Bounds for functions of dependent risks

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Abstract The problem of finding the best-possible lower bound on the distribution of a non-decreasing function of n dependent risks is solved when $n = 2$ and a lower bound on the copula of the portfolio is provided. The problem gets much more complicated in arbitrary dimensions. When no information on the structure of dependence of the random vector is available, we provide a bound on the distribution function of the sum of risks which we prove to be better than the one generally used in the literature.

Key words copulas – dependent risks – dependency bounds – Fréchet bounds

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1 Introduction

Consider a n-variate real function ψ and a random vector $X := (X_1, \dots, X_n)$. In this paper we study the problem of finding the best-possible lower bound on the distribution function (df) of $\psi(X)$ when the marginal distributions of the individual risks X_i are given and the structure of dependence of X is partially or completely unknown. This problem has a long history. Makarov (1981), in response to a question formulated by A.N. Kolmogorov, provided the first result for $n = 2$ and $\psi = +$, the sum operator. Some years later Frank et al. (1987) restated Makarov's result, using the well-known formulation of the problem based on copulas. Independently from this *geometric* approach, Rüschendorf (1982) gave a much more

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elegant proof of the same theorem using a dual result proved for a more general purpose. The *dual* approach of Rüschendorf was related to a much earlier issue, dating back to 1871: the so-called *Monge mass-transportation problem*; in particular, he solved a special case of its *Kantorovich version*. A complete analysis of this kind of problems is given in Rachev and Rüschendorf (1998). Introducing the use of dependence information, Williamson and Downs (1990) gave the best-possible bound for non-decreasing and continuous functionals $\psi(X_1, X_2)$ in the presence of a lower bound on the copula of a two-dimensional portfolio. Denuit et al. (1999) gave a bound for the sum in arbitrary dimensions and provided some applications; see also Embrechts et al. (2002) for the relevance of these techniques within quantitative risk management. Finally, Embrechts et al. (2003) stated sharpness of the bound in the presence of information for n-dimensional portfolios. The latter article however contains a gap in the proof of Theorem 3.2 and the problem remains open for $n > 2$, even for the case of the sum. In the no-information scenario, a bound for the df of the sum of risks can be obtained by the previously cited theorems, but it fails to be sharp whenever $n > 2$. Exploiting the dual result of Rüschendorf we give a better bound which, though not proved to be sharp, improves considerably the previous estimate for the df of the sum of identically distributed risks. Numerical computations of this bound are also provided.

1.1 Notation

We first fix some notation. Given a vector $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ we write $x_{-i} := (x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$ to indicate the $(n-1)$-valued vector obtained from x by deleting the i-th component. The indicator function of the set $B \subset \mathbb{R}$ is denoted by 1_B. We denote by F^- the left-continuous version of the df F of a random variable (rv) X, i.e. $F^-(x) := \mathbb{P}[X < x]$. We also use the notation $F^-(-\infty)$ and $F^-(+\infty)$ to indicate its left and right limit, respectively.

2 Definitions and preliminaries

In this section we introduce the main mathematical problem and recall some well-known concepts about copulas.

2.1 Copulas as dependence structures

Let $X_1, \dots, X_n$ be n real-valued rvs on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, with given dfs $F_i(x) = \mathbb{P}[X_i \leq x], i = 1, \dots, n$. The random vector $X := (X_1, \dots, X_n)$ can be seen as a portfolio of one-period financial or insurance risks. For some function $\psi : \mathbb{R}^n \to \mathbb{R}$, we consider the problem of bounding from below the df of the rv $\psi(X)$, over the class of possible dfs for X having fixed marginals. In fact, we search for

$$m_\psi(s) := \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1, \dots, n\}. \tag{2.1}$$

Of course, the df of $\psi(X)$ can be computed once the function $F(x_1,\dots,x_n) = \mathbb{P}[X_1 \le x_1,\dots,X_n \le x_n]$ is known. The latter is uniquely defined through the marginal dfs and their interdependence. The tool for modelling these dependencies is offered by the concept of copula.

Definition 2.1 *A* copula *is any function* $C : [0,1]^n \to [0,1]$ *which has the following three properties:*

(i) C *is non-decreasing in each argument.*
(ii) $C(1,\dots,1,u_i,1,\dots,1) = u_i$ *for all* $u_i \in [0,1], i = 1,\dots,n$.
(iii) C *is* n-increasing, *i.e. for all* $a = (a_1,\dots,a_n), b = (b_1,\dots,b_n) \in [0,1]^n$ *with* $a_i \le b_i, i = 1,\dots,n$, *we have*

$$\sum_{j_1=1}^{2} \cdots \sum_{j_n=1}^{2} (-1)^{j_1+\cdots+j_n} C(u_{1j_1},\dots,u_{nj_n}) \ge 0,$$

where $u_{i1} = a_i, u_{i2} = b_i$ *for all* $i = 1,\dots,n$.

