ALGEBRAIC $\mathbb{Z}^d$-ACTIONS OF ENTROPY RANK ONE

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ABSTRACT. We investigate algebraic $\mathbb{Z}^d$-actions of entropy rank one, namely those for which each element has finite entropy. Such actions can be completely described in terms of diagonal actions on products of local fields using standard adelic machinery. This leads to numerous alternative characterizations of entropy rank one, both geometric and algebraic. We then compute the measure entropy of a class of skew products, where the fiber maps are elements from an algebraic $\mathbb{Z}^d$-action of entropy rank one. This leads, via the relative variational principle, to a formula for the topological entropy of continuous skew products as the maximum of a finite number of topological pressures. We use this to settle a conjecture concerning the relational entropy of commuting toral automorphisms.

CONTENTS

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1. Introduction

An algebraic $\mathbb{Z}^d$-action is an action of $\mathbb{Z}^d$ by automorphisms of a compact abelian group. The action has entropy rank one if every element has finite entropy. Examples include commuting toral automorphisms, multiplication by 2 and by 3 on the 6-adic solenoid, and Ledrappier's example on a totally disconnected group [16]. Such actions share many dynamical properties with those of a single group automorphism, yet also exhibit striking rigidity phenomena (see, for example, [12] and [13]).

We give here a systematic account of all algebraic $\mathbb{Z}^d$-actions of entropy rank one. Each such action can be built up from prime actions of entropy rank one. Our main method is a modification of standard adelic machinery to show that each prime action of entropy rank one is algebraically conjugate to a diagonal action on a finite product of locally compact fields modulo an invariant cocompact discrete subgroup.

There are just three types of locally compact fields: finite extensions of the reals $\mathbb{R}$, of the p-adics $\mathbb{Q}_p$, or of Laurent power series $\mathbb{F}_p(\!(t)\!)$ over a finite field $\mathbb{F}_p$. Actions of commuting toral automorphisms use the reals, and actions on solenoids combine the reals and the p-adics. As we will see, Ledrappier's example uses the third and last type of locally compact field. More precisely, it is algebraically conjugate to a diagonal action on the product of three isomorphic copies of $\mathbb{F}_2(\!(t)\!)$ modulo an invariant cocompact discrete subgroup. Thus Ledreppier's example can be viewed as being generated by two commuting "toral" automorphisms, where $\mathbb{R}$ has been replaced by $\mathbb{F}_2(\!(t)\!)$. This new viewpoint perhaps explains why Ledrappier's example has played such a central role in the development of algebraic $\mathbb{Z}^d$-actions.

This structure theory for prime actions allows us to easily compute entropy for each element of a general algebraic $\mathbb{Z}^d$-action with entropy rank one. The generators of the action modify Haar measure in each locally compact factor by a multiplicative constant, analogous to the absolute value of an eigenvalue for a toral automorphism. We assemble this information into a finite set of Lyapunov vectors for the action. Then the entropy for a particular direction vector is just the sum of the positive dot products of this direction vector and the Lyapunov vectors.

Skew product transformations have been continual sources of interesting examples in dynamics. One important instance is the so-called "T-T^{-1}" transformation, which is a skew product of the 2-shift and its inverse with base transformation also the 2-shift. This simply defined transformation is Kolmogorov but not Bernoulli with respect to the direct product of Haar measure on the base and the fiber [11]. Its entropy with respect to this measure is $\log 2$. This transformation is also continuous. As shown by Marcus and Newhouse [21], its topological entropy is $\log(5/2)$ and there are exactly two invariant measures of maximal entropy. Marcus and Newhouse compute the entropy of similar skew products, where the fiber maps are powers of a single transformation, in other words drawn from a $\mathbb{Z}$-action. They ask "What happens if one skews into other groups?"

We answer this question for skewing with elements from an algebraic $\mathbb{Z}^d$-action of entropy rank one. The measure entropy of such a skew product has a simple expression in terms the Lyapunov vectors of the action. Using the relative variational principle of Ledrappier and Walters [17], we then show that the topological entropy of a continuous skew product is the largest of a finite number of topological pressures, analogous to the result of Marcus and Newhouse [21, Thm. B]. When the base transformation is a shift of finite type, these pressures can be explicitly computed in terms of the Lyapunov vectors.

Finally, we apply our results to compute the "relational entropy" of commuting group automorphisms, settling in the negative a conjecture made by Geller and Pollicott [9].

2. Statement of results

Let X be a compact abelian group. An *algebraic $\mathbb{Z}^d$-action* on X is a homomorphism $\alpha\colon \mathbb{Z}^d \to \operatorname{aut}(X)$ from $\mathbb{Z}^d$ to the group of (continuous) automorphisms of X. Denote the image of $\mathbf{n} \in \mathbb{Z}^d$ under α by $\alpha^{\mathbf{n}}$, so that $\alpha^{\mathbf{m}+\mathbf{n}} = \alpha^{\mathbf{m}} \circ \alpha^{\mathbf{n}}$ and $\alpha^{\mathbf{0}} = \mathrm{Id}_X$. Let $\mathbf{e}_j = (0, \dots, 1, \dots, 0)$ be the jth standard basis vector of $\mathbb{Z}^d$, so that α is generated by the d commuting automorphisms $\alpha^{\mathbf{e}_j}$. For a detailed account of algebraic $\mathbb{Z}^d$-actions, see Schmidt's comprehensive book [28].

Let μ be Haar measure on X, normalized so that $\mu(X) = 1$. Then every automorphism of X preserves μ. By [3, Prop. 7] the topological entropy of $\alpha^{\mathbf{n}}$ coincides with its entropy with respect to μ, and we denote both by $\mathsf{h}(\alpha^{\mathbf{n}})$. Say that α has *entropy rank one* if $\mathsf{h}(\alpha^{\mathbf{n}}) < \infty$ for all $\mathbf{n} \in \mathbb{Z}^d$.

Denote by R_d the ring $\mathbb{Z}[u_1^{\pm 1}, \dots, u_d^{\pm 1}]$ of Laurent polynomials in d commuting variables with integer coefficients. As explained in Section 3, duality theory provides a one-to-one correspondence between R_d-modules M and algebraic $\mathbb{Z}^d$-actions α_M on compact abelian groups $X_M = \widehat{M}$. Actions of the form $\alpha_{R_d/\mathfrak{p}}$, where $\mathfrak{p}$ is a prime ideal in R_d, are called *prime actions.* They form the basic building blocks for algebraic $\mathbb{Z}^d$-actions.

Our first main result is a structure theorem for prime actions of entropy rank one, which extends earlier work of Schmidt [27] for connected groups.

Theorem 2.1. *Let $\alpha_{R_d/\mathfrak{p}}$ be a prime action of entropy rank one on an infinite group $X_{R_d/\mathfrak{p}}$. Then there is a finite product $\mathbb{A} = \Bbbk^{(1)} \times \dots \times \Bbbk^{(m)}$ of locally compact fields, a diagonal action β of $\mathbb{Z}^d$ on $\mathbb{A}$ for which $\beta^{\mathbf{e}_i}$ multiplies the jth factor $\Bbbk^{(j)}$ by $\xi_i^{(j)} \in \Bbbk^{(j)}$, and a discrete cocompact β-invariant subgroup $\Lambda \subset \mathbb{A}$ such that $\alpha_{R_d/\mathfrak{p}}$ is algebraically conjugate to the quotient action of β on $\mathbb{A}/\Lambda$.*

If $\Bbbk$ is a locally compact field, $\mu_\Bbbk$ is a Haar measure on $\Bbbk$, and $\xi \in \Bbbk$, then $\mu_\Bbbk(\xi E) = \operatorname{mod}_\Bbbk(\xi)\mu_\Bbbk(E)$ for all compact subsets $E \subset \Bbbk$. Here $\operatorname{mod}_\Bbbk(\xi) \in$

$[0,\infty)$ is called the *module* of ξ, and plays the role of the modulus of an eigenvalue.

For a prime action $\alpha_{R_d/\mathfrak{p}}$, let β be the diagonal action of $\mathbb{Z}^d$ on $\mathbb{A} = \Bbbk^{(1)} \times \cdots \times \Bbbk^{(m)}$ described in the previous theorem. For each factor $\Bbbk^{(j)}$ define the jth *Lyapunov vector* $\mathbf{v}^{(j)}$ for $\alpha_{R_d/\mathfrak{p}}$ to be

$$\mathbf{v}^{(j)} = \bigl(\log \operatorname{mod}_{\Bbbk^{(j)}}(\xi_1^{(j)}), \dots, \log \operatorname{mod}_{\Bbbk^{(j)}}(\xi_d^{(j)})\bigr).$$

Define the *set of Lyapunov vectors* of $\alpha_{R_d/\mathfrak{p}}$ to be $\mathcal{L}(\alpha_{R_d/\mathfrak{p}}) = \{\mathbf{v}^{(1)}, \dots, \mathbf{v}^{(m)}\}$ (if $X_{R_d/\mathfrak{p}}$ is finite, put $\mathcal{L}(\alpha_{R_d/\mathfrak{p}}) = \varnothing$). The Lyapunov vectors, for example, can be used to compute $\mathsf{h}(\alpha^{\mathbf{n}})$ to be $\sum_{j=1}^{m} \max\{\mathbf{n}\cdot\mathbf{v}^{(j)}, 0\}$.

Next, consider a general algebraic $\mathbb{Z}^d$-action $\alpha = \alpha_M$ corresponding to an R_d-module M. For reasons explained in Section 4, we will confine our attention to Noetherian R_d-modules M, and call such actions *Noetherian.* In this case, the corresponding group $X = X_M$ has a filtration

$$X_0 = \{0\} \subset X_1 \subset X_2 \subset \cdots \subset X_{r-1} \subset X_r = X$$

of α-invariant compact subgroups X_j for which the restriction of α to each X_j/X_{j-1} is algebraically conjugate to a prime action $\alpha_{R_d/\mathfrak{p}_j}$. Also, α has entropy rank one if and only if each of these prime actions does. In this case we define the Lyapunov vectors of α to be $\mathcal{L}(\alpha) = \mathcal{L}(\alpha_{R_d/\mathfrak{p}_1}) \cup \cdots \cup \mathcal{L}(\alpha_{R_d/\mathfrak{p}_r})$, with multiplicity taken into account. This set turns out to be independent of the particular filtration used. The addition formula for entropy shows that $\mathsf{h}(\alpha^{\mathbf{n}}) = \sum_{\mathbf{v}\in\mathcal{L}(a)} \max\{\mathbf{n}\cdot\mathbf{v}, 0\}$.

We now turn to skew product transformations. Let (Y,ν) be a measure space and $T\colon Y \to Y$ be a measurable transformation preserving ν. To construct a skew product with base transformation T using fiber maps from an algebraic $\mathbb{Z}^d$-action α, let $\mathbf{s}\colon Y \to \mathbb{Z}^d$ be a measurable skewing function. Define the *skew product* $T \times^{\mathbf{s}} \alpha$ on $Y \times X$ by

$$(T \times^{\mathbf{s}} \alpha)(y,x) = \bigl(T(y), \alpha^{\mathbf{s}(y)}(x)\bigr).$$

Clearly $T \times^{\mathbf{s}} \alpha$ preserves the product measure $\nu \times \mu$.

To obtain useful results, we need to assume that $\mathbf{s}$ is ν*-integrable*, namely that

$$\int_Y \|\mathbf{s}(y)\|\, d\nu(y) < \infty,$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$. Hence $\mathbf{s}$ has an *average value* $\nu(\mathbf{s}) = \int_Y \mathbf{s}(y)\, d\nu(y) \in \mathbb{R}^d$. In addition, we often need to assume that $\mathbf{s}$ is T*-ergodic*, namely that the ergodic averages

$$\frac{1}{n}[\mathbf{s}(y) + \mathbf{s}(Ty) + \cdots + \mathbf{s}(T^{n-1}y)] \to \int_Y \mathbf{s}\, d\nu = \nu(\mathbf{s})$$

for ν-almost every $y \in Y$. Of course this condition automatically holds if T itself is assumed ergodic, but the extra flexibility turns out to be needed.

