. If $a'(t)$ vanishes for $t \to \infty$ then $|\log a(t)| = o(\psi(t))$.*

Proof By the remarks above, the positive part, $\log_+ a(t)$, is $o(\psi(t))$ for $t \to \infty$. However nothing prevents the scale function from becoming very small. We shall now show that a decrease in $a(t)$ by a factor e yields a larger increase in ψ eventually. Suppose $|a'(t)| \le \epsilon$ for $t \ge t_0$. Let $t_0 < t_1 < t_2$ and suppose $a(t_2) = a(t_1)/e$. Then $\psi(t_2) - \psi(t_1) \ge 1/\epsilon$. (The increase in ψ is minimal if a is maximal over the interval. So let a increase with slope ϵ, then decrease with slope $-\epsilon$ until it reaches its initial value, and finally decrease with slope $-\epsilon$ from the value $q = a(t_1)$ to the value $q/e = a(t_2)$. The increase of ψ over the final interval equals

$$\int_0^{s_0} \frac{ds}{q - \epsilon s} = -\frac{1}{\epsilon}\log(q - \epsilon s)\Big|_0^{s_0} = \frac{1}{\epsilon}\log\frac{q}{q - \epsilon s_0} = \frac{1}{\epsilon}$$

since $q - \epsilon s_0 = q/e$.) So, if beyond t_0 the scale function attains the value $a(t_0)/e^m$ in a point t then ψ will have increased by at least m/ϵ at that point. Hence $\psi'(t) \ge \psi'(t_0)$ implies $\log_+ \psi'(t) - \log_+ \psi'(t_0) - 1 \le \epsilon(\psi(t) - \psi(t_0))$. Since $\epsilon > 0$ is arbitrary it follows that $\log_+ \psi'(t) = o(\psi(t))$ for $t \to \infty$. ¶

Lemma A.3. *Let g be a positive continuous symmetric density which is asymptotic to a von Mises function $e^{-\psi}$. There exists a continuous unimodal symmetric density g_1 such that for all $c \in (0, 1)$*

$$g_1(s)/g(s) \to 0 \qquad g_1(cs)/g(s) \to \infty \qquad s \to \infty.$$

Proof Let $M_n(s) = \psi(s) - \psi(s - s/n)$, and let $M_n^*(s) = \min_{t>s} M_n(t)$ for $n \ge 2$. Each function M_n^* is increasing, continuous and unbounded (since $t/a(t) \to \infty$), and for each $s > 0$ the sequence $M_n^*(s)$ is decreasing. There exists a continuous increasing unbounded function b such that

$$\lim_{s\to\infty} M_n(s) - b(s) = \infty \qquad n = 1, 2, \ldots. \tag{A.1}$$

Indeed, define $M^*(s) = M_n^*(s)$ on $[a_n, b_n] = \{M_n^* \in [n, n+1]\}$, and $M^*(s) = n$ on $[b_{n-1}, a_n]$. Then $M^*(s)$ is increasing and $M^*(s) \le M_n^*(s)$ eventually for each $n \ge 2$. Set $b(s) = M^*(s)/2$ to obtain (A.1). The function $g_1 = e^{-\psi_1}$ with $\psi_1(s) = \psi(s) + b(s)$ is decreasing on $[0, \infty)$ and continuous, and $b(s) \to \infty$ implies $g_1(s)/g(s) \to 0$ for $s \to \infty$. For $c = 1 - 1/m$ the relations

$$\psi(s) - \psi_1(cs) = \psi(s) - \psi(cs) - b(cs) \ge M_m(s) - b(s) \to \infty$$