It is equivalent to say that a copula is a n-dimensional df restricted to $[0,1]^n$ having standard uniform marginals. Given a copula C and a set of n univariate marginals $F_1,\dots,F_n$, one can always define a df F on $\mathbb{R}^n$ having these marginals by

$$F(x_1,\dots,x_n) := C(F_1(x_1),\dots,F_n(x_n)). \tag{2.2}$$

Sklar's theorem (see Sklar (1973, Th. 1)) states conversely that we can always find a copula C coupling the marginals of a fixed df F trough (2.2). For continuous marginal dfs, this copula is unique. In our set-up it is convenient to identify the df F of X with the copula C merging the given marginals into the df $C(F_1(x_1),\dots,F_n(x_n))$. Denote by μ_C the corresponding probability measure on $\mathbb{R}^n$ and define:

$$\begin{aligned} \sigma_{C,\psi}(F_1,\dots,F_n)(s) :&= \mu_C[\psi(X) < s] \\ &= \int_{\{\psi<s\}} dC(F_1(x_1),\dots,F_n(x_n)), \end{aligned} \tag{2.3}$$

$$\begin{aligned} \tau_{C,\psi}(F_1,\dots,F_n)(s) := \sup_{x_1,\dots,x_{n-1}\in\mathbb{R}} & C(F_1(x_1),\dots, \\ & F_{n-1}(x_{n-1}), F_n^-(\psi^{\frown}_{x_{-n}}(s))), \end{aligned} \tag{2.4}$$

where $\psi^{\frown}_{x_{-n}}(s) := \sup\{x_n \in \mathbb{R} : \psi(x_{-n},x_n) < s\}$ for fixed $x_{-n} \in \mathbb{R}^{n-1}$.

By the above discussion, problem (2.1) can be equivalently expressed as

$$m_\psi(s) = \inf\{\sigma_{C,\psi}(F_1,\dots,F_n)(s) : C \in \mathfrak{C}_n\}, \tag{2.5}$$

where $\mathfrak{C}_n$ denotes the set of all n-dimensional copulas.

2.2 Dependency information

If we don't have the perfect knowledge of the copula C coupling the fixed marginal dfs of the portfolio X, the df of $\psi(X)$ cannot be determined exactly and problem (2.5) arises. However, it can be the case that partial information regarding C is known.

Given two copulas C_1 and C_2, we say that $C_1 \geq$ (resp. $\leq$) C_2 if and only if $C_1(u) \geq$ (resp. $\leq$) $C_2(u)$ for all $u \in [0,1]^n$. Using the properties of a copula it can be easily shown that any copula C lies between the so-called lower and upper Fréchet bounds $W, M : [0,1]^n \to [0,1]$; $W(u_1,\dots,u_n) := [\sum_{i=1}^n u_i - n + 1]^+$, $M(u_1,\dots,u_n) := \min\{u_1,\dots,u_n\}$, namely $W \leq C \leq M$. A third copula of interest is the product copula $\Pi : [0,1]^n \to [0,1], \Pi(u) := \prod_{i=1}^n u_i$ which represents independence among coupled rvs.

The copula of a df F contains all the dependency information of F. Hence putting a lower bound on the copula C of the portfolio can be interpreted as having partial information regarding its dependence structure. For instance, assuming that $C = M$ characterizes the risks of our portfolio as *comonotonic*, i.e. as being increasing functions of a common rv. See Dhaene et al. (2002) for more details on comonotonicity. Moreover, assuming that $C \geq \Pi$ identifies the risks as *positive lower orthant dependent* (PLOD); see Nelsen (1999, Def. 5.6.1).

If we assume that a lower bound $C_L : [0,1]^n \to [0,1]$ on C is known, we can reduce our search to

$$m_{C_L,\psi}(s) := \inf\{\sigma_{C,\psi}(F_1,\dots,F_n)(s) : C \in \mathfrak{C}_n, C \geq C_L\}. \tag{2.6}$$

Note that $m_{W,\psi}(t) = m_\psi(t)$, since assuming that $C \geq W$ corresponds to the situation in which we are completely ignorant about the dependence structure of the random vector X. Obviously, $m_{C_L,\psi}(t) \geq m_\psi(t)$ but we warn the reader that the last inequality is often strict even for a non-decreasing function ψ. Due to the fact that $\geq$ is not a complete ordering on $\mathfrak{C}_n$, letting $C \geq C_L$ is not necessarily a prudent assumption. In fact, for any $C_L \neq W$, we neglect all copulas which are not comparable to C_L with respect to $\geq$. By doing so we possibly exclude the riskiest copula, i.e. the one possibly solving (2.5). Finally note that, contrary to M, W is not a copula for $n > 2$.