Theorem 2.2. *Let α be a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one with Lyapunov vector set $\mathcal{L}(\alpha)$. Let T be a measure-preserving transformation of (Y,ν) and $\mathbf{s}\colon Y \to \mathbb{Z}^d$ be a ν-integrable and T-ergodic skewing function with average value $\nu(\mathbf{s}) \in \mathbb{R}^d$. Then*

$$\mathsf{h}_{\nu\times\mu}(T \times^{\mathbf{s}} \alpha) = \mathsf{h}_\nu(T) + \sum_{\mathbf{v}\in\mathcal{L}(\alpha)} \max\{\, \nu(\mathbf{s})\cdot\mathbf{v}, 0\,\}.$$

Suppose now that Y is compact, that T is a homeomorphism, and that the skewing function $\mathbf{s}\colon Y \to \mathbb{Z}^d$ is continuous. Then $T\times^{\mathbf{s}}\alpha$ is also a homeomorphism, and we ask for its topological entropy in terms of the Lyapunov vectors in $\mathcal{L}(\alpha)$. For each subset $E \subset \mathcal{L}(\alpha)$ define $f_E(y) = \sum_{\mathbf{v}\in E}\mathbf{s}(y)\cdot\mathbf{v}$, which is a continuous function on Y. By convention we put $f_\varnothing(y) \equiv 0$. Denote the topological pressure of a continuous function $f\colon Y \to \mathbb{R}$ with respect to T by $\mathsf{P}(f,T)$ (see Walter's book [30] for a lucid account of topological pressure and its properties, especially the variational principle).

Theorem 2.3. *Let α be a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one with Lyapunov vector set $\mathcal{L}(\alpha)$. Suppose that Y is a compact space, and that $T\colon Y\to Y$ and $\mathbf{s}\colon Y\to\mathbb{Z}^d$ are continuous. For every $E\subset\mathcal{L}(\alpha)$ define $f_E(y) = \sum_{\mathbf{v}\in E}\mathbf{s}(y)\cdot\mathbf{v}$. Then the topological entropy of $T\times^{\mathbf{s}}\alpha$ is given by*

$$\mathsf{h}(T\times^{\mathbf{s}}\alpha) = \max_{E\,\subset\,\mathcal{L}(\alpha)} \mathsf{P}(f_E, T).$$

We remark that when T is a shift of finite type, each of the pressures $\mathsf{P}(f_E,T)$ can be computed explicitly. Hence in this case the topological entropy of the skew product is easily calculated.

For example, let A and B be commuting automorphisms of $\mathbb{T}^m$ with real eigenvalues $\xi_1, \dots, \xi_m$ and $\eta_1, \dots, \eta_m$, respectively, on their common eigenspaces. The corresponding Lyapunov vectors for the $\mathbb{Z}^2$-action α they generate are $\mathbf{v}^{(j)} = (\log|\xi_j|, \log|\eta_j|)$ for $1 \leqslant j \leqslant m$. Let $Y = \{1,2\}^{\mathbb{Z}}$, T be the 2-shift on Y, and $\mathbf{s}(y) = \mathbf{e}_{y_0}$. Thus $T\times^{\mathbf{s}}\alpha$ is the skew product of A and B over the 2-shift. Computing pressures, we find that

$$\mathsf{h}(T\times^{\mathbf{s}}\alpha) = \max_{E\subset\{1,\dots,m\}} \log\Big(\prod_{j\in E}|\xi_j| + \prod_{j\in E}|\eta_j|\Big). \tag{2.1}$$

Let Z be a compact metric space, and let $\mathcal{R} \subset Z\times Z$ be an arbitrary closed subset, or *relation*. Friedland [8] defined a "relational entropy" $\mathsf{h}(\mathcal{R})$ for $\mathcal{R}$ to be the entropy of the shift map on $Z^{\mathbb{N}}$ restricted to the compact subset $\{(z_i)\in Z^{\mathbb{N}} : (z_i, z_{i+1}) \in \mathcal{R} \text{ for all } i\in\mathbb{N}\}$. If $\mathcal{R}_S$ is the graph of a continuous transformation $S\colon Z\to Z$, then $\mathsf{h}(\mathcal{R}_S)$ coincides with the usual topological entropy $\mathsf{h}(S)$, so in this sense relational entropy generalizes topological entropy.

Geller and Pollicott [9] studied an entropy $\mathsf{e}(A,B)$ for a pair of commuting transformations A and B, by using the union $\mathcal{R}_{A,B}$ of the graphs of A and of B and putting $\mathsf{e}(A,B) = \mathsf{h}(\mathcal{R}_{A,B})$. They showed that if $Z = \mathbb{T}$ and A and B are multiplication by p and by q, respectively, with $p \neq q$, then $\mathsf{e}(A,B) = \log(p+q)$, confirming a conjecture of Friedland. They also conjectured a formula for $\mathsf{e}(A,B)$ when A and B are commuting toral automorphisms.

Relational entropy is closely related to skew products. Let A and B be commuting group automorphisms, and α be the algebraic $\mathbb{Z}^2$-action defined by $\alpha^{\mathbf{e}_1} = A$ and $\alpha^{\mathbf{e}_2} = B$. The analogue of the condition $p \neq q$ above is that $\mu(\{x : Ax = Bx\}) = 0$. We can then compute $\mathsf{e}(A,B)$ using Theorem 2.3 as follows.

Theorem 2.4. *Let A and B be commuting automorphisms of a compact abelian group X that generate a Noetherian $\mathbb{Z}^2$-action α. Assume that $\mu(\{x : Ax = Bx\}) = 0$. Let $Y = \{1,2\}^{\mathbb{Z}}$ and T be the shift on Y. Define $\mathsf{s}\colon Y \to \mathbb{Z}^2$ by $\mathsf{s}(y) = \mathbf{e}_{y_0}$. Then $\mathsf{e}(A,B) = \mathsf{h}(T \times^{\mathsf{s}} \alpha)$.*

For example, suppose that A and B are commuting automorphisms of $\mathbb{T}^m$ with real eigenvalues $\xi_1, \ldots, \xi_m$ and $\eta_1, \ldots, \eta_m$, respectively, on their common eigenspaces. Suppose that $\mu(\{x : Ax = Bx\}) = 0$ or, equivalently here, that $A \neq B$. Applying Theorem 2.3 we see that $\mathsf{e}(A,B)$ is given by the formula (2.1). This shows that the formula for $\mathsf{e}(A,B)$ conjectured in [9] is not correct.

3. Algebraic $\mathbb{Z}^d$-actions

We begin with a brief description of algebraic $\mathbb{Z}^d$-actions and their relationships, via duality, with commutative algebra.

Let X be a compact abelian group, which we assume henceforth to be metrizable. Then its dual group $M = \widehat{X}$ is discrete, and is also countable by metrizability of X.

Denote by R_d the ring $\mathbb{Z}[u_1^{\pm 1}, \ldots, u_d^{\pm 1}]$ of Laurent polynomials in d commuting variables with integer coefficients. An element $f \in R_d$ has the form

$$f = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} \, \mathbf{u}^{\mathbf{n}},$$

where $f_{\mathbf{n}} \in \mathbb{Z}$ for all $\mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$, $f_{\mathbf{n}} = 0$ for all but finitely many $\mathbf{n}$, and $\mathbf{u}^{\mathbf{n}} = u_1^{n_1} \cdots u_d^{n_d}$.

We use α and duality to make M into an R_d-module as follows. For $\mathbf{n} \in \mathbb{Z}^d$ and $a \in M$ put $\mathbf{u}^{\mathbf{n}} \cdot a = \widehat{\alpha^{\mathbf{n}}}(a)$, where $\widehat{\alpha^{\mathbf{n}}}$ is the automorphism of M dual to $\alpha^{\mathbf{n}}$. This extends naturally to all $f \in R_d$ by putting

$$f \cdot a = \sum_{\mathbf{n} \in \mathbb{Z}^d} f_{\mathbf{n}} (\mathbf{u}^{\mathbf{n}} \cdot a).$$

The R_d-module M is called the *dual module* of α.

This process can be reversed. Suppose that M is a countable R_d-module. Then $X_M = \widehat{M}$ is a compact metrizable group. The R_d-module structure on M gives an algebraic $\mathbb{Z}^d$-action α_M on X_M, in which $\alpha_M^{\mathbf{n}}$ is dual to the automorphism of M given by multiplication by $\mathbf{u}^{\mathbf{n}}$.

Thus via duality there is a one-to-one correspondence between algebraic $\mathbb{Z}^d$-actions and R_d-modules.

An R_d-module is said to be Noetherian if it satisfies the ascending chain condition on submodules. We call an algebraic $\mathbb{Z}^d$-action *Noetherian* if its dual module is Noetherian over R_d. Duality shows that α is Noetherian if and only if whenever $X_1 \supset X_2 \supset \dots$ is a descending chain of closed α-invariant subgroups, then there is an m for which $X_k = X_m$ for all $k \geqslant m$.

An ideal $\mathfrak{p} \subset R_d$ is *prime* if it is a proper ideal with the property that if $f \cdot g \in \mathfrak{p}$, then either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$. A prime ideal $\mathfrak{p} \subset R_d$ is *associated to* an R_d-module M if there is an $a \in M$ such that $\mathfrak{p} = \{f \in R_d : f \cdot a = 0\}$. If M is Noetherian over R_d, then the set $\operatorname{asc}(M)$ of associated prime ideals is finite.

Algebraic $\mathbb{Z}^d$-actions of the form $\alpha_{R_d/\mathfrak{p}}$ with $\mathfrak{p}$ a prime ideal in R_d play a fundamental role. We call such an action a *prime action.* If α_M is an algebraic $\mathbb{Z}^d$-action with dual module M, then the prime actions $\alpha_{R_d/\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{asc}(M)$ are the *associated prime actions* of α_M. The associated prime actions of an algebraic $\mathbb{Z}^d$-action carry much of the information about its dynamical behavior.

To illustrate this point, let us characterize those algebraic $\mathbb{Z}^d$-actions having the important finiteness property of expansiveness, a result due to Schmidt [28, Thm. 6.5]. Recall that α is called *expansive* if there is a neighborhood U of the identity 0_X in X such that

$$\bigcap_{\mathbf{n} \in \mathbb{Z}^d} \alpha^{\mathbf{n}}(U) = \{0_X\}.$$

Introduce the notations $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$, $\mathbb{C}^\times = \mathbb{C} \smallsetminus \{0\}$, and

$$V_{\mathbb{C}}(\mathfrak{p}) = \{\mathbf{z} \in (\mathbb{C}^\times)^d : f(\mathbf{z}) = 0 \text{ for every } f \in \mathfrak{p}\}.$$

Theorem 3.1. *Let M be an R_d-module and α_M be the corresponding algebraic $\mathbb{Z}^d$-action. If α_M is expansive then it is Noetherian.*

Assume now that α_M is Noetherian, and let $\operatorname{asc}(M)$ be the finite set of its associated prime ideals. Then the following are equivalent:

(1) *α_M is expansive.*
(2) *$\alpha_{R_d/\mathfrak{p}}$ is expansive for every $\mathfrak{p} \in \operatorname{asc}(M)$.*
(3) *$V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{S}^d = \varnothing$ for every $\mathfrak{p} \in \operatorname{asc}(M)$.*

The following result shows that expansiveness is "exact." One direction is proved in [28, Cor. 6.15], and the other uses a simple argument in the proof of [6, Lemma 4.8].

