Credit Risk Models: An Overview

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A. Multivariate Models for Portfolio Credit Risk

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A1. Motivation

- Focus in credit risk research has mainly been on modelling of default of individual firm.
- Modelling of joint defaults in standard models (KMV, CreditMetrics) is relatively simplistic (based on multivariate normality).
- In large balanced loan portfolios main risk is occurrence of many joint defaults – this might be termed extreme credit risk.
- For determining tail of loss distribution, the specification of dependence between defaults is at least as important as the specification of individual default probabilities.

Modelling of Default – Overview

Consider portfolio of m firms and a time horizon T (typically 1 year). For $i \in \{1, \ldots, m\}$ define Y_i to be default indicator of company i, i.e. $Y_i = 1$ if company defaults by time T, $Y_i = 0$ otherwise. (Reduction to two states for simplicity.)

Model Types

- Latent variable models. Default occurs, if a latent variable X_i (often interpreted as asset value at horizon T) lies below some threshold D_i (liabilities). Examples: Merton model (1974), CreditMetrics, KMV.

- Mixture Models. Bernoulli default probabilities are made stochastic. $Y_i \mid Q_i \sim Be(Q_i)$ where Q_i is a random variable taking values in $[0, 1]$ and $Q_1, \ldots, Q_m$ are dependent. Example: CreditRisk$^+$.

Simplifications

- We consider only a two-state model (default/no-default). All of the ideas generalise to more-state models with different credit-quality classes. Probabilistic ideas are more easily understood in two-state setting.

- We neglect the modelling of exposures. The basic messages do not change when different exposures and losses-given-defaults are introduced.

A2. Latent Variable Models

Given random vector $\mathbf{X} = (X_1, \ldots, X_m)'$ with continuous marginal distributions F_i and thresholds $D_1, \ldots, D_m$, define $Y_i := 1_{\{X_i \leq D_i\}}$. Default probability of counterparty i given by

$$p_i := P\left(Y_i = 1\right) = P\left(X_i \leq D_i\right) = F_i\left(D_i\right) \ .$$

Notation: $(X_i, D_i)_{1 \leq i \leq m}$ denotes a latent variable model.

Examples

- Classical Merton-model.
  X_i is interpreted as asset value of company i at T. D_i is value of liabilities. Assume $\mathbf{X} \sim N(\mu, \Sigma)$.

Industry Examples of Latent Variable Models

- KMV-model. As Merton but D_i is now chosen so that default probability p_i equals average default probability of companies with same "distance-to-default" as company i.

- CreditMetrics. We assume $\mathbf{X} \sim N(0, \Sigma)$. Threshold D_i is chosen so that p_i equals average default probability of companies with same rating class as company i.

- Model of Li. (CreditMetrics Monitor 1999) X_i interpreted as survival time of company i. Assume X_i exponentially distributed with parameter λ_i chosen so that $P(X_i \leq T) = p_i$, with p_i determined as in CreditMetrics. Multivariate distribution of $\mathbf{X}$ specified using Gaussian copula.

Model Calibration

In both KMV and CreditMetrics, μ_i, Σ_{ii} and D_i are chosen so that p_i equals average historical default frequency for companies with a similar credit quality.
To determine further structure of Σ (i.e. correlations) both models assume a classical linear factor model for $p < m$.

$$X_i = \mu_i + \sum_{j=1}^{p} a_{i,j}\Theta_j + \sigma_i\varepsilon_i$$

for $\mathbf{\Theta} \sim N_p(\mathbf{0}, \Omega)$, independent standard normally distributed rv's $\varepsilon_1, \ldots, \varepsilon_m$, which are also independent of $\mathbf{\Theta}$.

$\mathbf{\Theta}$	global, country and industry effects impacting all companies.
$a_{i,j}$	weights for company i, factor j; $\boldsymbol{\epsilon}$ idiosyncratic effects.

Equivalent Latent Variable Models and Copulas

Definition: Two latent variable models $(X_i, D_i)_{1 \le i \le m}$ and $(\widetilde{X}_i, \widetilde{D}_i)_{1 \le i \le m}$ generating multivariate Bernoulli vectors $\mathbf{Y}$ and $\widetilde{\mathbf{Y}}$ are said to be equivalent if $\mathbf{Y} \stackrel{\mathrm{d}}{=} \widetilde{\mathbf{Y}}$.

Proposition: $(X_i, D_i)_{1 \le i \le m}$ and $(\widetilde{X}_i, \widetilde{D}_i)_{1 \le i \le m}$ are equivalent if:

1. $P(X_i \le D_i) = P\left(\widetilde{X}_i \le \widetilde{D}_i\right)$, $\forall i$.
2. $\mathbf{X}$ and $\widetilde{\mathbf{X}}$ have the same copula.

CreditMetrics and KMV are equivalent, as are all latent variable models that use the Gaussian dependence structure for latent variables, such as the model of Li, regardless of how marginals are modelled.