3 Main result with partial information

When partial information on the copula of a vector X is known, it is easy to find a general lower bound on $\sigma_{C,\psi}(F_1,\dots,F_n)(s)$.

Theorem 3.1 *Let $X = (X_1,\dots,X_n)$ be a random vector on $\mathbb{R}^n$ $(n > 1)$ having marginal dfs $F_1,\dots,F_n$ and copula C. Assume that there exists a copula C_L such that $C \geq C_L$. If $\psi : \mathbb{R}^n \to \mathbb{R}$ is non-decreasing in each coordinate, then for every real s we have*

$$\sigma_{C,\psi}(F_1,\dots,F_n)(s) \geq \tau_{C_L,\psi}(F_1,\dots,F_n)(s). \tag{3.1}$$

Proof First observe that for arbitrary $x \in \mathbb{R}^n$, the uniform continuity of a copula C implies that

$$\begin{aligned}
&\mu_C[X_1 \le x_1, \ldots, X_{n-1} \le x_{n-1}, X_n < x_n] \\
&= \lim_{k\to\infty} \mu_C[X_1 \le x_1, \ldots, X_{n-1} \le x_{n-1}, X_n \le x_n - 1/k] \\
&= C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), \lim_{k\to\infty} F_n(x_n - 1/k)) \\
&= C(F_1(x_1), \ldots, F_{n-1}(x_{n-1}), F_n^-(x_n)).
\end{aligned}$$

Now fix $(\bar{x}_1, \ldots, \bar{x}_{n-1}) \in \mathbb{R}^{n-1}$ and assume that $\bar{x}_n := \psi^{\wedge}_{\bar{x}_{-n}}(s)$ is finite. Then

$$\{X_1 \le \bar{x}_1, \ldots, X_{n-1} \le \bar{x}_{n-1}, X_n < \bar{x}_n\} \subset \{\psi(X) < s\}$$

and hence

$$\begin{aligned}
\mu_C[\psi(X) < s] \ge& \mu_C[X_1 \le \bar{x}_1, \ldots, X_{n-1} \le \bar{x}_{n-1}, X_n < \bar{x}_n] \\
=& C(F_1(\bar{x}_1), \ldots, F_{n-1}(\bar{x}_{n-1}), F_n^-(\bar{x}_n)) \\
\ge& C_L(F_1(\bar{x}_1), \ldots, F_{n-1}(\bar{x}_{n-1}), F_n^-(\psi^{\wedge}_{\bar{x}_{-n}}(s))).
\end{aligned}$$

If $\bar{x}_n = +\infty$, then $\psi(\bar{x}_{-n}, x_n) < s$ for all $x_n \in \mathbb{R}$, and hence

$$\begin{aligned}
\mu_C[\psi(X) < s] \ge& \mu_C[X_1 \le \bar{x}_1, \ldots, X_{n-1} \le \bar{x}_{n-1}, X_n \in \mathbb{R}] \\
=& C(F_1(\bar{x}_1), \ldots, F_{n-1}(\bar{x}_{n-1}), 1) \\
\ge& C_L(F_1(\bar{x}_1), \ldots, F_{n-1}(\bar{x}_{n-1}), F_n^-(+\infty)).
\end{aligned}$$

Analogously, if $\bar{x}_n = -\infty$ then $\psi(\bar{x}_{-n}, x_n) \ge s$ for all $x_n \in \mathbb{R}$, so that

$$\begin{aligned}
\mu_C[\psi(X) < s] \ge 0 &= C_L(F_1(\bar{x}_1), \ldots, F_{n-1}(\bar{x}_{n-1}), 0) \\
&= C_L(F_1(\bar{x}_1), \ldots, F_{n-1}(\bar{x}_{n-1}), F_n^-(-\infty)).
\end{aligned}$$

The theorem follows by taking the supremum over all $(\bar{x}_1, \ldots, \bar{x}_{n-1}) \in \mathbb{R}^{n-1}$. □

Remark 3.1 We stress the following points regarding this theorem.

(i) Though the function W is not a copula for $n > 2$, the bound (3.1) is valid even when the lower copula-bound C_L is replaced by W. In fact, the inequality $C \ge W$ holds for all $C \in \mathfrak{C}^n$ and is best-possible for all integers n; see Nelsen (1999, Th. 2.10.12). However, when $n > 2$, the bound (3.1) may fail to be sharp, as we show in Section 5 below.

(ii) Bound (3.1) improves the corresponding bound given in Williamson and Downs (1990, Th. 1) for $n = 2$ and extends it to arbitrary n. Moreover, a proof analogous to the one above can be used to improve the bound on $\mathbb{P}[\psi(X) \le s]$, which is given in Denuit et al. (1999, (12)) for $\psi = +$, and to extend Prop. 2 and Prop. 5 in the same paper to more general functionals ψ.