Proposition 3.2. *Let α be an algebraic $\mathbb{Z}^d$-action on X, and let K be a closed α-invariant subgroup of X. Then the action α is expansive if and only if the restriction α_K of α to K is expansive and the induced action $\alpha_{X/K}$ of α on X/K is expansive.*

It is often informative to examine a notion of expansiveness along subspaces of $\mathbb{R}^d$ (see [4] for details). Let H be a hyperplane of dimension $d-1$

in $\mathbb{R}^d$. Say that α is *expansive along* H if there is a neighborhood U of 0_X and a ball $B(r)$ around 0 in $\mathbb{R}^d$ such that

$$\bigcap_{\mathbf{n}\in(H+B(r))\cap\mathbb{Z}^d} \alpha^{\mathbf{n}}(U) = \{0_X\}.$$

We let $\mathsf{N}_{d-1}(\alpha)$ denote the set of all hyperplanes along which α is *not* expansive. According to [4], if X is infinite then $\mathsf{N}_{d-1}(\alpha)$ is a closed nonempty subset of the compact Grassman manifold of hyperplanes, and it determines all lower dimensional expansive behavior. For algebraic actions, this set is computed explicitly in [6].

There is another place where prime actions arise. If M is a Noetherian R_d-module, then it is easy to find a chain of submodules

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{r-1} \subset M_r = M \tag{3.1}$$

such that $M_j/M_{j-1} \cong R_d/\mathfrak{q}_j$ for $1 \leqslant j \leqslant r$, where each $\mathfrak{q}_j$ is a prime ideal containing one of the associated prime ideals of M (see [28, Prop. 6.1]). Dual to this filtration is a reversed chain of closed α_M-invariant subgroups

$$X_M = X_0 \supset X_1 \supset X_2 \supset \cdots \supset X_{r-1} \supset X_r = \{0\}, \tag{3.2}$$

where X_j is the annihilator of M_j in X, and the induced action of α_M on X_{j-1}/X_j is isomorphic to the prime action $\alpha_{R_d/\mathfrak{q}_j}$. In this sense an arbitrary Noetherian algebraic $\mathbb{Z}^d$-action can be built up as a finite succession of extensions by prime actions.

Although the prime ideals $\mathfrak{q}_j$ appearing in the successive quotients in (3.1) are not necessarily unique, we will see in Proposition 8.3 that there is a strong relation between them and $\operatorname{asc}(M)$.

4. Rank one actions

We introduce two notions of rank one for algebraic $\mathbb{Z}^d$-actions together with a closely related notion of irreducibility.

Definition 4.1. Let α be an algebraic $\mathbb{Z}^d$-action.

(i) α has *entropy rank one* if $\mathsf{h}(\alpha^{\mathbf{n}}) < \infty$ for all $\mathbf{n} \in \mathbb{Z}^d$.
(ii) α has *expansive rank one* if there exists an $\mathbf{n} \in \mathbb{Z}^d$ such that $\alpha^{\mathbf{n}}$ is an expansive transformation.
(iii) α is *irreducible* if every proper closed α-invariant subgroup is finite.

Remarks 4.2. (1) See [6] for a more general discussion of expansive rank and entropy rank (with a slightly different definition of entropy rank which is equivalent in the expansive case). For more information about irreducible actions and their properties see [7], [13], [15], and [28, Section 29].

(2) More generally say that α has *entropy rank* k if the restriction of α to every subgroup of $\mathbb{Z}^d$ of rank k has finite entropy, and k is minimal with this property. Then entropy rank zero corresponds to X being finite. Thus the property defined in Definition 4.1(i) should really be termed "entropy

rank at most one," but it is convenient to use the briefer term here. An analogous remark applies to expansive rank.

Proposition 4.3. *If an algebraic $\mathbb{Z}^d$-action has expansive rank one, then it also has entropy rank one.*

Proof. Choose $\mathbf{n}$ so that $\alpha^{\mathbf{n}}$ is expansive. Then by [4, Thm. 6.3] or [29] it follows that $\mathsf{h}(\alpha^{\mathbf{m}}) < \infty$ for every $\mathbf{m} \in \mathbb{Z}^d$, so that α has entropy rank one. □

The converse of Proposition 4.3 is false. For example, the identity automorphism on an infinite compact group has entropy rank one but not expansive rank one. Less trivially, so does an ergodic toral automorphism which has some eigenvalues of modulus one. Example 7.4 of [6] gives an interesting algebraic $\mathbb{Z}^3$-action of expansive rank three and entropy rank two.

We next characterize rank one in terms of the associated prime actions. The case when X is connected is treated in [7, Theorem 4.4]; the argument here for the general case is similar.

Proposition 4.4. *Let α_M be a Noetherian algebraic $\mathbb{Z}^d$-action. Then α_M has entropy rank one if and only if each of its associated prime actions $\alpha_{R_d/\mathfrak{p}}$ for $\mathfrak{p} \in \operatorname{asc}(M)$ has entropy rank one. Similarly, α_M has expansive rank one if and only if each associated prime action $\alpha_{R_d/\mathfrak{p}}$ has expansive rank one.*

Proof. First suppose that α_M has entropy rank one. Let $\mathfrak{p} \in \operatorname{asc}(M)$. Then $\mathfrak{p} = \{f \in R_d : f \cdot a = 0\}$ for some $a \in M$, and so $R_d/\mathfrak{p} \cong R_d \cdot a \subset M$. By duality, $\alpha_{R_d/\mathfrak{p}}$ is a quotient of α_M. Hence $\mathsf{h}(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}) \leqslant \mathsf{h}(\alpha^{\mathbf{n}}_M) < \infty$ for all $\mathbf{n} \in \mathbb{Z}^d$, so that $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one.

Conversely, suppose that for each $\mathfrak{p} \in \operatorname{asc}(M)$ the associated prime action $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one. The restriction α_{X_{j-1}/X_j} of α_M to a partial quotient X_{j-1}/X_j from the filtration (3.2) is isomorphic to the prime action $\alpha_{R_d/\mathfrak{q}_j}$, where $\mathfrak{q}_j$ contains some $\mathfrak{p} \in \operatorname{asc}(M)$. The surjection $R_d/\mathfrak{p} \to R_d/\mathfrak{q}_j$ dualizes to an inclusion $X_{R_d/\mathfrak{q}_j} \to X_{R_d/\mathfrak{p}}$. Hence for every $\mathbf{n} \in \mathbb{Z}^d$ we have that

$$\mathsf{h}(\alpha^{\mathbf{n}}_{X_{j-1}/X_j}) = \mathsf{h}(\alpha^{\mathbf{n}}_{R_d/\mathfrak{q}_j}) \leqslant \mathsf{h}(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}) < \infty.$$

Repeated use of Yuzvinsky's addition formula (see [19] or [28, Thm. 14.1]) then shows that

$$\mathsf{h}(\alpha^{\mathbf{n}}_M) = \sum_{j=1}^{r} \mathsf{h}(\alpha^{\mathbf{n}}_{X_{j-1}/X_j}) < \infty$$

for every $\mathbf{n} \in \mathbb{Z}^d$, so that α_M has entropy rank one.

Now suppose that α_M has expansive rank one, so that $\alpha^{\mathbf{n}}_M$ is expansive for some $\mathbf{n} \in \mathbb{Z}^d$. Let $\mathfrak{p} \in \operatorname{asc}(M)$. As before, $\alpha_{R_d/\mathfrak{p}}$ is a quotient of α_M. By Proposition 3.2, $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is expansive, and so $\alpha_{R_d/\mathfrak{p}}$ has expansive rank one for every $\mathfrak{p} \in \operatorname{asc}(M)$.

Conversely, suppose that $\alpha_{R_d/\mathfrak{p}}$ has expansive rank one for every $\mathfrak{p} \in \operatorname{asc}(M)$. We will see in Propositions 7.1 and 7.2 that $\alpha^{\mathbf{m}}_{R_d/\mathfrak{p}}$ is expansive except for those $\mathbf{m}$ lying in a finite union of hyperplanes in $\mathbb{R}^d$. It follows that there is an $\mathbf{n} \in \mathbb{Z}^d$ for which $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ is expansive for all $\mathfrak{p} \in \operatorname{asc}(M)$. Then repeated application of Proposition 3.2 to the filtration (3.2) shows that $\alpha^{\mathbf{n}}_M$ is expansive, so that α_M has expansive rank one. □

The analogue of Proposition 4.4 for entropy rank greater than one can fail, because the set of nonexpansive hyperspaces can have nonempty interior.

Example 4.5. Consider the $\mathbb{Z}^3$-action $\alpha_{R_3/\mathfrak{p}}$, where $\mathfrak{p} = \langle 1+u_1+u_2, u_3-2\rangle$, treated in [6, Example 5.8], which the reader should consult for details. This action has expansive rank two, yet the set $\mathsf{N}_2(\alpha_{R_3/\mathfrak{p}})$ of nonexpansive 2-planes has nonempty interior. Hence there are a finite number of prime ideals $\mathfrak{p}_j$, each obtained from $\mathfrak{p}$ by a coordinate change of monomials in R_3, such that every 2-plane is nonexpansive for at least one of the $\alpha_{R_3}/\mathfrak{p}_j$. Let $M = \bigoplus_j R_3/\mathfrak{p}_j$. Then α_M has expansive rank three, but all of its associated prime actions $\alpha_{R_3/\mathfrak{p}_j}$ have expansive rank two.

There exist non-Noetherian actions having entropy rank one.

Example 4.6. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$, and consider $M = \mathbb{Q}^2$ as an R_1-module via $u_1 \cdot \mathbf{q} = A\mathbf{q}$. Then $M_n = (n!)^{-1}\mathbb{Z}^2$ is a strictly increasing sequence of R_1-submodules whose union is M, showing that M is not Noetherian over R_1. However, $X_{M_{j+1}}$ is a finite extension of X_{M_j} for $j \geqslant 1$, and so α_M has entropy rank one.

This example works because each intermediate group is a zero entropy extension of its predecessor. But are there examples of non-Noetherian actions of entropy rank one where the action on successive quotients has at least one element with positive entropy? Answering this question turns out to be equivalent to answering Lehmer's Problem, which has been open for almost 70 years. According to [18], the original number-theoretic version of Lehmer's Problem can be reformulated as follows.

Problem 4.7 (Lehmer)**.** For every $\epsilon > 0$ is there an automorphism ϕ of a compact abelian group for which $0 < \mathsf{h}(\phi) < \epsilon$?

To see the equivalence between these problems, first suppose that $M_1 \subset M_2 \subset \dots$ is an increasing chain of Noetherian R_d-modules such that for every j there is an $\mathbf{n} \in \mathbb{Z}^d$ for which $\mathsf{h}(\alpha^{\mathbf{n}}_{M_{j+1}/M_j}) > 0$. Using prime filtrations of the form (3.1), we may assume that $M_{j+1}/M_j \cong R_d/\mathfrak{q}_j$ for prime ideals $\mathfrak{q}_j$. Anticipating our results on entropy, our assumption that $\mathsf{h}(\alpha^{\mathbf{n}}_{R_d/\mathfrak{q}_j}) > 0$ for some $\mathbf{n}$ is equivalent to the existence of a nonzero Lyapunov vector $\mathbf{v}_j \in \mathcal{L}(\alpha_{R_d/\mathfrak{q}_j})$ for all $j \geqslant 1$. It is then easy to see that there is an $\mathbf{n} \in \mathbb{Z}^d$ for which $\mathbf{n} \cdot \mathbf{v}_j > 0$ for infinitely many j. The addition formula for entropy shows that

$$\mathsf{h}(\alpha_M^{\mathbf{n}}) = \mathsf{h}(\alpha_{M_1}^{\mathbf{n}}) + \sum_{j=1}^{\infty} \mathsf{h}(\alpha_{R_d/\mathfrak{q}_j}^{\mathbf{n}}) < \infty,$$

and $\mathsf{h}(\alpha_{R_d/\mathfrak{q}_j}^{\mathbf{n}}) \geqslant \mathbf{n}\cdot\mathbf{v}_j > 0$ for infinitely many j. Hence for every $\epsilon > 0$ there is a j for which $0 < \mathsf{h}(\alpha_{R_d/\mathfrak{q}_j}^{\mathbf{n}}) < \epsilon$, showing that the answer to Lehmer's Problem would be affirmative. Conversely, if Lehmer's problem has an affirmative answer, it is easy to use a direct product of a countable number of automorphisms with summable positive entropies having the desired non-Noetherian and entropy properties.