Special Case: Homogeneous Groups

It is common to group obligors together to form homogeneous groups. This corresponds to the mathematical concept of exchangeability.
A random vector $\mathbf{X}$ is exchangeable if

$$(X_1, \ldots, X_m) \stackrel{\mathsf{d}}{=} \left(X_{p(1)}, \ldots, X_{p(m)}\right) \ ,$$

for any permutation $(p(1), \ldots, p(m))$ of $(1, \ldots, m)$.

Exchangeable Default Model

We talk of an exchangeable default model if the default indicator vector $\mathbf{Y}$ is exchangeable.
If a latent variable vector $\mathbf{X}$ is exchangeable (or has an exchangeable copula) and all individual default probabilities $P(X_i \leq D_i)$ are equal, then $\mathbf{Y}$ is exchangeable.

Exchangeability allows a simplified notation for default probabilities:

$$\begin{aligned}\pi_k &:= P\left(Y_{i_1} = 1, \dots, Y_{i_k} = 1\right) , \\ &\qquad \{i_1, \dots, i_k\} \subset \{1, \dots, m\}, \quad 1 \leq k \leq m \,, \\ \pi &:= \pi_1 = P\left(Y_i = 1\right) , \quad i \in \{1, \dots, m\} \,.\end{aligned}$$

The Copula is Critical

To see this consider special case of exchangeable default model.

Consider any subgroup of k companies $\{i_1, \ldots, i_k\} \subset \{1, \ldots, m\}$.

$$\begin{aligned}\pi_k = P\left(Y_{i_1} = 1, \ldots, Y_{i_k} = 1\right) &= P\left(X_{i_1} \le D_{i_1}, \ldots, X_{i_k} \le D_{i_k}\right) \\ &= C_{1,\ldots,k}(\pi, \ldots, \pi)\,,\end{aligned}$$

where $C_{1,\ldots,k}$ is the k–dimensional margin of C.

The copula C crucially determines higher order joint default probabilities and thus extreme risk that many companies default. For π small, copulas with lower tail dependence will lead to higher π_k's and more joint defaults.

Comparison of Exchangeable Gaussian and t Copulas

If $\mathbf{X}$ is given an asset value interpretation large (downward) movements of the X_i might be expected to occur together; therefore tail dependence may be realistic.

Two cases: (extensions such as generalized hyperbolic distributions can be considered analogously).

1. $\mathbf{X} \sim N_m(\mathbf{0}, R)$
2. $\mathbf{X} \sim t_{m,\nu}(\mathbf{0}, R)$.

R is an equicorrelation matrix with off–diagonal element $\rho > 0$, so that $\mathbf{X}$ is exchangeable with correlation matrix R in both cases. We also fix thresholds so that $\mathbf{Y}$ is exchangeable in both cases and $P\left(Y_i = 1\right) = \pi$, $\forall i$, in both models. We vary the value for ν.