Theorem 3 in Williamson and Downs (1990) states that, when $n = 2$ and ψ is also continuous, there will always be a copula $C^{(t)}$ attaining bound (3.1), i.e. that bound cannot be tightened; see also Rüschendorf (1982, Prop. 1) in the case $C_L = W$. Note that in Williamson and Downs (1990) the marginal dfs F_1 and F_2 are defined to be left-continuous and are not allowed to be both discontinuous at the real values x_1 and x_2, respectively, such that $x_1 + x_2 = s$.

Theorem 3.2 in Embrechts et al. (2003) claims sharpness of the bound (3.1) also for $n > 2$. Firstly note that, contrary to Embrechts et al. (2003, p. 151), (2.3) is the correct way of defining the operator σ, if one wants to correctly state sharpness of the bound (3.1) even for $n = 2$. In fact, $\sigma_{C,\psi}(F_1, \dots, F_n)(s+) := \mu_C[\psi(X) \leq s]$ may have no minimum over the set $\mathfrak{C}_2$; see Nelsen (1999, p. 187) for more details in the case of the sum of risks. Moreover, in Embrechts et al. (2003) the authors define the function $C^t : [0,1]^n \to [0,1]$ as follows:

$$C^t(u) := \begin{cases} \max\{t, C_L(u)\} & \text{if } u = (u_1, \dots, u_n) \in [t,1]^n, \\ \min\{u_1, \dots, u_n\} & \text{otherwise,} \end{cases}$$

and state that C^t is a copula attaining the bound given in Theorem 3.1 under the extra-assumption of ψ being left-continuous in the last argument. Also earlier available copies of this paper contained a revised proof of the same result. Unfortunately, the following counterexample shows that the function C^t is not a copula for $n > 2$ and, therefore, sharpness of the bound (3.1) in arbitrary dimensions still needs settling.

Example 3.1 (Geiss and Päivinen) We show that the function C^t is not a copula for $n > 2$. Let for instance $C_L = \Pi$ and $t = \alpha^2$ for some $\alpha \in (0,1)$. It is possible to choose α such that the function C^t does not satisfy property (iii) in Definition 2.1. In fact, substituting $a = (\alpha, \dots, \alpha), b = (1, \dots, 1) \in [0,1]^n$ in the latter, we have

$$\begin{aligned} &\sum_{j_1=1}^{2} \cdots \sum_{j_n=1}^{2} (-1)^{j_1+\cdots+j_n} C^t(u_{1j_1}, \dots, u_{nj_n}) \\ =&\sum_{j_1=1}^{2} \cdots \sum_{j_n=1}^{2} (-1)^{j_1+\cdots+j_n} \max\left\{t, \prod_{i=1}^{n} u_{ij_i}\right\} \\ =&\sum_{k=0}^{n} (-1)^k \binom{n}{k} \max\{\alpha^2, \alpha^k\} = 1 - n\alpha + (n-1)\alpha^2. \end{aligned}$$

It is immediate to check that, for $\alpha = \frac{n}{2(n-1)}$,

$$1 - n\alpha + (n-1)\alpha^2 = -\frac{(n-2)^2}{4(n-1)} < 0, \text{ for all } n > 2. \tag{3.2}$$

Since $a_i \leq b_i, i = 1, \dots, n$, (3.2) shows that, for $n > 2$, the function C^t is not n-increasing on its domain and therefore it is not a copula in general.

The reader can easily check that, for $n = 2$, the function C^t is a copula and coincides with the copula $C^{(t)}$ defined in Williamson and Downs (1990, Th. 3). For applications of Theorem 3.1, including how to calculate numerically the bound for every choice of $F_1, \dots, F_n$ and C_L, see Embrechts et al. (2003).

4 Main result without information on dependence: non-negative, continuous and identically distributed risks

Throughout the rest of the paper we will consider $C_L = W$. In this situation, the bound (3.1) is not sharp if $n > 2$, and it is convenient to express (2.5) by a duality result given in Rüschendorf (1982).