hold, and yield the desired result. ¶

Proposition A.4. *Let g_d be a continuous positive symmetric density which is asymptotic to a von Mises function $e^{-\psi}$. Choose r_n such that $\int_{r_n}^{\infty} g_d(s)ds \sim 1/n$. Let g_1 be the probability density in the lemma above. There exists a unimodal density $g(\mathbf{x}) = g_0(\|\mathbf{x}\|_\infty)$ on $\mathbb{R}^d$ with cubic level sets and marginals g_1. The sample clouds from the density g, scaled by r_n converge onto the standard cube $C = [-1,1]^d$. The functions $h_n(\mathbf{u}) = nr_n^d g(r_n\mathbf{u})$ are unimodal with cubic level sets. They satisfy*

$$h_n(\mathbf{u}) \to \begin{cases} \infty & \mathbf{u} \in (-1,1)^d \\ 0 & \mathbf{u} \notin [-1,1]^d. \end{cases}$$

Let E be a closed subset of C, containing the origin as interior point, star-shaped with continuous boundary, see (2.3)*. Set $c_E = |E|/2^d$. Then $g_E(\mathbf{x}) = g_0(n_E(\mathbf{x}))/c_E$ is a probability density, and the sample clouds from g_E scaled by r_n converge onto the set E.*

Proof Existence of g follows from Proposition A.5 below. Let G_1 be the df with density g_1. Then $n(1-G_1(cr_n)) \to 0$ for $c > 1$, and $n(1-G_1(cr_n)) \to \infty$ for $c \in (0,1)$. Let π be the probability distribution with the unimodal density g. The limit relations on the marginal dfs G_1 imply that $n\pi(B_n) \to \infty$ for the block $B_n = [-2r_n, 2r_n]^{d-1} \times [cr_n, 2r_n]$ for any $c \in (0,1)$. Since h_n is unimodal with cubic level sets it follows that $h_n \to \infty$ uniformly on $[-c,c]^d$ for any $c \in (0,1)$. (Since $h_n(c\mathbf{1}) \le k$ implies $n\pi(B_n) \le (2c)^d k$.) The area of a horizontal slice of the density g_E at level $y/c_E > 0$ is less than the area of the horizontal slice of g at level y, but the height of the slice is proportionally more by the factor c_E. So the slices have the same volume. The level sets of the scaled densities are related:

$$\{h_E \ge t/c_E\} = rE \quad \iff \quad \{h \ge t\} = rC.$$

So the function h_E mimics the behaviour of $h = h_C$. ¶

A.2 Probability densities with cubic level sets and given marginals

It is simple to construct continuous unimodal densities with convex level sets all of the same shape. Take a continuous decreasing function f_0 on $[0,\infty)$, the generator, and a bounded open convex set $D \subset \mathbb{R}^d$ containing the origin, and write $f(\mathbf{z}) = f_0(n_D(\mathbf{z}))$, see (2.1). If $t^{d-1}f_0(t)$ is integrable then so is f. The marginals $f_1, \ldots, f_d$ may be evaluated by integration. They will be continuous and unimodal.

It is much harder to determine for a given set D what densities may occur as marginals, and to reconstruct the density f from its marginal f_d, even in the case where D is the unit ball in l^p norm. If D is the Euclidean unit ball in $\mathbb{R}^3$ then the marginals of spherically symmetric probability distributions which do not charge the origin are precisely the unimodal densities, since the uniform distribution on the unit sphere projects onto the uniform distribution on the interval $(-1,1)$ on the vertical axis. See [14] for results when D is the Euclidean ball in arbitrary dimension, and [17] for the case where D is the unit ball in l^1.

The l^∞ theory is quite simple. The projection of the uniform distribution on the interior of the unit cube on the vertical axis is the uniform distribution on $(-1,1)$. So there is a one to one correspondence between continuous (or lower semi-continuous) unimodal densities f on $\mathbb{R}^d$ with cubic level sets and continuous (or lower semi-continuous) unimodal symmetric marginals f_d.

If a probability distribution on $\mathbb{R}^d$ with cubic symmetry charges the boundary of a cube $(-c,c)^d$ the marginal distribution will have an atom in the points $\pm c$.