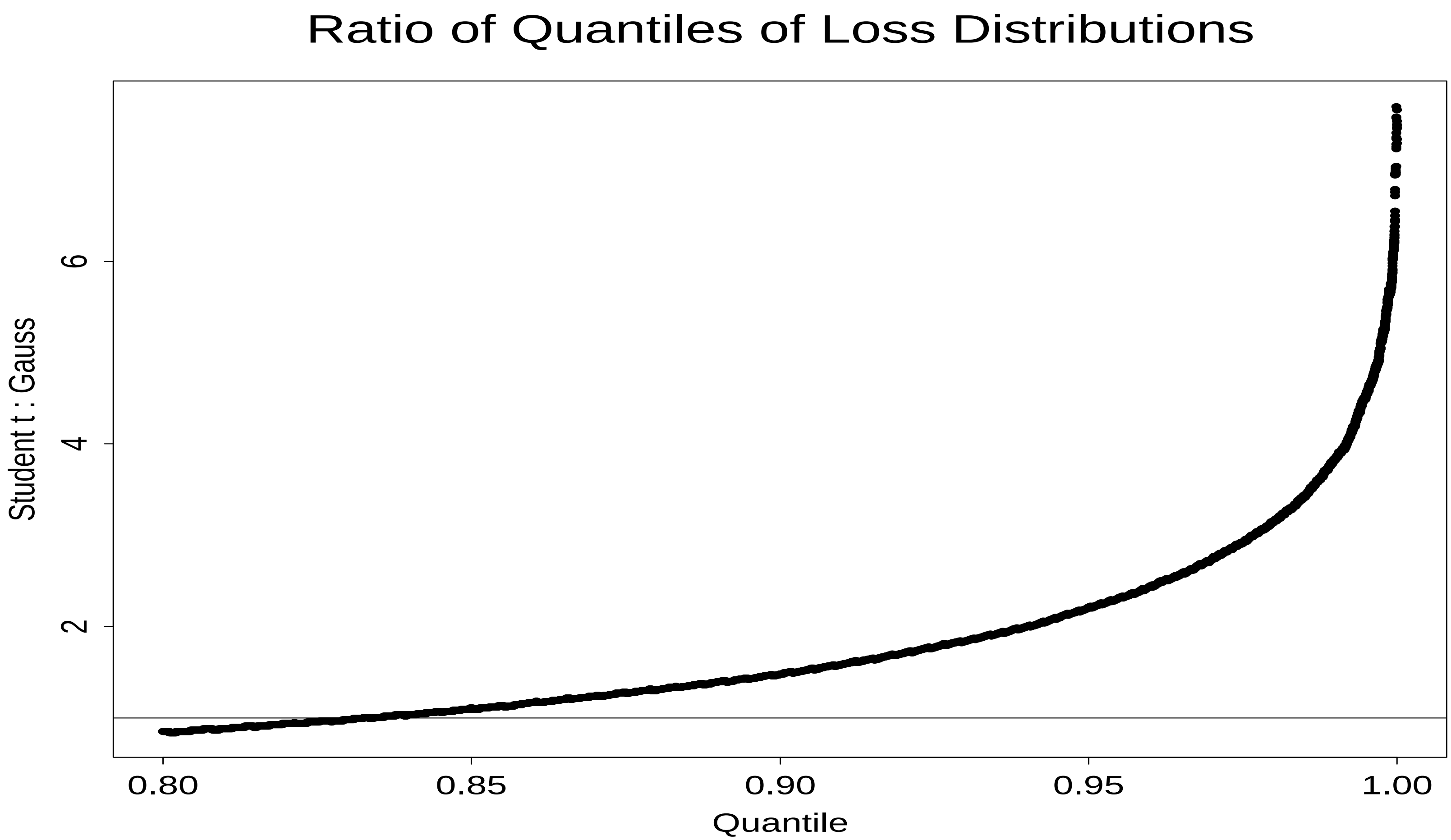
Simulation Study

We consider $m = 10000$ companies. All losses given default are one unit; total loss is number of defaulting companies. Set $\pi = 0.005$ and $\rho = 0.038$, these being values corresponding to a homogeneous group of "medium" credit quality in the KMV/CreditMetrics Gaussian approach. We set $\nu = 10$ in t–model and perform 100000 simulations to determine loss distribution.

The risk is compared by comparing high quantiles of the loss distributions (the so–called Value–at–Risk approach to measuring risk).

Results	Min	25%	Med	Mean	75%	90%	95%	Max
Gauss	1	28	43	49.8	64	90	109	331
t	0	1	9	49.9	42	132	235	3238

Ratio of quantiles of loss distributions (t:Gaussian)



$m = 10000$, $\pi = 0.005$, $\rho = 0.038$ and $\nu = 10$.

A3. Exchangeable Bernoulli Mixture Models

The default indicator vector $(Y_1, \ldots, Y_m)$ follows an exchangeable Bernoulli mixture model if there exists a rv Q taking values in $(0, 1)$ such that, given Q, $Y_1, \ldots, Y_m$ are iid Be(Q) rvs.

$$\pi = P(Y_i = 1) = E(Y_i) = E(E(Y_i \mid Q)) = E(Q)$$

$$\pi_k = P\left(Y_{i_1} = 1, \ldots, Y_{i_k} = 1\right) = E\left(Q^k\right) = \int_0^1 q^k dG(q)\,,$$

where $G(q)$ is the mixture distribution function of Q. Unconditional default probabilities and higher order joint default probabilities are moments of the mixing distribution.

It follows that, for $i \neq j$, $\mathrm{cov}\,(Y_i, Y_j) = \pi_2 - \pi^2 = \mathrm{var}\; Q \geq 0$.
Default correlation is given by $\rho_Y := \mathrm{corr}\,(Y_i, Y_j) = \frac{\pi_2 - \pi^2}{\pi - \pi^2}$.

Examples of Mixing Distributions

- Beta $\quad Q \sim \text{Beta}(a,b),\ g(q) = \beta(a,b)^{-1} q^{a-1}(1-q)^{b-1},\ a,b > 0$
- Probit–Normal $\quad \Phi^{-1}(Q) \sim N\left(\mu, \sigma^2\right)$ (CreditMetrics/KMV)
- Logit–Normal $\quad \log\left(\frac{Q}{1-Q}\right) \sim N\left(\mu, \sigma^2\right)$ (CreditPortfolioView)