Theorem 4.1 *For any measurable function ψ and real threshold s, we have that*

$$m_\psi(s) = 1 - \inf\left\{ \sum_{i=1}^n \int f_i dF_i : f_i \text{ are bounded measurable functions on } \mathbb{R} \text{ s.t.} \right.$$
$$\left. \sum_{i=1}^n f_i(x_i) \geq 1_{[s,+\infty)}(\psi(x_1,\ldots,x_n)) \text{ for all } x_i \in \mathbb{R}, i = 1,\ldots,n \right\}. \tag{4.1}$$

This dual optimization problem is very difficult to solve. The only explicit results known in the literature are given in Rüschendorf (1982) for the case of the sum of marginals being all uniformly or binomially distributed. Unfortunately, the dependence structure which solves (4.1) in the case of the sum of uniform marginals does not work in the general case, where the solution may depend upon the marginals chosen. For that reason, below we restrict our attention to $\psi(x) = \sum_{i=1}^n x_i$ and set all marginal dfs equal to a common df F, which we assume to be non-negative and continuous. In this situation (4.1) reads as

$$m_\psi(s) = 1 - \inf\left\{ n \int f dF : f \text{ bounded measurable function on } \mathbb{R} \text{ s.t.} \right.$$
$$\left. \sum_{i=1}^n f(x_i) \geq 1_{[s,+\infty)} \left(\sum_{i=1}^n x_i \right) \text{ for all } x_i \in [0,+\infty)^n, i = 1,\ldots,n \right\}. \tag{4.2}$$

It is easy to show that the bound stated in (3.1), which we call *standard bound* in the following, reduces to

$$\tau_{W,+}(F,\ldots,F)(s) = [nF(s/n) - n + 1]^+ \tag{4.3}$$

for every $s \geq nF^{-1}\left(\frac{F(x_F^*)+n-1}{n}\right)$, $x_F^* := \inf\{x \geq 0 : F' \text{ is decreasing on } [x,+\infty)\}$. For the numerical example given in Sect. 5 we obtain that (4.3) holds when $s \geq 4.39$ (resp. 12.26) for a Log-Normal$(-0.2, 1)$ (resp. $\Gamma(3,1)$) df. Since every Pareto density is decreasing on its domain, (4.3) is valid for every non-negative s in case of a Pareto portfolio; this example is for instance relevant for the measurement of operational risk, as can be seen from Moscadelli (2004).

We use (4.2) to provide a bound which is better (i.e. $\geq$) than the standard one (4.3).

Theorem 4.2 *Let F be a non-negative, continuous df. If $F_i = F, i = 1,\ldots,n$, then for every $s \geq 0$,*

$$m_+(s) \geq 1 - n \inf_{r \in [0, s/n)} \frac{\int_r^{s-(n-1)r} (1 - F(x))dx}{s - nr}. \tag{4.4}$$

Proof For $r \in [0, s/n)$ define $\hat{f}_r : \mathbb{R} \to \mathbb{R}$ as follows:

$$\hat{f}_r(x) := \begin{cases} 0 & \text{if } x < r, \\ \frac{x-r}{s-nr} & \text{if } r \le x \le s-(n-1)r, \\ 1 & \text{otherwise.} \end{cases}$$

We prove that $\hat{f}_r$ is an admissible function in (4.2). Since $\hat{f}_r$ is non-negative, it is sufficient to show that we have $\sum_{i=1}^n \hat{f}_r(x_i) \ge 1$ when $\sum_{i=1}^n x_i \ge s$. If $x_i \ge s-(n-1)r$ for some $i = 1, \dots, n$, this trivially follows, so take $x_1, \dots, x_n \in [0, s-(n-1)r]$ with $\sum_{i=1}^n x_i \ge s$. Define

$$I := \{i \le n : x_i \ge r\}, \quad \bar{I} := \{1, \dots, n\} \setminus I$$

and observe that we have

$$\sum_{i \in I} x_i \ge s - \sum_{i \in \bar{I}} x_i \ge s - (\#\bar{I})r.$$

By definition of $\hat{f}_r$ it follows that

$$\begin{aligned} \sum_{i=1}^n \hat{f}_r(x_i) = \sum_{i \in I} \hat{f}_r(x_i) = \sum_{i \in I} \frac{x_i - r}{s-nr} &= \frac{\sum_{i \in I} x_i - (\#I)r}{s-nr} \\ &\ge \frac{s - ((\#I) + (\#\bar{I}))r}{s-nr} \ge 1. \end{aligned}$$

The theorem follows by checking that

$$\int \hat{f}_r(x) dF(x) = 1 - \frac{\int_r^{s-(n-1)r} F(x)dx}{s-nr}$$

and taking the infimum over all $r \in [0, s/n)$. □

Remark 4.1 (i) Note that

$$\lim_{r \to s/n} \left\{ 1 - n \frac{\int_r^{s-(n-1)r} (1-F(x))dx}{s-nr} \right\} = nF(s/n) - n + 1,$$

hence it follows that (4.4) is greater or equal than the standard lower bound given in (4.3) and (3.1) for every threshold s at which (4.3) is valid. In Sect. 5 we actually show that (4.4) is strictly greater than (4.3) in several cases of interest.

(ii) For $n = 2$, (4.4) gives the sharp bound already stated in (3.1).

(iii) For $n > 2$, the infimum in (4.4) can be easily calculated numerically by finding the zero-derivative points of its argument in the specified interval.