Rather than formulate our results as conditional on Lehmer's Problem, which may not confer any essentially new generality, we will confine our attention to Noetherian actions.

5. Algebraic preliminaries

We sketch here the algebraic ideas needed to describe the structure of algebraic $\mathbb{Z}^d$-actions of entropy rank one. For more algebraic background see [5] and [10]. Detailed accounts of global fields and local fields are contained in [25] and [31].

An integral domain D has *characteristic zero* if $n \cdot 1_D \neq 0_D$ for all $n \geqslant 1$, in which case we write $\operatorname{char} D = 0$. It has *characteristic* p if $p \cdot 1_D = 0_D$ for some prime number $p \geqslant 2$, denoted by $\operatorname{char} D = p$. In the latter case we also say that D has *positive characteristic.*

By definition we require that all prime ideals $\mathfrak{p}$ in R_d be proper. Observe that $\mathfrak{p}$ is prime if and only if $R_d/\mathfrak{p}$ is an integral domain. If $\mathfrak{p} \cap \mathbb{Z} = \{0\}$, then $\operatorname{char} R_d/\mathfrak{p} = 0$, and $X_{R_d/\mathfrak{p}}$ is a connected topological group whose topological dimension we denote by $\dim X_{R_d/\mathfrak{p}} \geqslant 1$. If $\mathfrak{p} \cap \mathbb{Z} = p\mathbb{Z}$ for a prime p, then $\operatorname{char} R_d/\mathfrak{p} = p$, and $X_{R_d/\mathfrak{p}}$ is totally disconnected, or equivalently, $\dim X_{R_d/\mathfrak{p}} = 0$.

If $\mathbb{K}$ is an extension field of $\mathbb{F}$, the *transcendence degree* $\operatorname{trdeg}_{\mathbb{F}} \mathbb{K}$ is the maximal number of elements in $\mathbb{K}$ that are algebraically independent over $\mathbb{F}$. For a prime ideal $\mathfrak{p}$ in R_d, let $\mathbb{K}$ denote the fraction field of $R_d/\mathfrak{p}$. If $\operatorname{char} R_d/\mathfrak{p} = 0$, we define $\operatorname{trdeg}_{\mathbb{Q}} R_d/\mathfrak{p}$ to be $\operatorname{trdeg}_{\mathbb{Q}} \mathbb{K}$, while if $\operatorname{char} R_d/\mathfrak{p} = p$ we put $\operatorname{trdeg}_{\mathbb{F}_p} R_d/\mathfrak{p}$ to be $\operatorname{trdeg}_{\mathbb{F}_p} \mathbb{K}$.

The *Krull dimension* of a ring R is the length r of the longest chain $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_r$ of prime ideals in R. The following result clarifies the relationship between transcendence degree and Krull dimension for quotients $R_d/\mathfrak{p}$. Roughly speaking, it says that the set of prime ideals $\mathfrak{p}$ in R_d consists of $d+1$ layers with respect to inclusion, where the kth layer consists of those $\mathfrak{p}$ for which $\operatorname{kdim} R_d/\mathfrak{p} = d+1-k$.

Proposition 5.1. *The ring R_d has Krull dimension* $\operatorname{kdim} R_d = d+1$. *Every prime ideal $\mathfrak{p}$ is contained in a maximal chain*

$$\langle 0\rangle \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_k = \mathfrak{p} \subsetneq \cdots \subsetneq \mathfrak{p}_{d+1}$$

of prime ideals $\mathfrak{p}_j$*. Its position* k *is the same for all such chains and is given by* $k = d + 1 - \operatorname{kdim} R_d/\mathfrak{p}$.

If $\operatorname{char} R_d/\mathfrak{p} = 0$ *then* $\operatorname{kdim} R_d/\mathfrak{p} = 1 + \operatorname{trdeg}_{\mathbb{Q}} R_d/\mathfrak{p}$*, while if* $\operatorname{char} R_d/\mathfrak{p} = p$ *then* $\operatorname{kdim} R_d/\mathfrak{p} = \operatorname{trdeg}_{\mathbb{F}_p} R_d/\mathfrak{p}$.

Every maximal ideal $\mathfrak{m} \subset R_d$ *has finite index, and* $R_d/\mathfrak{m}$ *is a finite field. A prime ideal* $\mathfrak{p}$ *satisfies* $\operatorname{kdim} R_d/\mathfrak{p} = 1$ *if and only if* $|R_d/\mathfrak{p}| = \infty$ *and* $|R_d/\mathfrak{a}| < \infty$ *for every ideal* $\mathfrak{a} \supsetneq \mathfrak{p}$.

Proof. Almost all of this is standard commutative algebra for polynomial rings. The only differences are the use of integer coefficients (which contributes the extra 1 in $\operatorname{kdim} R_d/\mathfrak{p}$ in characteristic zero), and the use of Laurent polynomials (easily handled by forming fractions from $\mathbb{Z}[u_1, \dots, u_d]$ using the multiplicative set of monomials for denominators).

Although there are elementary arguments for the statements in the first two paragraphs, they also follow from the observation that R_d is a Cohen-Macaulay ring [5, Prop. 18.9], and therefore universally catenary [5, Cor. 18.10].

For the last paragraph, first observe that if $\mathfrak{m}$ is a maximal ideal, then $R_d/\mathfrak{m}$ is a field that is finitely generated over $\mathbb{Z}$ as a ring, and is therefore finite. Next, suppose that $\operatorname{kdim} R_d/\mathfrak{p} = 1$, and let $\mathfrak{a} \supsetneq \mathfrak{p}$ be an ideal. Considered as an $R_d/\mathfrak{p}$-module, $R_d/\mathfrak{a}$ has a prime filtration as in (3.1), where each quotient $M_j/M_{j-1} \cong R_d/\mathfrak{m}_j$ for some maximal ideal $\mathfrak{m}_j$. Hence M_j/M_{j-1} is finite for every j, and therefore so is $R_d/\mathfrak{a}$. Conversely, suppose that $\mathfrak{p}$ is a prime ideal such that $|R_d/\mathfrak{a}| < \infty$ for every ideal $\mathfrak{a} \supsetneq \mathfrak{p}$. Let $\mathfrak{q}$ be a prime ideal with $\mathfrak{q} \supsetneq \mathfrak{p}$. Then $R_d/\mathfrak{q}$ is a finite integral domain, hence a field, so that $\mathfrak{q}$ is maximal. Hence $\operatorname{kdim} R_d/\mathfrak{p} = 1$. □

An *absolute value* $|\cdot|$ on a field $\mathbb{K}$ is a function $|\cdot|\colon \mathbb{K} \to \mathbb{R}$ such that there is a constant C so that for all a, b in $\mathbb{K}$ we have that

(i) $|a| \geq 0$ and $|a| = 0$ if and only if $a = 0$,
(ii) $|ab| = |a|\,|b|$, and
(iii) $|a + b| \leq C \max(|a|, |b|)$.

If instead of (iii) the stronger property $|a + b| \leq \max(|a|, |b|)$ holds, we say that $|\cdot|$ is a *nonarchimedean* absolute value; otherwise $|\cdot|$ is *archimedian*. We will always assume that $|\cdot|$ is non-trivial, namely that $|a| \neq 0, 1$ for some $a \in \mathbb{K}$. Two absolute values $|\cdot|_1$ and $|\cdot|_2$ are called *equivalent* if the metrics they induce on $\mathbb{K}$ give the same topology. This is the case exactly when there is a positive constant κ such that $|\cdot|_1 = |\cdot|_2^{\kappa}$. An equivalence class of absolute values is called a *place* on $\mathbb{K}$. Places are denoted by letters like v and w, and the set of all places on $\mathbb{K}$ is denoted by $\mathcal{P}(\mathbb{K})$. If $v \in \mathcal{P}(\mathbb{K})$, we let $\mathbb{K}_v$ denote the completion of $\mathbb{K}$ with respect to any absolute value in v; this is well-defined since absolute values in v give equivalent metrics on $\mathbb{K}$.

A *global field* $\mathbb{K}$ is a finite field extension of either $\mathbb{Q}$, in which case $\mathbb{K}$ is also called an *algebraic number field*, or of $\mathbb{F}_p(t)$, where $\mathbb{K}$ is called a *function field over* $\mathbb{F}_p$. A *local field* $\Bbbk$ is the completion $\Bbbk = \mathbb{K}_v$ of a global field with respect to a place $v \in \mathcal{P}(\mathbb{K})$.

Ostrowski's Theorem [25, Thm. 4-30 (i)] states that every place on $\mathbb{Q}$ is either the *infinite place* ∞ corresponding to the usual absolute value $|\cdot|_\infty$, or the place p corresponding to the p-adic absolute value $|\cdot|_p$ for some prime number p, defined by $|m\,p^k/n|_p = p^{-k}$ where $p \nmid mn$. Places of the second type are called *finite places*. The local fields for $\mathbb{Q}$ are therefore $\mathbb{Q}_\infty = \mathbb{R}$ and the p-adic fields $\mathbb{Q}_p$.

Similarly, by [25, Theorem 4-30 (ii)] every place on $\mathbb{F}_p(t)$ is either the infinite place ∞ defined by

$$\left|\frac{f}{g}\right|_\infty = p^{\deg f - \deg g}, \text{ where } f, g \in \mathbb{F}_p[t],$$

or the place r defined for an irreducible polynomial $r \in \mathbb{F}_p[t]$ by

$$\left|\frac{f}{g} r^k\right|_r = q^{-k}, \text{ where } f, g \in \mathbb{F}_p[t],\ r \nmid fg, \text{ and } q = p^{\deg r}. \tag{5.1}$$

In this case the infinite place ∞ is determined by some choice of transcendental element in $\mathbb{F}_p(t)$, and we choose this element to be t. Then $|\cdot|_\infty$ is defined by (5.1) where $r = 1/t$, and so we say that $1/t$ is the infinite prime in $\mathbb{F}_p(t)$. The completion $\mathbb{F}_p(t)_t$ of $\mathbb{F}_p(t)$ with respect to the place defined by using $r = t$ in (5.1) is isomorphic to the field $\mathbb{F}_p(\!(t)\!)$ of Laurent series in t defined by

$$\mathbb{F}_p(\!(t)\!) = \Big\{ \sum_{j=n}^{\infty} a_j t^j : n \in \mathbb{Z}, a_j \in \mathbb{F}_p \Big\}.$$

If $r \in \mathbb{F}_p[t]$ is irreducible, then $\mathbb{F}_p(t)_r$ is isomorphic to $\mathbb{F}_q(\!(u)\!)$, where $q = p^{\deg r}$, while $\mathbb{F}_p(t)_\infty = \mathbb{F}_p(t)_{t^{-1}} \cong \mathbb{F}_p(\!(t^{-1})\!)$.