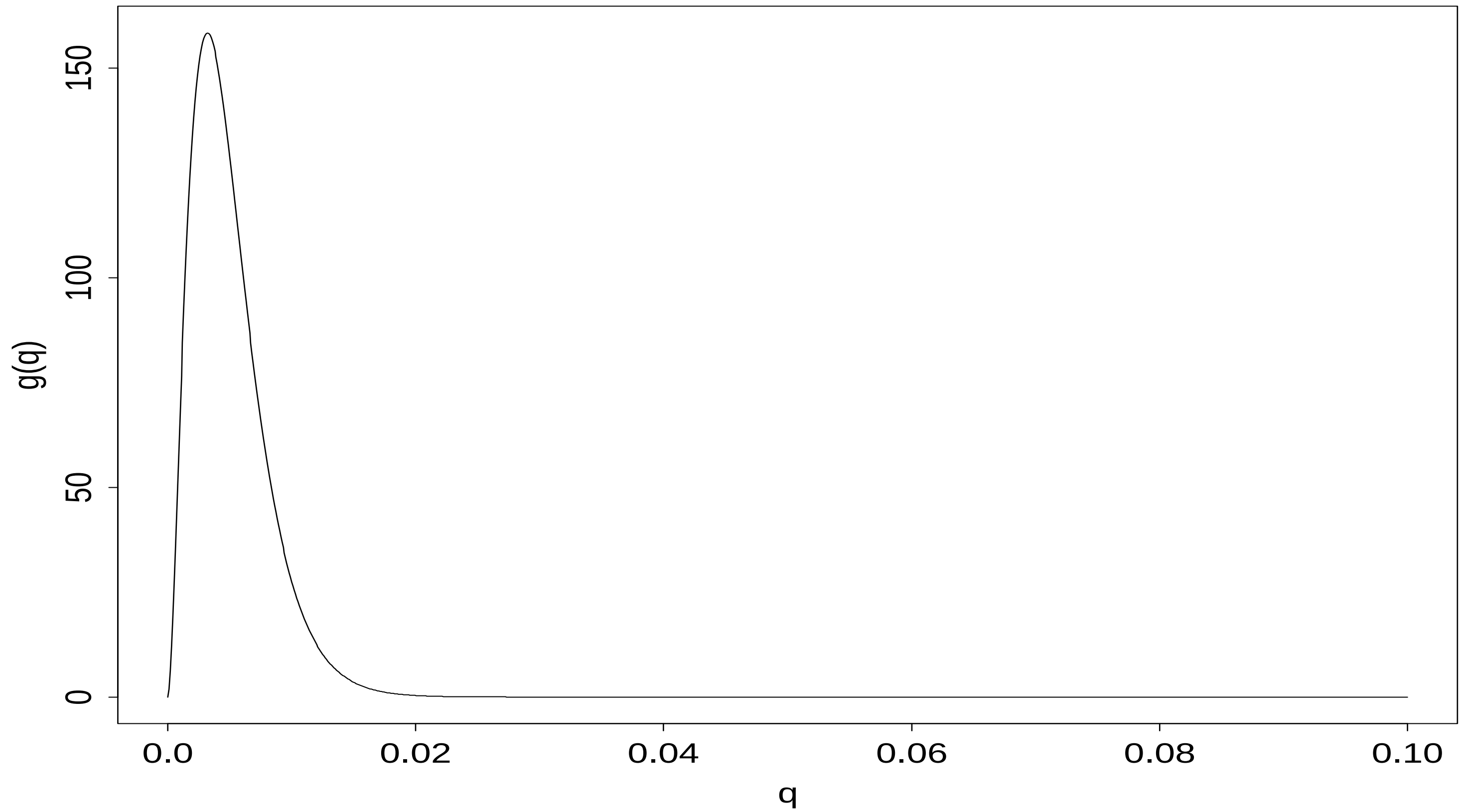
Parameterizing Mixing Distributions

These examples all have two parameters. If we fix the default probability π and default correlation ρ_Y (or equivalently the first two moments of the mixing distribution π and π_2) then we fix these two parameters and fully specify the model.

Example: Exchangeable Beta–Bernoulli Mixture Model

$\pi = a/(a+b)$, $\pi_2 = \pi(a+1)/(a+b+1)$.

Beta Mixing Distribution



Beta Density $g(q)$ of mixing variable Q in exchangeable Bernoulli mixture model with $\pi = 0.005$ and $\rho_Y = 0.0018$.

Extreme Risk in Large Balanced Portfolios

In exchangeable models for large homogeneous groups with similar exposures the tail of the loss distribution is proportional to the tail of the mixing distribution (Frey & McNeil 2001).