(iv) The assumptions under which Theorem 4.2 is valid are consistent with most dfs F and thresholds s of actuarial/financial interest.

5 Numerical results

In this section we numerically compare the *dual* bound (4.4) with the *standard* bound (4.3).

5.1 Computing numerically the best-possible bound

When the exact value is not available, a good approximation for $m_+(s)$ can be found by solving two linear problems (LPs). We follow Williamson and Downs (1990) in defining the two dfs

$$\underline{F}_N(x) := \frac{1}{N}\sum_{i=1}^{N} 1_{[q_r,+\infty)}(x), \overline{F}_N(x) := \frac{1}{N}\sum_{i=0}^{N-1} 1_{[q_r,+\infty)}(x);$$

the jump points $q_0, \ldots, q_N$ are the quantiles of F defined by $q_0 := \inf \text{supp}(F)$, $q_N := \sup \text{supp}(F)$ and $q_r := F^{-1}(r/N), r = 1, \ldots, N-1$. In the applications to follow we will always take $q_0 = 0$ and $q_N = +\infty$. We have $\underline{F}_N \leq F \leq \overline{F}_N$, from which it follows that, for every real s

$$\sigma_{C,+}(\underline{F}_N, \ldots, \underline{F}_N)(s) \leq \sigma_{C,+}(F, \ldots, F)(s) \leq \sigma_{C,+}(\overline{F}_N, \ldots, \overline{F}_N)(s).$$

Therefore

$$\underline{m}_+(s) \leq m_+(s) \leq \overline{m}_+(s), \tag{5.1}$$

where $\underline{m}_+(s)$ and $\overline{m}_+(s)$ are naturally defined as:

$$\underline{m}_+(s) := \inf\left\{\mathbb{P}\left[\sum_{i=1}^{n} X_i < t\right] : X_i \backsim \underline{F}_N, i = 1, \ldots, n\right\},$$
$$\overline{m}_+(s) := \inf\left\{\mathbb{P}\left[\sum_{i=1}^{n} X_i < t\right] : X_i \backsim \overline{F}_N, i = 1, \ldots, n\right\}.$$

Given that $\underline{F}_N$ is a (possibly defective) discrete df, $\underline{m}_+(s)$ is the solution of the following LP:

$$\underline{m}_+(s) = \min_{p_{j_1,\ldots,j_n}} \sum_{j_1=1}^{N} \cdots \sum_{j_n=1}^{N} p_{j_1,j_2,\ldots,j_n} 1_{(-\infty,t)}\left(\sum_{i=1}^{n} q_{j_i}\right) \text{ subject to}$$
$$\begin{cases} \sum_{j_2=1}^{N}\sum_{j_3=1}^{N}\cdots\sum_{j_n=1}^{N} p_{j_1,\ldots,j_n} = \frac{1}{N} & j_1 = 1, \ldots, N, \\ \sum_{j_1=1}^{N}\sum_{j_3=1}^{N}\cdots\sum_{j_n=1}^{N} p_{j_1,\ldots,j_n} = \frac{1}{N} & j_2 = 1, \ldots, N, \\ \qquad\qquad \ldots, & \\ \sum_{j_1=1}^{N}\sum_{j_2=1}^{N}\cdots\sum_{j_{n-1}=1}^{N} p_{j_1,\ldots,j_n} = \frac{1}{N} & j_n = 1, \ldots, N, \\ 0 \leq p_{j_1,\ldots,j_n} \leq 1 & j_i = 1, \ldots, N, \\ & i = 1, \ldots, n. \end{cases} \tag{5.2}$$

The function $\overline{m}_+(s)$ is the solution of an analogous LP. Since for N tending to infinity the dfs $\overline{F}_N$ and $\underline{F}_N$ converge to the original df F, calculating $m_+(s)$ with

any given level of accuracy is a matter of solving (5.2) and the corresponding LP for $\overline{m}_+(s)$ with N large enough. Unfortunately, that is not a trivial task. The dimension of the two LPs is N^n rows (variables) per nN columns (constraints) and, while the length of the interval $[\underline{m}_+(s), \overline{m}_+(s)]$ asymptotically decreases as $1/N$, the computational time and the memory needed to solve the two LPs increase exponentially. Finally note that a numerical solution will truncate $\underline{F}_N$ at a certain finite value. The software used automatically sets this upper limit so that (5.1) is maintained.