Proposition A.5. *Let f_d be a symmetric density on $\mathbb{R}$. There exists a density f on $\mathbb{R}^d$ which is constant on the boundaries of cubes and with marginals equal to f_d if and only if*

$$\int_r^\infty \frac{f_d(r)-f_d(t)}{t^d}dt \geq 0 \qquad r>0. \tag{A.2}$$

Proof Suppose f has the form above with $D=(-1,1)^d$ the open unit cube, and $f_0 \geq 0$ a Borel function such that $t^{d-1}f_0(t)$ is integrable. Since f is constant on the faces of the cube $(-y,y)^d$, with value $f_0(y)$, one finds

$$f_d(y) = \int_{\mathbb{R}^{d-1}} f(\mathbf{x},y)d\mathbf{x} = (2y)^{d-1}f_0(y) + (2d-2)\int_y^\infty (2t)^{d-2}f_0(t)dt \qquad y>0. \tag{A.3}$$

If f_0 is C^1 one may take the derivative on the right to find the elegant relation:

$$df_d(y) = (2y)^{d-1}df_0(y) \quad y>0. \tag{A.4}$$

This describes the relation between the generator and the marginal for unimodal densities.

Given a symmetric density m, define the function H by

$$\int_r^\infty \frac{H(t)}{2t}dt = (2r)^{d-1}\int_r^\infty \frac{m(t)}{(2t)^d}dt \qquad r>0. \tag{A.5}$$

Differentiation gives

$$H(r) = m(r) - (2d-2)(2r)^{d-1}\int_r^\infty \frac{m(t)}{(2t)^d}dt. \tag{A.6}$$

Now use (A.5) to obtain

$$H(r) = m(r) - (2d-2)\int_r^\infty \frac{H(t)}{2t}dt.$$

If we compare this to (A.3) we see that the last equation states that m is the marginal of the density $f(\mathbf{z}) = f_0(\|\mathbf{z}\|_\infty)$ with $f_0(r) = H(r)/(2r)^{d-1}$, provided H is non-negative. The latter condition is equivalent to (A.2) by (A.6). ¶

The next result shows that regular variation of the marginal density for a unimodal multivariate density with cubic level sets implies regular variation of the generator f_0 with the same slowly varying function but a different constant. The converse result also holds.

Proposition A.6. *Let $f(\mathbf{z}) = f_0(\|\mathbf{z}\|_\infty)$ be a density on $\mathbb{R}^d$ with marginals f_d. Let $\lambda > 0$. The marginal density varies regularly with exponent $-(\lambda+1)$ if and only if the function f_0 varies regularly with exponent $-(\lambda+d)$. Their asymptotic behaviour is related: The marginal density f_d has the form $f_d(t) \sim \frac{L(t)}{(\lambda+1)t^{\lambda+1}}$, $t \to \infty$ with $\lambda > 0$ and $L(t)$ a slowly varying function, if and only if*

$$f_0(r) \sim \frac{1}{2^{d-1}} \frac{L(r)}{(\lambda+d)r^{\lambda+d}} \qquad r \to \infty.$$

Proof Suppose $f_0(r) = \frac{1}{2^{d-1}} \frac{L(r)}{(\lambda+d)r^{\lambda+d}}$ as $r \to \infty$. The relation (see e.g. Theorem 0.6 in [18])

$$\int_r^\infty \frac{L(t)}{t^{c+1}} dt \sim L(r) \int_r^\infty t^{-c-1} dt = L(r) \frac{1}{ct^c} \qquad r \to \infty,\ c > 0 \tag{A.7}$$

implies

$$\int_r^\infty t^{d-2} f_0(t) dt \sim \frac{L(r)}{\lambda+d} \int_r^\infty t^{-\lambda-2} dt = \frac{L(r)}{(\lambda+d)(\lambda+1)r^{\lambda+1}}.$$