Let $\mathbb{K}$ be a global field, and let $\mathbb{F}$ be the field $\mathbb{Q}$ or $\mathbb{F}_p(t)$ according to the characteristic of $\mathbb{K}$. Let w be a place on $\mathbb{K}$ and let $|\cdot|$ be an absolute value from w. The restriction of $|\cdot|$ to $\mathbb{F}$ defines an absolute value and therefore a place v for $\mathbb{F}$. We say that w *lies above* v. For each place on $\mathbb{F}$, there is at least one but only finitely many places lying above it. We put

$$\mathcal{P}_\infty(\mathbb{K}) = \{w \in \mathcal{P}(\mathbb{K}) : w \text{ lies above } \infty\},$$

and call elements of $\mathcal{P}_\infty(\mathbb{K})$ the *infinite places* of $\mathbb{K}$. We also define $\mathcal{P}_0(\mathbb{K}) = \mathcal{P}(\mathbb{K}) \smallsetminus \mathcal{P}_\infty(\mathbb{K})$, whose elements are the *finite places* of $\mathbb{K}$. If $\operatorname{char}\mathbb{K} = 0$ then a place w lies above ∞ if and only if the corresponding local field $\mathbb{K}_w$ is isomorphic to $\mathbb{R}$ or $\mathbb{C}$, or, equivalently, w is archimedian. Note that when $\operatorname{char}\mathbb{K} > 0$, all places, including ∞, are nonarchimedian.

Every local field is locally compact and nondiscrete. In fact, the classification theorem [25, Thm. 4-12] shows that every nondiscrete locally compact field is isomorphic to either a finite extension of $\mathbb{R}$, or of $\mathbb{Q}_p$, or of $\mathbb{F}_p(\!(t)\!)$. Thus the class of nondiscrete locally compact fields coincides with that of local fields.

Therefore a local field $\Bbbk$ has a Haar measure $\mu_\Bbbk$. For $0 \neq a \in \Bbbk$ the automorphism $x \mapsto ax$ multiplies Haar measure by a fixed number denoted $\operatorname{mod}_\Bbbk(a)$, so that $\mu_\Bbbk(aE) = \operatorname{mod}_\Bbbk(a)\mu_\Bbbk(E)$ for every compact set $E \subset \Bbbk$.

We define $\operatorname{mod}_{\Bbbk}(0) = 0$. It turns out that $|a|_{\Bbbk} = \operatorname{mod}_{\Bbbk}(a)$ is an absolute value on $\Bbbk$, which is the one we will always use. This choice agrees with the absolute values on $\mathbb{Q}$ and $\mathbb{F}_p(t)$ defined above. It also provides the correct normalization of absolute values for the *product formula for global fields* $\mathbb{K}$, which asserts that

$$\prod_{v\in\mathcal{P}(\mathbb{K})} |a|_v = 1 \quad \text{for every } a \in \mathbb{K}^{\times}. \tag{5.2}$$

Let $\mathbb{K}$ be a global field and $\mathcal{P}(\mathbb{K})$ be its set of places. Define the *adele group* $\mathbb{A}_{\mathbb{K}}$ of $\mathbb{K}$ to be

$$\mathbb{A}_{\mathbb{K}} = \Bigl\{(a_v) \in \prod_{v\in\mathcal{P}(\mathbb{K})} \mathbb{K}_v : |a_v|_v \leqslant 1 \text{ for almost every } v \in \mathcal{P}_0(\mathbb{K})\Bigr\}.$$

There is a restricted direct product topology on $\mathbb{A}_{\mathbb{K}}$ making it a locally compact group with coordinate-wise operations [25, Sec. 5.1]. The *diagonal embedding* $i\colon \mathbb{K} \to \mathbb{A}_{\mathbb{K}}$ is defined by $i(a)_v = a$ for all $v \in \mathcal{P}(\mathbb{K})$. It turns out that $i(\mathbb{K})$ is discrete and cocompact in $\mathbb{A}_{\mathbb{K}}$ [25, Thm. 5-11]. This will be the key to using adeles to determine the group $X_{R_d/\mathfrak{p}}$.

6. Structure theorem for prime actions

In this section we show that a prime action $\alpha_{R_d/\mathfrak{p}}$ of entropy rank one is algebraically conjugate to a diagonal action β on a finite product of local fields modulo a β-invariant discrete cocompact subgroup. This structure results from the crucial observation that the quotient field $\mathbb{K}$ of $R_d/\mathfrak{p}$ is a global field, and then applying the adelic machinery described in the previous section to the finite set $\mathcal{S}_{\mathfrak{p}}$ of places v on $\mathbb{K}$ for which $R_d/\mathfrak{p}$ is unbounded in $\mathbb{K}_v$. The required conjugacy from $\alpha_{R_d/\mathfrak{p}}$ to β is then dual to the diagonal embedding $R_d/\mathfrak{p} \to \prod_{v\in\mathcal{S}_{\mathfrak{p}}} \mathbb{K}_v$, and we invoke self-duality for local fields to complete the description. Several examples show how this conjugacy works for actions on tori and solenoids. In positive characteristic, it allows us to locally decompose examples like Ledrappier's into a direct product of Laurent power series local fields over a finite field, leading to explicit "eigenspaces" for such actions.

We first relate entropy rank one, Krull dimension, and global fields.

Proposition 6.1. *Let $\mathfrak{p}$ be a prime ideal in R_d and $\mathbb{K}$ be the quotient field of $R_d/\mathfrak{p}$.*

1. $\operatorname{kdim} R_d/\mathfrak{p} = 0$ *iff $\mathfrak{p}$ is a maximal ideal in R_d iff $R_d/\mathfrak{p}$ is a finite field.*
2. $\operatorname{kdim} R_d/\mathfrak{p} = 1$ *iff $\mathbb{K}$ is a global field.*
3. *$\alpha_{R_d/\mathfrak{p}}$ has entropy rank one iff* $\operatorname{kdim} R_d/\mathfrak{p} = 0$ *or* 1.

Proof. (1) The definitions show that $\operatorname{kdim} R_d/\mathfrak{p} = 0$ iff $\mathfrak{p}$ is a maximal ideal in R_d iff $R_d/\mathfrak{p}$ is a field. Any field that is finitely generated as an algebra over $\mathbb{Z}$ is finite, and any finite integral domain is a field.

(2) First consider the case $\operatorname{char} R_d/\mathfrak{p} = 0$. By Proposition 5.1, $\operatorname{kdim} R_d/\mathfrak{p} = 1$ iff $\operatorname{trdeg}_{\mathbb{Q}} \mathbb{K} = 0$ iff $\mathbb{K}$ is global. Now assume that $\operatorname{char} R_d/\mathfrak{p} = p > 0$. Then by Proposition 5.1, $\operatorname{kdim} R_d/\mathfrak{p} = 1$ iff $\operatorname{trdeg}_{\mathbb{F}_p} \mathbb{K} = 1$, and Noether normalization shows that the latter is equivalent to the existence of a transcendental element $u \in \mathbb{K}$ such that $\mathbb{K}$ is a finite integral extension of $\mathbb{F}_p(u)$, which is the same as $\mathbb{K}$ being global.

(3) The proof that $\operatorname{kdim} R_d/\mathfrak{p} = 1$ implies that $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one follows immediately from the adelic framework we will develop. However, we give here a direct proof of both directions.

First consider the case $\operatorname{char} R_d/\mathfrak{p} = p > 0$. Then $\operatorname{kdim} R_d/\mathfrak{p} = \operatorname{trdeg}_{\mathbb{F}_p} \mathbb{K}$. If $\operatorname{kdim} R_d/\mathfrak{p} \geqslant 2$, there are distinct monomials $\mathbf{u^m}$ and $\mathbf{u^n}$ in $R_d/\mathfrak{p}$ that are algebraically independent. It follows that the subring N that they generate, considered as a module over $\mathbb{F}_p[\mathbf{u}^{\pm\mathbf{n}}]$, is isomorphic to a direct sum of countably many copies of $\mathbb{F}_p[\mathbf{u}^{\pm\mathbf{n}}]$. Thus $\alpha_N^{\mathbf{n}}$ is the product of infinitely many p-shifts, and so $\mathsf{h}(\alpha_N^{\mathbf{n}}) = \infty$. Since X_N is a quotient of $X_{R_d/\mathfrak{p}}$, we see that $\mathsf{h}(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}) = \infty$, and so $\alpha_{R_d/\mathfrak{p}}$ does not have entropy rank one.

Continuing with the case $\operatorname{char} R_d/\mathfrak{p} = p > 0$, suppose that $\operatorname{trdeg}_{\mathbb{F}_p} \mathbb{K} \leqslant 1$. Fix $\mathbf{n} \in \mathbb{Z}^d$. If $\mathbf{u^n}$ is algebraic in $\mathbb{K}$ over $\mathbb{F}_p$, then $R_d/\mathfrak{p}$ is an increasing union of finite subgroups, each invariant under multiplication by $\mathbf{u^n}$. Therefore $X_{R_d/\mathfrak{p}}$ is the inverse limit of finite quotients by subgroups invariant under $\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}$, so that $\mathsf{h}(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}) = 0$. Next, suppose that $\mathbf{u^n}$ is transcendental in $\mathbb{K}$ over $\mathbb{F}_p$. Since $\operatorname{trdeg}_{\mathbb{F}_p} \mathbb{K} \leqslant 1$, the image of every monomial $\mathbf{u^k}$ in $\mathbb{K}$ is algebraic over the subfield $\mathbb{F}_p(\mathbf{u^n})$. Hence $\mathbb{K}$ has finite dimension m over $\mathbb{F}_p(\mathbf{u^n})$. Pick $f_1, \ldots, f_m \in R_d/\mathfrak{p}$ that are linearly independent over $\mathbb{F}_p(\mathbf{u^n})$, and let N be the $\mathbb{F}_p[\mathbf{u}^{\pm 1}]$-submodule of $\mathbb{K}$ that they generate. Then $\alpha_N^{\mathbf{n}}$ is isomorphic to a product of m copies of the full p-shift, so that $\mathsf{h}(\alpha_N^{\mathbf{n}}) = m \cdot p$. Now $\mathbb{K}$ is the increasing union of multiples $N_j = a_j N$ of N with finite quotients N_{j+1}/N_j, so that $\mathsf{h}(\alpha_{\mathbb{K}}^{\mathbf{n}}) = m \cdot p$. Hence $\mathsf{h}(\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}) \leqslant \mathsf{h}(\alpha_{\mathbb{K}}^{\mathbf{n}}) < \infty$, showing that $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one.

Next, consider the case $\operatorname{char} R_d/\mathfrak{p} = 0$, so that by Proposition 5.1 we have $\operatorname{trdeg}_{\mathbb{Q}} \mathbb{K} = \operatorname{kdim} R_d/\mathfrak{p} - 1$. First suppose that $\operatorname{kdim} R_d/\mathfrak{p} \leqslant 1$. The case $\operatorname{kdim} R_d/\mathfrak{p} = 0$ cannot arise in characteristic zero, so assume that $\operatorname{kdim} R_d/\mathfrak{p} = 1$. Then $\mathbb{K}$ is algebraic over $\mathbb{Q}$. The images c_j of u_j in the quotient field $\mathbb{K}$ can therefore be considered as algebraic numbers,

$$R_d/\mathfrak{p} \cong \mathbb{Z}[c_1^{\pm 1}, \ldots, c_d^{\pm 1}] \subset \mathbb{K} = \mathbb{Q}(c_1, \ldots, c_d),$$

and multiplication by $\mathbf{u^n}$ corresponds to multiplication by $\mathbf{c^n} = c_1^{n_1} \cdots c_d^{n_d}$ in $\mathbb{K}$. Let $k = \dim_{\mathbb{Q}} \mathbb{K}$ and choose a basis for $\mathbb{K}$ over $\mathbb{Q}$. Multiplication on $\mathbb{K}$ by $\mathbf{c^n}$ has a rational matrix A with respect to this basis. Then $\alpha_{R_d/\mathfrak{p}}^{\mathbf{n}}$ is a quotient of the dual $\widehat{A}$ on $\widehat{\mathbb{K}} \cong \widehat{\mathbb{Q}}^k$, and $\mathsf{h}(\widehat{A}) < \infty$ by [20]. Thus $\alpha_{R_d/\mathfrak{p}}$ has entropy rank one.