5.2 Plots of the best-possible bound

In this section we illustrate the quality of the estimate of the sharp bound on $\sigma_{C,\psi}(F_1, \dots, F_n)$ provided by the dual bound (4.4). Some dfs of actuarial and financial interest are considered for F. In Figure 5.1, standard (4.3) and dual (4.4) bounds for a portfolio of three Pareto-distributed risks are given. It is relevant to note that the dual bound is strictly greater than the standard one, in accordance with Remark 4.1 (i). Most importantly, the dual value always falls within the range $[\underline{m}_+(s), \overline{m}_+(s)]$, which we plot for some thresholds of interest. This range has been calculated by setting $N = 180$. The two linear problems have been solved using ILOG CPLEX® C Callable Libraries. Note that switching to $n = 4$ drastically lowers the quality of approximation to $N < 60$. In Figure 5.1, the values of $\mu_C[X_1 + X_2 + X_3 < s]$ in case of independent ($C = \Pi$) and comonotonic ($C = M$) scenarios are also given. For the calculation of the distribution of the sum of comononotic rvs note that, in case of a common marginal F, we have that $\mu_M[\sum_{i=1}^n X_i < s] = F(s/n)$, while the convolution is computed by iterated conditioning, i.e.

$$\mu_\Pi \left[\sum_{i=1}^{n} X_i < s \right] = \int dF(x_n) \dots \int dF(x_2) F(s - \sum_{i=2}^{n} x_i).$$

In Figs. 5.2 and 5.3 we do the same for Log-Normal and Γ portfolios. We remark that the time of computation of (4.4) is weakly affected by the dimension of the portfolio.

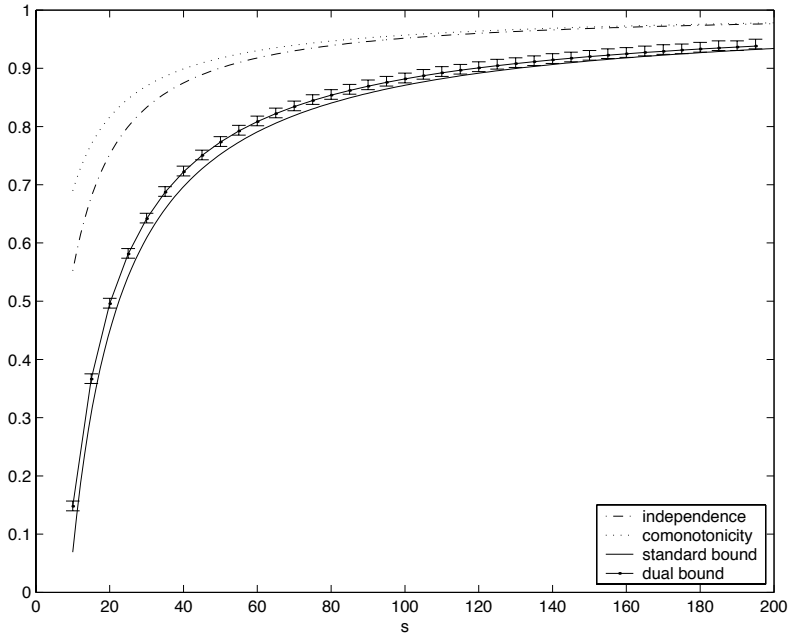

Fig. 5.1 Range for $\mathbb{P}[X_1 + X_2 + X_3 < s]$ for a Pareto(1.5,1)-portfolio under independence and comonotonic scenarios. We also give the standard (4.3) and dual (4.4) bounds. Numerical bounds for the true value of $m_+(s)$ are also provided