The definition of the marginal density f_d in (A.3) and the above result give

$$\begin{aligned} f_d(r) &= (2r)^{d-1} f_0(r) + 2(d-1) \int_r^\infty (2t)^{d-2} f_0(t) dt \\ &\sim \frac{L(r)}{(\lambda+d)r^{\lambda+1}} + (d-1) \frac{L(r)}{(\lambda+d)(\lambda+1)r^{\lambda+1}} = \frac{L(r)}{(\lambda+1)r^{\lambda+1}} \qquad r \to \infty. \end{aligned}$$

To prove the opposite direction, we use relation (A.6) instead of (A.3). ¶

A.3 Summary of Notation

We shall use the following notation to distinguish between the original and the meta variables, and between the basic variables and the normalized limit variables:

$\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{Z}$, $f(\mathbf{z})$, $z_i = t$, $\mathbf{w}$ for the original vector, etc.

$\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{X}$, $g(\mathbf{x})$, $x_i = s$, $\mathbf{u}$ for the meta vector, etc.

Frequently used symbols:

$a_n \ll b_n$ or $a_n = o(b_n)$	$a_n/b_n \to 0$ for $n \to \infty$
$a_n \sim b_n$	$a_n/b_n \to 1$ for $n \to \infty$ (asymptotic equality)
$a_n \asymp b_n$	ratios a_n/b_n and b_n/a_n are bounded eventually (weak asymptotic equality)
int(E) and cl(E)	the interior and the closure of a set E
B and C	denote the open centered unit ball and the cube $[-1,1]^d$

References

[1] G. Balkema and P. Embrechts. *High Risk Scenarios and Extremes. A Geometric Approach.* European Mathematical Society, Zurich, 2007.

[2] T. Bedford and R. Cooke. *Probabilistic risk analysis: foundations and methods.* Cambridge University Press, Cambridge, 2001.

[3] N.H. Bingham, C.M. Goldie, and J.L. Teugels. *Regular Variation.* Cambridge University Press, Cambridge, 1987.

[4] R.A. Davis, E. Mulrow, and S.I. Resnick. Almost sure limit sets of random samples in $\mathbb{R}^d$. *Adv. Appl. Prob.*, 20:573–599, 1988.

[5] L. Duchateau and P. Janssen. *The Frailty Model.* Springer, 2008.

[6] P. Embrechts. Copulas: A personal view. *J. Risk and Insurance*, (to appear), 2008.

[7] H.-B. Fang, K.-T. Fang, and S. Kotz. The meta-elliptical distributions with given marginals. *Journal of Multivariate Analysis*, 82:1–16, 2002.

[8] R. Frey, A.. J. McNeil, and M. Nyfeler. Copulas and credit models. *RISK*, 14:111–114, 2001.

[9] J. Galambos. *The asymptotic theory of extreme order statistics.* Wiley, New York, 1978.

[10] C. Genest, M. Gendron, and M. Bourdeau-Brien. The advent of copulas in finance. *European J. of Finance*, (to appear), 2008.

[11] C. Genest and J. Nešlehová. A primer on copulas for count data. *The Astin Bulletin*, 37:475–515, 2007.

[12] A.K. Gupta and D. Song. l_p-norm spherical distribution. *Journal of Statistical Planning and Inference*, 60:241–260, 1997.

[13] P. Hougaard. *Analysis of Multivariate Survival Data.* Springer, 2000.

[14] H. Joe. *Multivariate models and dependence concepts.* Chapman & Hall, 1997.

[15] K. Kinoshita and S.I. Resnick. Convergence of scaled random samples in $\mathbb{R}^d$. *The Annals of Probability*, 19:1640–1663, 1991.

[16] D. Li. On default correlation: a copula function approach. *Journal of Fixed Income*, 9:43–54, 2001.

[17] A. J. McNeil and J. Nešlehová. Multivariate archimedean copulas, d–monotone functions and l_1–norm symmetric distributions. *Annals of Statistics*, To appear, 2008.

[18] S.I. Resnick. *Extreme Values, Regular Variation, and Point Processes.* Springer, 1987.