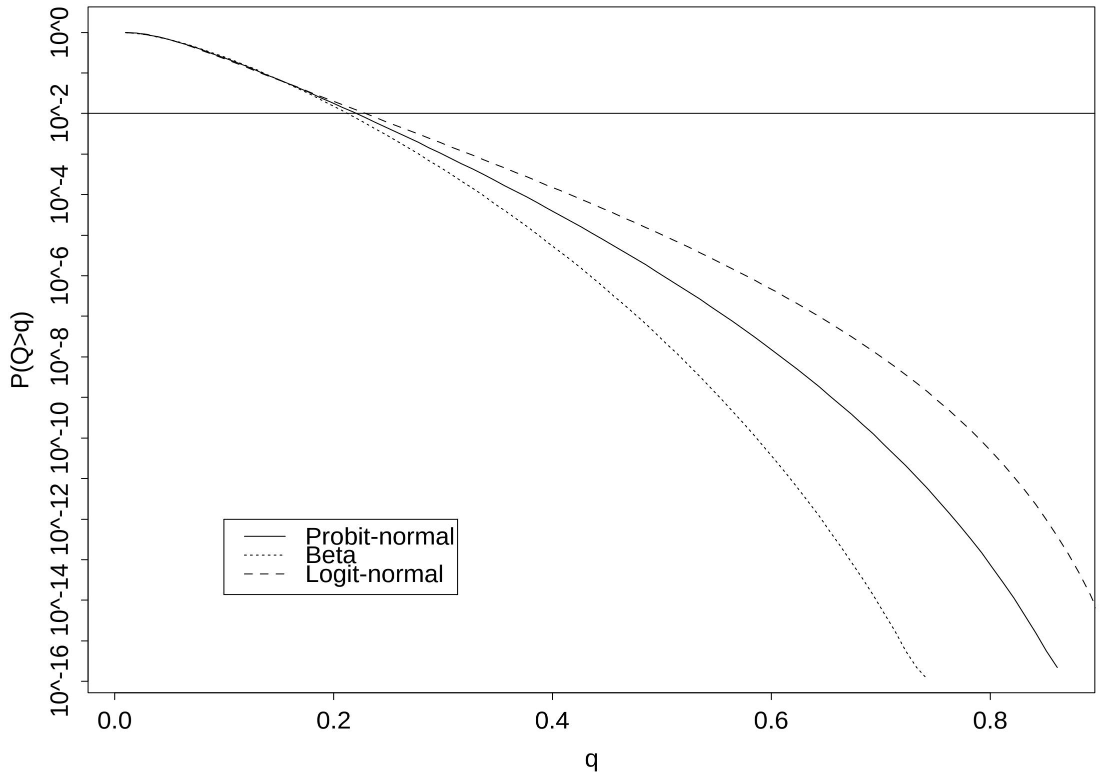
For portfolio size m large

$$\mathrm{VaR}_\alpha(\mathsf{Loss}) \approx m\,\overline{e}\,\mathrm{VaR}_\alpha(Q).$$

where $\overline{e}$ is mean exposure.

This result underlies loss distribution approximation in KMV and scaling rule in Basel II.

Tail of mixing distribution with first two moments fixed



Tail of the mixing distribution G in three exchangeable Bernoulli mixture models: probit–normal; logit–normal; beta.

More General Bernoulli Mixture Models

Definition: (Mixture Model with Factor Structure)
$(Y_1, \ldots, Y_m)$ follow a Bernoulli mixture model with p–factor structure if there is a random vector $\mathbf{\Psi} = (\Psi_1, \ldots, \Psi_p)$ with $p < m$ and continuous functions $f_i : \mathbb{R}^p \to (0,1)$, such that

1. $Y_i \mid \mathbf{\Psi} \sim Be(Q_i)$, $i = 1, \ldots, m$, where

$$Q_i = f_i\,(\Psi_1, \ldots, \Psi_p) \quad \text{for all } 1 \leq i \leq m\,.$$

2. $(Y_1, \ldots, Y_m)$ are conditionally independent given $\mathbf{\Psi}$.

Remark: Poisson mixture models with factor structure can be defined analogously, by making the Poisson rate parameters dependent on $\mathbf{\Psi}$.

Example: CreditRisk$^+$ has this kind of structure.

A4. Mapping Latent Variable to Mixture Models

It is often possible to transform a latent variable model to obtain an equivalent Bernoulli mixture model with factor structure. This is useful in Monte Carlo simulation, since Bernoulli mixture models are generally easier to simulate than latent variable models.

Example: KMV/Creditmetrics

$\mathbf{X}$ is Gaussian and follows a classical linear p–factor model.

$$X_i = \sum_{j=1}^{p} a_{i,j}\Theta_j + \sigma_i\varepsilon_i = \mathbf{a}_i'\mathbf{\Theta} + \sigma_i\varepsilon_i$$

for a p–dimensional random vector $\mathbf{\Theta} \sim N_p(\mathbf{0}, \Omega)$, independent standard normally distributed rv's $\varepsilon_1, \ldots, \varepsilon_m$, which are also independent of $\mathbf{\Theta}$.

CreditMetrics/KMV as a Bernoulli Mixture Model

For the mixing factors take $\mathbf{\Psi} = \mathbf{\Theta}$.

$$P(Y_i = 1 \mid \mathbf{\Psi}) = P(X_i \leq D_i \mid \mathbf{\Psi}) = P(\varepsilon_i \leq (D_i - \mathbf{a}_i'\mathbf{\Psi})/\sigma_i \mid \mathbf{\Psi})$$
$$= \Phi\left((D_i - \mathbf{a}_i'\mathbf{\Psi})/\sigma_i\right) .$$

Clearly $Y_i \mid \mathbf{\Psi} \sim \mathrm{Be}(Q_i)$ where $Q_i = \Phi\left((D_i - \mathbf{a}_i'\mathbf{\Psi})/\sigma_i\right)$.

Thus Q_i has a probit–normal distribution.

Moreover, conditional on $\mathbf{\Psi}$, the Y_i are independent.

Mapping Other L.V. Models to Mixture Models

A similar mapping is possible when the latent variables follow a multivariate normal mixture model, as in the case of t or generalised hyperbolic latent variables.

$\mathbf{X}$ has a normal mixture distribution if $X_i = g_i(W) + WZ_i$ where $W \geq 0$ is independent of $\mathbf{Z}$, $g_i : (0, \infty) \to \mathbb{R}$, and $\mathbf{Z}$ is Gaussian vector with $E(\mathbf{Z}) = \mathbf{0}$.

If Gaussian vector $\mathbf{Z}$ follows a linear factor model as before then it is possible to derive explicitly an equivalent Bernoulli mixture model.