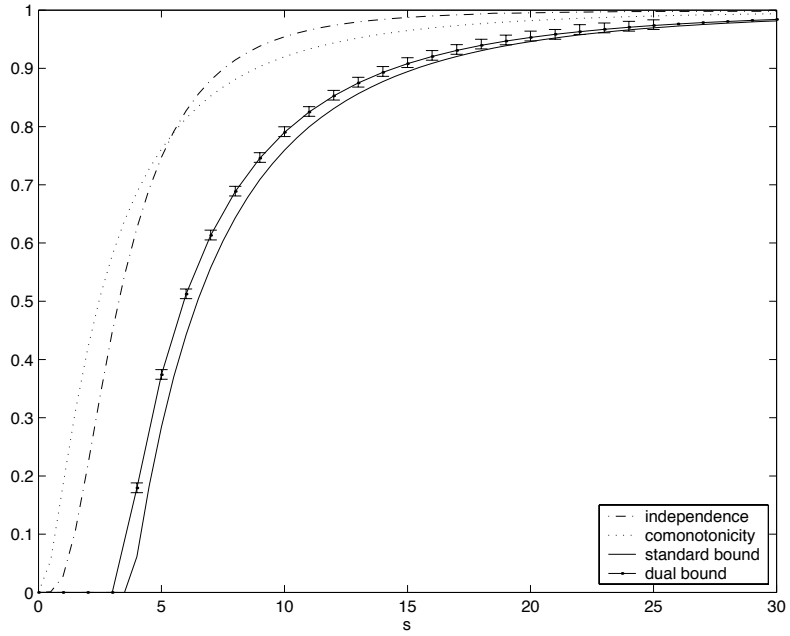

Fig. 5.2 The same as Figure 1 for a Log-Normal(-0.2,1)-portfolio

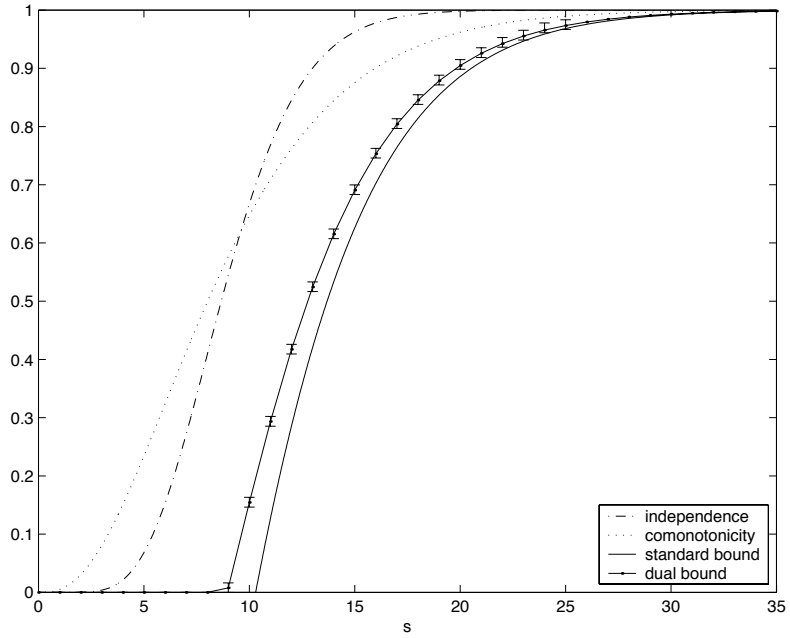

Fig. 5.3 The same as Figure 1 for a $\Gamma(3,1)$-portfolio

6 Conclusions

The problem of finding the best-possible lower bound for the distribution of a non-decreasing function of dependent risks is solved when the portfolio is two-dimensional and this also if some information on the dependence structure of the portfolio is provided. The problem gets much more complicated in arbitrary dimensions. When no information on the copula of the random vector is given, we provide a new bound which we prove to be better than the standard one generally used in the literature.

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References

Dall'Aglio, G. (1972). Fréchet classes and compatibility of dfs. *Symposia Math. 9*, 131–150.

Denuit, M., C. Genest, and É. Marceau (1999). Stochastic bounds on sums of dependent risks. *Insurance Math. Econ. 25*(1), 85–104.

Dhaene, J. and M. Denuit (1999). The safest dependence structure among risks. *Insurance Math. Econ. 25*(1), 11–21.

Dhaene, J., M. Denuit, M. J. Goovaerts, R. Kaas, and D. Vyncke (2002). The concept of comonotonicity in actuarial science and finance: theory. *Insurance Math. Econom. 31*(1), 3–33.

Embrechts, P., A. Höing, and A. Juri (2003). Using copulae to bound the Value-at-Risk for functions of dependent risks. *Finance Stoch. 7*(2), 145–167.

Embrechts, P., A. J. McNeil, and D. Straumann (2002). Correlation and dependence in risk management: properties and pitfalls. In M. Dempster (Ed.), *Risk Management: Value at Risk and Beyond*, pp. 176–223. Cambridge: Cambridge Univ. Press.

Frank, M. J., R. B. Nelsen, and B. Schweizer (1987). Best-possible bounds for the distribution of a sum—a problem of Kolmogorov. *Probab. Theory Related Fields 74*(2), 199–211.

Joe, H. (1997). *Multivariate Models and Dependence Concepts*. London: Chapman & Hall.

Makarov, G. D. (1981). Estimates for the distribution function of the sum of two random variables with given marginal distributions. *Theory Probab. Appl. 26*, 803–806.

Moscadelli, M. (2004). The modelling of operational risk: experience with the analysis of the data collected by the Basel Committee. Preprint, Banca d'Italia.

Nelsen, R. B. (1999). *An Introduction to Copulas*. Berlin Heidelberg New York: Springer.

Rachev, S. T. and L. Rüschendorf (1998). *Mass Transportation Problems. Vol. I-II*. Berlin Heidelberg New York: Springer.

Rüschendorf, L. (1982). Random variables with maximum sums. *Adv. in Appl. Probab. 14*(3), 623–632.

Sklar, A. (1973). Random variables, joint distribution functions, and copulas. *Kybernetika 9*, 449–460.

Williamson, R. C. and T. Downs (1990). Probabilistic arithmetic. I. Numerical methods for calculating convolutions and dependency bounds. *Internat. J. Approx. Reason. 4*(2), 89–158.