Finally, suppose that $\operatorname{kdim} R_d/\mathfrak{p} \geqslant 2$. Then some monomial $\mathbf{u^n}$ must be transcendental. Hence $\mathbb{Z}[\mathbf{u}^{\pm\mathbf{n}}] \subset R_d/\mathfrak{p}$ is invariant under $\widehat{\alpha}_{R_d/\mathfrak{p}}^{\mathbf{n}}$, which means via duality that $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ has a quotient that is the full shift on $\mathbb{T}^{\mathbb{Z}}$, and so $\mathsf{h}(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}) = \infty$. □

Let $\mathfrak{p}$ be a prime ideal in R_d such that $\operatorname{kdim} R_d/\mathfrak{p} = 1$. Then by the above the quotient field $\mathbb{K}$ of $R_d/\mathfrak{p}$ is a global field, and let $\mathcal{P}(\mathbb{K})$ denote its set of places. Recall that for each $v \in \mathcal{P}(\mathbb{K})$ we choose the absolute value $|\cdot|_v$ in v defined by $|a|_v = \operatorname{mod}_{\mathbb{K}_v}(a)$. For a given $v \in \mathcal{P}(\mathbb{K})$, we say that $R_d/\mathfrak{p}$ is *v-unbounded* if there is an $a \in R_d/\mathfrak{p}$ for which $|a|_v > 1$. Put

$$\mathcal{S}_{\mathfrak{p}} = \{v \in \mathcal{P}(\mathbb{K}) : R_d/\mathfrak{p} \text{ is } v\text{-unbounded}\}.$$

Since $R_d/\mathfrak{p}$ is finitely generated, $\mathcal{S}_{\mathfrak{p}}$ is a finite subset of $\mathcal{P}(\mathbb{K})$, and it always contains $\mathcal{P}_\infty(\mathbb{K})$.

Define the *adele group of $R_d/\mathfrak{p}$* to be

$$\mathbb{A}_{R_d/\mathfrak{p}} = \prod_{v \in \mathcal{S}_{\mathfrak{p}}} \mathbb{K}_v,$$

and the diagonal embedding $i\colon \mathbb{K} \to \mathbb{A}_{R_d/\mathfrak{p}}$ by $i(a)_v = a$ for all $a \in \mathbb{K}$ and $v \in \mathcal{S}_{\mathfrak{p}}$. Our goal is to show that $i(R_d/\mathfrak{p})$ is discrete and cocompact in $\mathbb{A}_{R_d/\mathfrak{p}}$.

To do this, introduce

$$T_{\mathfrak{p}} = \{a \in \mathbb{K} : |a|_v \leqslant 1 \text{ for all } v \in \mathcal{P}(\mathbb{K}) \smallsetminus \mathcal{S}_{\mathfrak{p}}\}, \tag{6.1}$$

sometimes called the ring of $\mathcal{S}_{\mathfrak{p}}$-units in $\mathbb{K}$.

Proposition 6.2. *Let $\mathfrak{p}$ be a prime ideal in R_d such that* $\operatorname{kdim} R_d/\mathfrak{p} = 1$*, let $\mathbb{K}$ be the quotient field of $R_d/\mathfrak{p}$, and define the ring $T_{\mathfrak{p}}$ of $\mathcal{S}_{\mathfrak{p}}$-units in $\mathbb{K}$ by* (6.1)*.*

(1) *$T_{\mathfrak{p}}$ is the integral closure of $R_d/\mathfrak{p}$ in $\mathbb{K}$.*
(2) *$T_{\mathfrak{p}}$ is finitely generated over $R_d/\mathfrak{p}$.*
(3) *$i(T_{\mathfrak{p}})$ is discrete and cocompact in $\mathbb{A}_{R_d/\mathfrak{p}}$, and therefore so is $i(R_d/\mathfrak{p})$.*

Proof. Statement (1) follows from the characterization of integral closure of a domain in terms of valuations [10, Thm. 10.8]. Statement (2) is a consequence of the finiteness of the integral closure for affine domains (see [5, Cor. 13.13], which also applies to finitely generated algebras over $\mathbb{Z}$).

Let $j\colon \mathbb{K} \to \mathbb{A}_{\mathbb{K}}$ denote the diagonal embedding of $\mathbb{K}$ into its adele group, and retain the notation $i\colon R_d/\mathfrak{p} \to \alpha_{R_d/\mathfrak{p}}$ for the restricted diagonal embedding defined above. By [25, Thm. 5-11], $j(\mathbb{K})$ is discrete and cocompact in $\mathbb{A}_{\mathbb{K}}$. Hence there is a compact set $C \subset \mathbb{A}_{\mathbb{K}}$ such that $C + j(\mathbb{K}) = \mathbb{A}_{\mathbb{K}}$. Let $B_{\mathbb{K}_v}(r)$ denote the ball of radius r around 0 in $\mathbb{K}_v$. By definition of the restricted product topology on $\mathbb{A}_{\mathbb{K}}$, there is a finite set $\mathcal{F} \supset \mathcal{P}_\infty(\mathbb{K})$ of places and an $r > 0$ such that

$$C \subset \prod_{v \in \mathcal{F}} B_{\mathbb{K}_v}(r) \times \prod_{v \in \mathcal{P}(\mathbb{K}) \smallsetminus \mathcal{F}} B_{\mathbb{K}_v}(1).$$

The Approximation Theorem [25, Thm. 5-8] shows that there is an $a \in \mathbb{K}^\times$ such that $|a|_v < 1/r$ for all $v \in \mathcal{F} \smallsetminus \mathcal{S}_{\mathfrak{p}}$ and $|a|_v \leqslant 1$ for all $v \in \mathcal{P}(\mathbb{K}) \smallsetminus (\mathcal{F} \cup \mathcal{S}_{\mathfrak{p}})$.

Hence there is an $s > 0$ such that

$$aC \subset \prod_{v\in\mathcal{S}_\mathfrak{p}} B_{\mathbb{K}_v}(s) \times \prod_{v\in\mathcal{P}(\mathbb{K})\smallsetminus\mathcal{S}_\mathfrak{p}} B_{\mathbb{K}_v}(1),$$

and clearly $aC + j(\mathbb{K}) = \mathbb{A}_\mathbb{K}$. Put $D = \prod_{v\in\mathbb{S}_p} B_{\mathbb{K}_v}(s)$, which is obviously compact. We claim it also has the property that $i(T_\mathfrak{p}) + D = \mathbb{A}_{R_d/\mathfrak{p}}$. For suppose that $x = (x_v)_{v\in\mathcal{S}_\mathfrak{p}} \in \mathbb{A}_{R_d/\mathfrak{p}}$. Extend x to an element $y \in \mathbb{A}_\mathbb{K}$ by putting $y_v = x_v$ for all $v \in \mathcal{S}_\mathfrak{p}$ and $y_v = 0$ for all $v \notin \mathcal{S}_\mathfrak{p}$. Since $aC + j(\mathbb{K}) = \mathbb{A}_\mathbb{K}$, there exists an element $b \in \mathbb{K}$ such that $y - j(b) \in aC$. Then $|b|_v \leqslant 1$ for all $v \notin \mathcal{S}_\mathfrak{p}$, so that $b \in T_\mathfrak{p}$. Hence $x - i(b) \in D$, showing that $i(T_\mathfrak{p}) + D = \mathbb{A}_{R_d/\mathfrak{p}}$, as claimed.

Finally, since $T_\mathfrak{p}$ is finitely generated over $R_d/\mathfrak{p}$, there is an $b \in \mathbb{K}^\times$ for which $bT_\mathfrak{p} \subset R_d/\mathfrak{p}$. Thus $i(R_d/\mathfrak{p})$ is trapped between the two cocompact discrete subgroups $i(bT_\mathfrak{p})$ and $i(T_\mathfrak{p})$, so itself must be discrete and cocompact. □

We next describe the self-duality of local fields.

Proposition 6.3. *Let $\Bbbk$ be a local field, and for $a \in \Bbbk$ define $\phi_a \colon \Bbbk \to \Bbbk$ by $\phi_a(x) = ax$. There is a topological isomorphism identifying $\widehat{\Bbbk}$ with $\Bbbk$ such that the dual map $\widehat{\phi_a}$ corresponds to ϕ_a.*

Proof. Fix a nonzero character $\chi \in \widehat{\Bbbk}$. For $b \in \Bbbk$ define the character χ_b by $\chi_b(a) = \chi(ba)$. Then the correspondence $b \leftrightarrow \chi_b$ is a topological isomorphism between $\Bbbk$ and $\widehat{\Bbbk}$ (see [31, Thm. II.5.3]). Clearly $\chi_b(\phi_a(x)) = \chi(bax) = \chi_{\phi_a(b)}(x)$, so that $\widehat{\phi_a}$ is identified with ϕ_a. □

Using Proposition 6.1, the following result implies Theorem 2.1, our main result on the structure of prime actions of entropy rank one.

Theorem 6.4. *Let $\mathfrak{p}$ be a prime ideal of R_d such that* kdim $R_d/\mathfrak{p} = 1$. *Then there is a diagonal action β on the adele group $\mathbb{A}_{R_d/\mathfrak{p}}$ and a β-invariant discrete cocompact subgroup $\Lambda \subset \mathbb{A}_{R_d/\mathfrak{p}}$, such that $\alpha_{R_d/\mathfrak{p}}$ is algebraically conjugate to the quotient action of β on $\mathbb{A}_{R_d/\mathfrak{p}}/\Lambda$.*

Proof. Abbreviate $\mathbb{A}_{R_d/\mathfrak{p}}$ by $\mathbb{A}$. By Proposition 6.2, the image $i(R_d/\mathfrak{p}) \subset \mathbb{A}$ is discrete and cocompact. Let Λ be the annihilator of $i(R_d/\mathfrak{p})$ in $\mathbb{A}$. Then Λ is also discrete and cocompact. The dual of the inclusion $i\colon R_d/\mathfrak{p} \to \mathbb{A}$ is the quotient $\widehat{\mathbb{A}} \to \widehat{\mathbb{A}}/\Lambda \cong \prod_{v\in\mathcal{S}_\mathfrak{p}} \widehat{\mathbb{K}_v}/\Lambda = X_{R_d/\mathfrak{p}}$. Finally, by Proposition 6.3, $\widehat{\mathbb{K}_v}$ is identified with $\mathbb{K}_v$, and under this identification $\alpha_{R_d/\mathfrak{p}}$ corresponds to a diagonal action β on $\widehat{\mathbb{A}}/\Lambda$. □

Example 6.5. (*Single toral automorphism*) Let $d = 1$ and $\mathfrak{p} = \langle u_1^2 - u_1 - 1\rangle$. The roots of the generator for $\mathfrak{p}$ are $\xi = (1+\sqrt{5})/2$ and $\xi' = (1-\sqrt{5})/2$. Hence $R_1/\mathfrak{p} \cong \mathbb{Z}[\xi]$, and its quotient field is $\mathbb{K} = \mathbb{Q}(\sqrt{5})$. Since ξ is an algebraic unit, $R_1/\mathfrak{p}$ is v-bounded for all finite places on $\mathbb{K}$. There are exactly two infinite places ∞_1 and ∞_2 on $\mathbb{K}$, corresponding to the two real embeddings of $\mathbb{K}$. These are given by $|a+b\sqrt{5}|_{\infty_1} = |a+b\sqrt{5}|_{\mathbb{R}}$ and $|a+b\sqrt{5}|_{\infty_2} = |a-b\sqrt{5}|_{\mathbb{R}}$, where $a,b \in \mathbb{Q}$ and $|\cdot|_{\mathbb{R}}$ is the usual absolute value. Thus here $\mathcal{S}_{\mathfrak{p}} = \{\infty_1, \infty_2\}$, and so

$$\mathbb{A}_{R_1/\mathfrak{p}} = \mathbb{K}_{\infty_1} \times \mathbb{K}_{\infty_2} \cong \mathbb{R}^2.$$

The diagonal embedding of $R_1/\mathfrak{p}$ into $\mathbb{A}_{R_1/\mathfrak{p}}$ has image corresponding to the lattice Λ in $\mathbb{R}^2$ generated by $(1,1)$ and (ξ,ξ'). Thus $\alpha_{R_1/\mathfrak{p}}$ corresponds to the $\mathbb{Z}$-action on the torus $\mathbb{A}_{R_1/\mathfrak{p}}/\Lambda \cong \mathbb{T}^2$ generated by the matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$.