Examples:

1. Student t model: $W = \sqrt{\nu/V}$, $V \sim \chi^2_\nu$ and $g_i(W) = \mu_i$.
2. Generalized hyperbolic: $W \sim$ NIG and $g_i(W) = \mu_i + \beta_i W$.

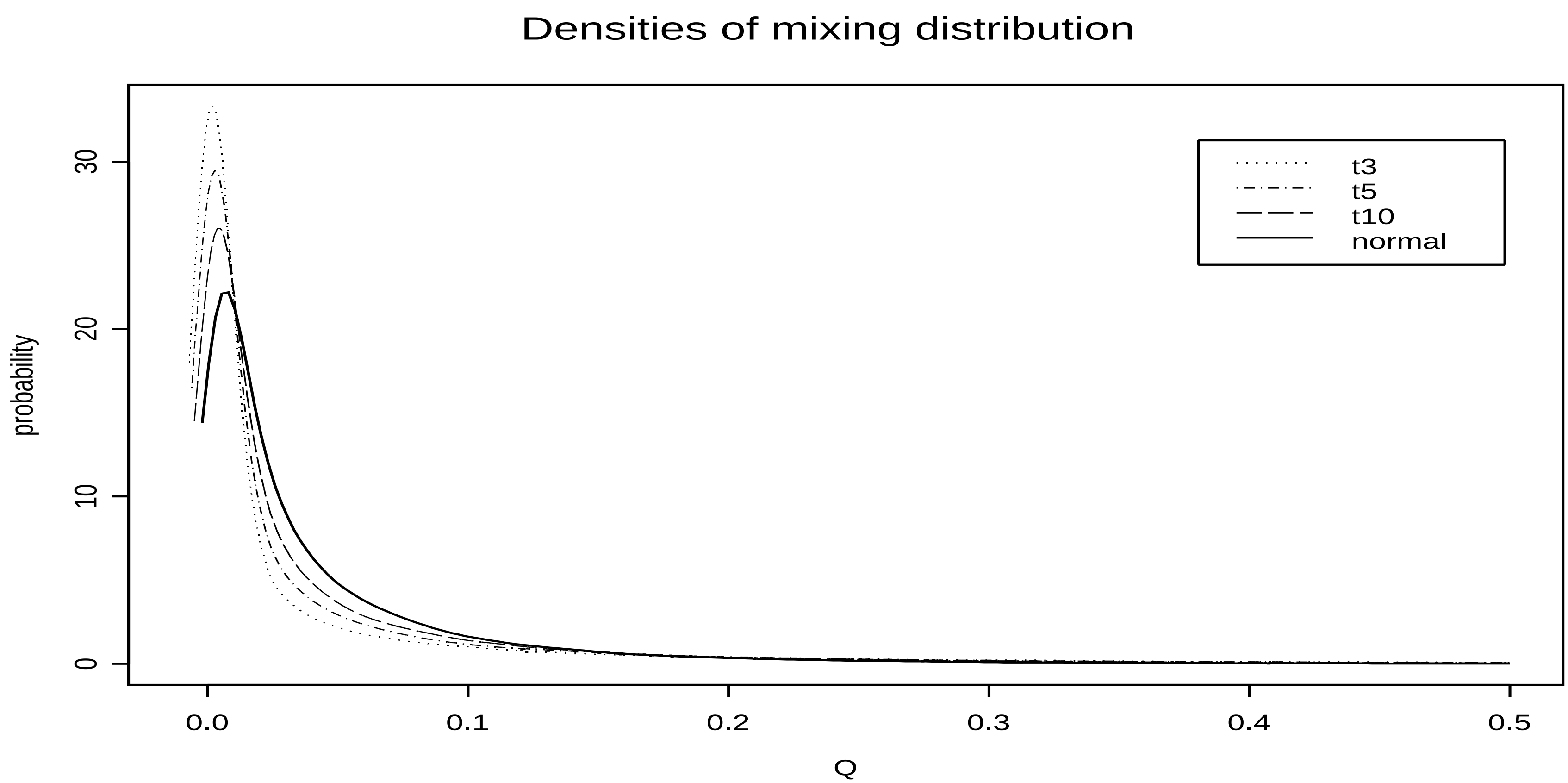
Normal and t: Equivalent Mixture Approach

The profound differences between the Gaussian and t copulas with similar asset correlation can be understood in terms of the differences between the corresponding mixture distributions. Consider two cases (again in exchangeable special case).

Case 1: Asset Correlation held fixed.
Here we observe clear differences between the densities of the equivalent mixing distributions as we vary degrees of freedom. These account for differences in distribution of number of defaults.

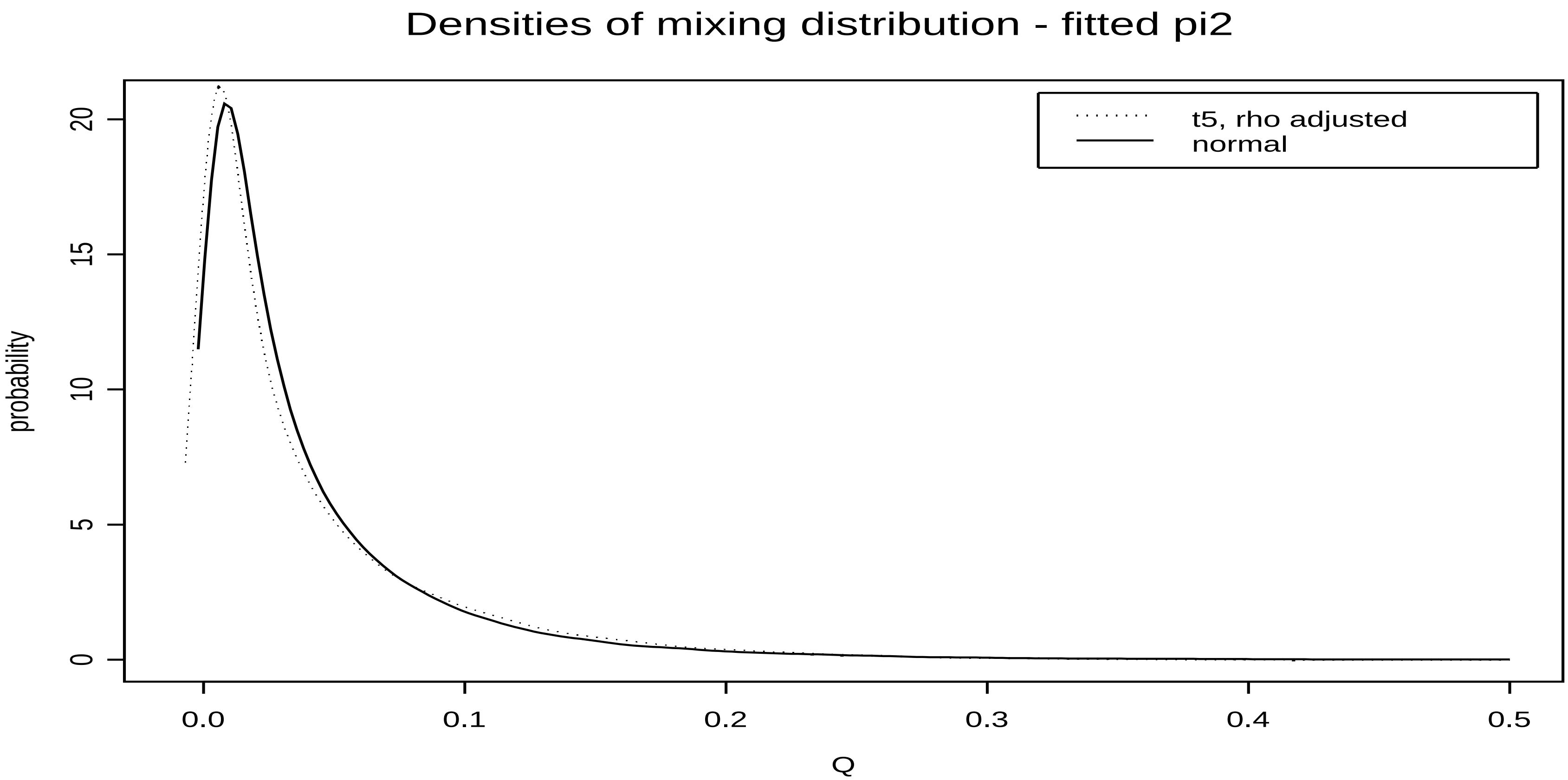
Case 2: Default Correlation held fixed.
Here differences between densities are much less obvious. Distributions of the number of defaults very similar 95th and 99th percentiles; differences visible only very far in the tail.

Mixing densities – similar asset correlation



Distribution of (Q) for exchangeable Gaussian and t copulas; $\pi = 0.04$ and $\rho = 0.3$.

Mixing densities – similar default correlation



Distribution of Q for exchangeable Gaussian and t copulas; $\pi = 0.04$ and in the normal model $\rho = 0.3$.

A5. Statistical Issues – Model Calibration

Methods of model calibration used in practice seem ad hoc. Very little actual statistical fitting of credit models to historical data takes place. Parameters, particularly those governing dependence, often chosen using rational economic arguments, rather than estimated.

Reasons: lack of quality historical data on historical default; feeling that the existing data (S&P or Moodys) not relevant for own portfolio, or not relevant for the future.

Historical Default Data

Typical Data Format:

Year	Rating	Companies	Defaults
2000	A	317	2
	B	500	25
	⋮	⋮	⋮
1999	A	280	1
	B	560	37

For illustration consider single homogeneous group (say B–rated). Heterogeneity can be modelled using covariates in various ways.

Suppose our time horizon of interest is one year and we have n years of historical data $\{(m_j, M_j)\,,\, j = 1, \ldots, n\}$, where m_j denotes the number of obligors observed in year j and M_j is the number of these that default.

Statistical Approaches

Assume an exchangeable Bernoulli mixture model in each year period with $Q_1, \ldots, Q_n$ identically distributed.