Example 6.6. (*Commuting toral automorphisms*) Let $d=2$ and $\mathfrak{p} = \langle u_1^2 - 2u_1 - 1, u_2^2 - 4u_2 + 1\rangle$. The roots of the first polynomial are $\xi = 1+\sqrt{2}$ and $\xi' = 1-\sqrt{2}$, and those of the second are $\eta = 2+\sqrt{3}$ and $\eta' = 2-\sqrt{3}$. All of these are algebraic units. Then $R_2/\mathfrak{p} \cong \mathbb{Z}[\xi,\eta] = \mathbb{Z}[\sqrt{2},\sqrt{3}]$, and $\mathbb{K} = \mathbb{Q}(\sqrt{2},\sqrt{3})$. As in the previous example, $R_2/\mathfrak{p}$ is v-bounded for all finite places v on $\mathbb{K}$. There are exactly four infinite places ∞_σ on $\mathbb{K}$, one for each element σ in the Galois group G of $\mathbb{K}$ over $\mathbb{Q}$, defined by $|a|_{\infty_\sigma} = |\sigma(a)|_{\mathbb{R}}$. Hence $\mathcal{S}_{\mathfrak{p}} = \{\infty_\sigma : \sigma \in G\}$, and

$$\mathbb{A}_{R_2/\mathfrak{p}} = \prod_{\sigma \in G} \mathbb{K}_{\infty_\sigma} \cong \mathbb{R}^4.$$

Then $i(R_2/\mathfrak{p})$ is a lattice in $\mathbb{A}_{R_2/\mathfrak{p}}$ and the quotient is isomorphic to $\mathbb{T}^4$. Using this lattice the $\mathbb{Z}^2$-action $\alpha_{R_2/\mathfrak{p}}$ is generated by the toral automorphisms A and B given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 4 \end{bmatrix}.$$

Example 6.7. (*Commuting solenoidal automorphisms*) Let $d=2$ and $\mathfrak{p} = \langle u_1 - 2, u_2 - 3\rangle$. Then $R_2/\mathfrak{p} \cong \mathbb{Z}[1/6]$, $\mathbb{K} = \mathbb{Q}$, and $\alpha_{R_2/\mathfrak{p}}$ is the natural extension of the $\mathbb{N}^2$-action on $\mathbb{T}$ generated by multiplication by 2 and by 3. Hence $R_2/\mathfrak{p}$ is unbounded exactly at the places 2, 3, and ∞ on $\mathbb{Q}$, so that $\mathcal{S}_{\mathfrak{p}} = \{2,3,\infty\}$ and $\mathbb{A}_{R_2/\mathfrak{p}} = \mathbb{Q}_2 \times \mathbb{Q}_3 \times \mathbb{R}$. Then $X_{R_2/\mathfrak{p}}$ is the quotient of $\mathbb{A}_{R_2/\mathfrak{p}}$ modulo the invariant lattice $i(R_2/\mathfrak{p})$, and so is locally the product of the 2-adics, the 3-adics, and the reals.

This local product structure for solenoids was first developed in [20] to explain Yuzvinsky's formula for the entropy of solenoidal automorphisms. Shortly thereafter, Katok and Spatzier [14] used these ideas to, among other things, give a geometric understanding of Rudolph's result [26] about measures on $\mathbb{T}$ simultaneously invariant under $\times 2$ and $\times 3$.

Example 6.8. (*Ledrappier's example*) Our adelic viewpoint allows us to take apart Ledrappier's example to see what makes it tick. Let $d=2$ and $\mathfrak{p} = \langle 2, 1+u_1+u_2\rangle$. Then

$$X_{R_2/\mathfrak{p}} \cong \{x \in (\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2} : x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0 \text{ for all } i,j \in \mathbb{Z}\}. \tag{6.2}$$

Here char $R_2/\mathfrak{p} = 2$. Then $R_2/\mathfrak{p} \cong \mathbb{F}_2[t^{\pm 1}, (1+t)^{-1}]$, where the isomorphism is defined by $u_1 \mapsto t$ and $u_2 \mapsto 1+t$. The quotient field is $\mathbb{K} = \mathbb{F}_2(t)$. The only places on $\mathbb{K}$ where $R_2/\mathfrak{p}$ is unbounded are the finite places corresponding to the polynomials t and $1+t$, together with the infinite place corresponding to t^{-1}, so that $\mathcal{S}_{\mathfrak{p}} = \{t, 1+t, t^{-1}\}$. Thus

$$\mathbb{A}_{R_2/\mathfrak{p}} = \mathbb{F}_2(t)_t \times \mathbb{F}_2(t)_{1+t} \times \mathbb{F}_2(t)_{t^{-1}} \cong \mathbb{F}_2(\!(t)\!) \times \mathbb{F}_2(\!(1+t)\!) \times \mathbb{F}_2(\!(t^{-1})\!).$$

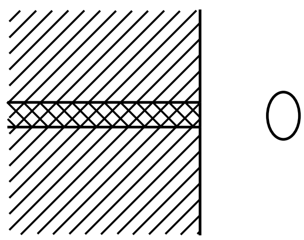
Each of these three completions of $\mathbb{F}_2(t)$ induces a subgroup of $X_{R_2/\mathfrak{p}}$. Let us first describe this subgroup explicitly for the place t. Since we are in characteristic 2, it is convenient to write characters on $\mathbb{F}_2(\!(t)\!)$ additively with values in $\mathbb{F}_2$, consistent with the isomorphism in (6.2). Define the basic character $\chi \in \mathbb{F}_2(\!(t)\!)\widehat{\ }$ by $\chi\Big(\sum_{j=-n}^{\infty} a_j t^j\Big) = a_0 \in \mathbb{F}_2$. For $f \in \mathbb{F}_2(\!(t)\!)$ define $\chi_f \in \mathbb{F}_2(\!(t)\!)\widehat{\ }$by $\chi_f(g) = \chi(fg)$. As in Proposition 6.3, the correspondence $f \leftrightarrow \chi_f$ identifies $\mathbb{F}_2(\!(t)\!)$ with its dual group. Thus each $f \in \mathbb{F}_2(\!(t)\!)$ corresponds to a point we call $x_f \in X_{R_2/\mathfrak{p}}$, defined by

$$(x_f)_{(m,n)} = \chi_f(u_1^m u_2^n) = \chi_f(t^m(1+t)^n) = \chi(t^m(1+t)^n f).$$

Note, for example, that when $n < 0$ we use the Laurent expansion $(1+t)^{-1} = 1 + t + t^2 + t^3 + \dots$ in $\mathbb{F}_2(\!(t)\!)$ when defining x_f. Explicitly, if $f = a_0 + a_1 t + a_2 t^2 + \dots$, a portion of the corresponding point x_f is shown in Figure 1(a). In Figure 1(b) we depict the overall structure of such points. There is a half-space of 0's on the right, bordered by a line of 1's, and the double-hatched half-line of coordinates, corresponding to the coefficients of f, determines the rest of the point in the single-hatched half-space.

$a_0+2a_1+a_2$	$2a_0+a_1$	a_0	0	0	0
$a_1 \quad a_2$	a_0+a_1	a_0	0	0	0
a_2	a_1	a_0	0	0	0
$a_0+a_1+a_2$	a_0+a_1	a_0	0	0	0
$3a_0+2a_1+a_2$	$2a_0+a_1$	a_0	0	0	0

(a)

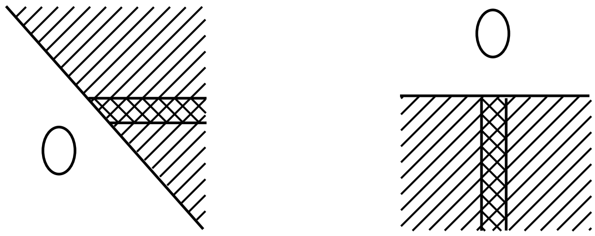


(b)

FIGURE 1. Points corresponding to the completion at the place t

Carrying out a similar analysis for the places $1+t$ and t^{-1} yields points in $X_{R_2/\mathfrak{p}}$ having structures depicted in Figure 2(a) and (b), respectively. For each there is a half-space of 0's, bordered by a line of 1's, and the double-hatched half-line of coordinates determines the rest of the coordinates.

This analysis shows that Ledrappier's example has the same formal structure as a $\mathbb{Z}^2$-action by automorphisms of $\mathbb{T}^3$, except that the local field $\mathbb{R}$ has been replaced by three isomorphic copies of the local field $\mathbb{F}_2(\!(t)\!)$. For example, let X_t denote the image of $\mathbb{F}_2(\!(t)\!) \times 0 \times 0 \subset \mathbb{A}_{R_2/\mathfrak{p}}$ in $\mathbb{A}_{R_2/\mathfrak{p}}/\Lambda$. Similarly



(a) (b)

FIGURE 2. Points corresponding to the completions at the places $1+t$ and t^{-1}

define X_{1+t} and $X_{t^{-1}}$ using the other two factors. Then $X_{R_2/\mathfrak{p}}$ is locally the direct product of these three subgroups, which play the role of "eigenspaces" for the action. Explicitly, suppose that $x \in X_{R_2/\mathfrak{p}}$ is close to 0, so that x contains a large triangle of 0's with the origin well within its interior. To find the projection of x to X_t, use the coordinates $x_{-n,0}$ of x for $n \geqslant 0$ together with the half-space of 0's bordered by 1's matching the left-hand boundary of the triangle of 0's, to construct a point $\pi_t(x) \in X_t$ having the form shown in Figure 1(b). Construct similar projections $\pi_{1+t}(x) \in X_{1+t}$ and $\pi_{t^{-1}}(x) \in X_{t^{-1}}$. A simple verification shows that

$$x = \pi_t(x) + \pi_{1+t}(x) + \pi_{t^{-1}}(x)$$

is the local product decomposition of x. In addition, we can easily recover the three directional homoclinic groups described in Example 9.5 of [6] as the three intersections $X_t \cap X_{1+t}$, $X_t \cap X_{t^{-1}}$, and $X_{1+t} \cap X_{t^{-1}}$.