Method 1: Maximum Likelihood (Assume independence of Q_i)

Parameters of mixing distribution (e.g. beta, logit–, or probit–normal) can be estimated by maximum likelihood. Particularly easy for beta: $M_1, \ldots, M_n$ have a beta–binomial distribution with probability function:

$$P(M = k) = \binom{m}{k} \beta(a + k, b + m - k) / \beta(a, b)\,.$$

Method 2: Moment Estimation

We have seen the importance of $\pi = E(Q)$ and $\pi_2 = E(Q^2)$ (or ρ_Y) in homogeneous groups. How do we estimate these moments?

Lemma. Let $\binom{M}{k}$ be (random) number of subgroups of k companies in those that default. Then $E\binom{M}{k} = \binom{m}{k}\pi_k$.

Proof. $\binom{M}{k} = \sum_{(i_1,\ldots,i_k)\subset(1,\ldots,m)} Y_{i_1}\cdots Y_{i_k}$.

An unbiased and consistent estimator of π_k is

$$\widehat{\pi}_k = \frac{1}{n}\sum_{j=1}^{n} \frac{M_j\,(M_j-1)\cdots(M_j-k+1)}{m_j\,(m_j-1)\cdots(m_j-k+1)}, \quad k = 1, 2, 3, \ldots .$$

A6. Implications for pricing basket credit derivatives

Insights on dependence–modelling for loan portfolios have also implications for pricing of basket credit derivatives. Consider portfolio with m obligors (the basket) held by bank A. We are interested in pricing of following stylized default swap:

Second to default swap: Fix horizon $\{T\}$. Bank A receives from counterparty B a fixed payment K at time T if at least two obligors in the basket have defaulted (i.e. had a credit event) until time T; otherwise it receives nothing. At $t = 0$ A pays to B a fixed premium.

Intuition: pricing sensitive to occurrence of joint defaults.

Remark: Real second–to–default swaps are more complicated. The payments depend on identities of defaulted counterparties; moreover, payment due at time of credit event.

A pricing model

Stylized version of reduced-form model à la Duffie–Singleton or Jarrow–Lando–Turnbull. Our simplifications:

– interest–rate r is deterministic

– default-intensities are rv's instead of processes.

Denote by τ_i the default–time of obligor i in the basket.

Assumption 1: The default–times τ_i, $1 \leq i \leq m$ follow a mixed exponential distribution, i.e. there is some p–dimensional random vector $\mathbf{\Psi}$ $(p < m)$ such that conditional on $\mathbf{\Psi}$ the τ_i are independent exponentially distributed rv's with parameter $\lambda_i(\mathbf{\Psi})$. In particular,

$$P\left(\tau_i < T \mid \mathbf{\Psi}\right) = 1 - \exp\left(-\lambda_i(\mathbf{\Psi})\, T\right) \approx \lambda_i(\mathbf{\Psi})T \,. \qquad (1)$$

Defaults then follow a Bernoulli–mixture model with π as in (1).

Pricing of credit–derivatives

Following standard–practice we assume that Assumption 1 holds under a pricing–measure Q. Hence for every claim H depending on $\tau_1, \ldots, \tau_m$ the price at $t = 0$ equals

$$P_0 = e^{-rT} E\left(H\left(\tau_1, \ldots, \tau_m\right)\right) .$$

In particular we get for our second–to–default swap

$$P_0 = e^{-rT} Q\left(\sum_{i=1}^{m} Y_i \geq 2\right).$$

Specific model:

We choose λ and $\boldsymbol{\Psi}$ so that the one–year default probability corresponds to the default–probability in the one–factor latent variable model with t copula, i.e.

$$\lambda_i = -\ln\left(1 - \Phi\left(\frac{t_\nu^{-1}(\pi)\sqrt{W/\nu} - \sqrt{\rho}\Theta}{\sqrt{1-\rho}}\right)\right),$$

$$\Theta \sim N(0,1), W \sim \chi^2(\nu)\,.$$

Simulations:

Homogeneous portfolio with $m = 14$, $T = 1$, and varying values for default probability π and asset correlation ρ.

$$\begin{array}{lll} \text{Portfolio A:} & \pi = 0.15\% & \rho = 0.38\% \\ \text{Portfolio B:} & \pi = 0.50\% & \rho = 3.80\% \end{array}$$

In the following table we give the ratio $P_0^t / P_0^{\text{normal}}$ of the price of stylized second–to–default swap in in t–model and normal model.

Portfolio	$\nu = 5$	$\nu = 10$	$\nu = 20$
A	11.0	7.3	4.4
B	3.3	2.6	2.0

Choice of the copula has again drastic effect!

Conclusions

- Extreme risk in latent variable models is driven by copula of $\mathbf{X}$.
- The assumption of a multivariate normal distribution and a calibration based on asset correlations alone may seriously underestimate the extreme risk in latent variable models.
- Extreme risk in Bernoulli mixture models with factor structure is driven by the mixing distribution of the factors.
- The two model types may often be mapped into one another. It is particularly useful (Monte Carlo simulation and also for fitting) to represent latent variable models as Bernoulli mixture models.
- Model calibration should use historical default data and not be based solely on assumptions about asset value correlations.

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