Example 6.9. (*Action defined by a point*) Let $\mathbb{K}$ be a global field, and let $\mathbf{c} = (c_1, \dots, c_d) \in (\mathbb{K}^\times)^d$. Define the *evaluation map* $\eta_{\mathbf{c}} \colon R_d \to \mathbb{K}$ by $\eta_{\mathbf{c}}(f) = f(\mathbf{c})$. The image of $\eta_{\mathbf{c}}$ is the subring of $\mathbb{Z}[c_1^{\pm 1}, \dots, c_d^{\pm 1}]$ of $\mathbb{K}$. We denote the kernel of $\eta_{\mathbf{c}}$ by $\mathfrak{p}_{\mathbf{c}}$, which is prime since $\eta_{\mathbf{c}}$ maps to a field. Here

$$\mathcal{P}_{\mathfrak{p}_{\mathbf{c}}} = \{w \in \mathcal{P}_0(\mathbb{K}) : |c_j|_w \neq 1 \text{ for some } j\} \cup \mathcal{P}_\infty(\mathbb{K})$$

When $\operatorname{char} R_d/\mathfrak{p}_{\mathbf{c}} = 0$, the adelic structure has already been used to provide a description of the action (see [7] or [28, Section II.7]).

7. Characterizations of entropy rank one

This section contains a number of different characterizations of entropy rank one for prime actions.

We begin with the connected case. Here $\overline{\mathbb{Q}}$ denotes the algebraic closure of $\mathbb{Q}$.

Theorem 7.1. *Suppose that* $\mathfrak{p}$ *is a prime ideal in* R_d *with* $\operatorname{char} R_d/\mathfrak{p} = 0$. *Then the following are equivalent.*

(1) $\alpha_{R_d/\mathfrak{p}}$ *has entropy rank one.*

(2) $\alpha_{R_d/\mathfrak{p}}$ *is irreducible.*
(3) $\dim X_{R_d/\mathfrak{p}} < \infty$.
(4) *There exists a finite product* $\mathbb{A} = \Bbbk^{(1)} \times \cdots \times \Bbbk^{(m)}$ *of local fields* $\Bbbk^{(j)}$ *of characteristic zero, a diagonal* $\mathbb{Z}^d$*-action* β *on* $\mathbb{A}$*, and a* β*-invariant cocompact discrete subgroup* Λ *of* $\mathbb{A}$*, such that* $\alpha_{R_d/\mathfrak{p}}$ *is conjugate to the action of* β *on* $\mathbb{A}/\Lambda$.
(5) $\operatorname{kdim} R_d/\mathfrak{p} = 1$.
(6) $\operatorname{trdeg}_{\mathbb{Q}} R_d/\mathfrak{p} = 0$.
(7) *The quotient field* $\mathbb{K}$ *of* $R_d/\mathfrak{p}$ *is global.*
(8) *The vector space* $R_d/\mathfrak{p} \otimes \mathbb{Q}$ *is finite dimensional over* $\mathbb{Q}$.
(9) *The variety* $V_{\mathbb{C}}(\mathfrak{p})$ *is finite.*
(10) *There exists* $\mathbf{c} \in (\overline{\mathbb{Q}}^{\times})^d$ *such that* $\mathfrak{p} = \mathfrak{p}_{\mathbf{c}} = \{f \in R_d : f(\mathbf{c}) = 0\}$.

If we assume furthermore that $\alpha_{R_d/\mathfrak{p}}$ *is expansive, we have three more equivalent conditions.*

(11) $\alpha_{R_d/\mathfrak{p}}$ *has expansive rank one.*
(12) *There exists a finite union* U *of hyperplanes in* $\mathbb{R}^d$ *such that* $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ *is expansive whenever* $\mathbf{n} \notin U$.
(13) *The set* $\mathsf{N}_{d-1}(\alpha_{R_d/\mathfrak{p}})$ *of nonexpansive hyperplanes is finite.*

Proof. By Proposition 6.1, (1), (5) and (7) are equivalent, and by Proposition 5.1 (6) and (7) are equivalent. Standard algebraic arguments show that (6), (8), (9), and (10) are equivalent. Equivalence of (1) and (2) follows from [7, Thm. 4.4]. By [7, Thm. 3.4], (2) implies (4), which obviously implies (3) since local fields have finite topological dimension. By [28, Cor. 7.4], (3) implies (10), completing the proof that the first ten statements are equivalent.

Assume furthermore that $\alpha_{R_d/\mathfrak{p}}$ is expansive. Equivalence of (12) and (13) follows from [4, Thm. 3.6], and clearly (12) implies (11). By Proposition 4.3, (11) implies (1). Now assume that (4) holds. Let β be the diagonal action on $\mathbb{A}$ having the form

$$\beta^{\mathbf{n}}(a^{(1)}, \dots, a^{(m)}) = (\mathbf{c}^{\mathbf{n}} a^{(1)}, \dots, \mathbf{c}^{\mathbf{n}} a^{(m)}). \tag{7.1}$$

Since $\alpha_{R_d/\mathfrak{p}}$ is assumed expansive, and expansiveness for algebraic actions is determined locally at 0, each vector

$$\mathbf{v}^{(j)} = (\log |c_1|_{\Bbbk^{(j)}}, \dots, \log |c_d|_{\Bbbk^{(j)}}) \neq \mathbf{0}. \tag{7.2}$$

Let $H^{(j)}$ be the orthogonal complement of $\mathbf{v}^{(j)}$ in $\mathbb{R}^d$. Then the union U of the hyperplanes $H^{(1)}, \dots, H^{(m)}$ satisfies (12). □

In the case of positive characteristic some conditions in the previous proposition do not make sense, but expansiveness is guaranteed. The case when $R_d/\mathfrak{p}$ is finite has already been dealt with in Proposition 6.1. In the following we let $\overline{\mathbb{F}_p(t)}$ denote the algebraic closure of $\mathbb{F}_p(t)$.

Theorem 7.2. *Suppose that* $\mathfrak{p}$ *is a prime ideal in* R_d *with* $\operatorname{char} R_d/\mathfrak{p} = p > 0$ *and* $|R_d/\mathfrak{p}| = \infty$. *Then the following are equivalent*

(1) $\alpha_{R_d/\mathfrak{p}}$ *has entropy rank one.*
(2) $\alpha_{R_d/\mathfrak{p}}$ *is irreducible.*
(3) *There exists a finite product* $\mathbb{A} = \Bbbk^{(1)} \times \cdots \times \Bbbk^{(m)}$ *of local fields* $\Bbbk^{(j)}$ *of characteristic* p*, a diagonal* $\mathbb{Z}^d$*-action* β *on* $\mathbb{A}$*, and a* β*-invariant cocompact discrete subgroup* Λ *of* $\mathbb{A}$*, such that* $\alpha_{R_d/\mathfrak{p}}$ *is conjugate to the action of* β *on* $\mathbb{A}/\Lambda$*.*
(4) $\alpha_{R_d/\mathfrak{p}}$ *has expansive rank one.*
(5) $\operatorname{kdim} R_d/\mathfrak{p} = 1$*.*
(6) $\operatorname{trdeg}_{\mathbb{F}_p} R_d/\mathfrak{p} = 1$*.*
(7) *The quotient field* $\mathbb{K}$ *of* $R_d/\mathfrak{p}$ *is global.*
(8) *There exists* $\mathbf{c} \in \big(\overline{\mathbb{F}_p(t)}^{\times}\big)^d$ *such that* $\mathfrak{p} = \{f \in R_d \mid f(\mathbf{c}) = 0\}$*.*
(9) *There exists a finite union* U *of hyperplanes in* $\mathbb{R}^d$ *such that* $\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}}$ *is expansive whenever* $\mathbf{n} \notin U$*.*
(10) *The set* $\mathsf{N}_{d-1}(\alpha_{R_d/\mathfrak{p}})$ *of non-expansive hyperplanes is finite.*

Proof. When $\operatorname{char} R_d/\mathfrak{p} > 0$ we know by [6, Prop. 7.3] that entropy rank, expansive rank, and Krull dimension coincide, so that (1), (4), and (5) are equivalent. Proposition 5.1 and standard algebraic arguments show that (5), (6), (7), and (8) are equivalent. Finite $\alpha_{R_d/\mathfrak{p}}$-invariant subgroups of $X_{R_d/\mathfrak{p}}$ correspond via duality to ideals of finite index in $R_d/\mathfrak{p}$, so that Proposition 5.1 also shows that (2) and (5) are equivalent. Theorem 6.4 shows that (5) implies (3). By [4, Thm. 3.6], (9) and (10) are equivalent, and (10) clearly implies (4). Finally, assume (3). Then since $\alpha_{R_d/\mathfrak{p}}$ is expansive, the diagonal action in (7.1) has nonzero vectors $\mathbf{v}^{(j)}$ defined by (7.2), and the argument that (10) follows is the same as there. □

8. Lyapunov vectors

The dynamical behavior of a toral automorphism is largely determined by the logarithms of the absolute values of its eigenvalues, or its *Lyapunov exponents*. For a $\mathbb{Z}^d$-action generated by d commuting toral automorphisms, we need to know the d Lyapunov exponents in each eigenspace, which together form the components of a *Lyapunov vector* for the eigenspace. Using our adelic machinery, we show that these notions make sense for all Noetherian algebraic $\mathbb{Z}^d$-actions of entropy rank one, and this can be used to easily compute entropy for individual elements of the action.

Before we start, it is convenient to introduce the notion of *list*, which is a collection of elements where multiplicity matters but order does not. The list containing $a_1, \dots, a_n$ is denoted by $\langle a_1, \dots, a_n\rangle$. Thus $\langle 0,1,1\rangle = \langle 1,0,1\rangle \neq \langle 0,1\rangle$. The union of lists is defined in the obvious way, by joining them together and preserving multiplicities.

Suppose that $\alpha_{R_d/\mathfrak{p}}$ is a prime action with entropy rank one. By Theorem 2.1, $\alpha_{R_d/\mathfrak{p}}$ is algebraically conjugate to a diagonal action β on a product $\Bbbk^{(1)} \times \cdots \times \Bbbk^{(m)}$ of local fields modulo a β-invariant discrete cocompact subgroup. Let β have the form

$$\beta^{\mathbf{e}_i}(a^{(1)},\dots,a^{(m)}) = (\xi_i^{(1)}a^{(1)},\dots,\xi_i^{(m)}a^{(m)}),$$

where $\xi_i^{(j)} \in \Bbbk^{(j)}$. Define

$$\mathbf{v}^{(j)} = \bigl(\log|\xi_1^{(j)}|_{\Bbbk^{(j)}},\dots,\log|\xi_d^{(j)}|_{\Bbbk^{(j)}}\bigr),$$

which we call the *Lyapunov vector* for β on $\Bbbk^{(j)}$. Then the *Lyapunov list* $\mathcal{L}(\alpha_{R_d/\mathfrak{p}})$ is defined to be

$$\mathcal{L}(\alpha_{R_d/\mathfrak{p}}) = \langle \mathbf{v}^{(1)},\dots,\mathbf{v}^{(m)}\rangle.$$

If $R_d/\mathfrak{p}$ is a field, we define $\mathcal{L}(\alpha_{R_d/\mathfrak{p}}) = \varnothing$.

Examples 8.1. (1) For the single toral automorphism in Example 6.5, $\mathcal{L}(\alpha_{R_1/\mathfrak{p}}) = \langle \log|\xi|_{\mathbb{R}}, \log|\xi'|_{\mathbb{R}}\rangle$.

(2) For the commuting toral automorphisms in Example 6.6, the Lyapunov list consists of four vectors $(\log|\sigma(\xi)|_{\mathbb{R}}, \log|\sigma(\eta)|_{\mathbb{R}})$ for σ in the Galois group G.

(3) The action generated by $\times 2$ and $\times 3$ in Example 6.7 has Lyapunov list

$$\begin{aligned}\mathcal{L}(\alpha_{R_2/\mathfrak{p}}) &= \langle (\log|2|_{\mathbb{Q}_2}, \log|3|_{\mathbb{Q}_2}),\ (\log|2|_{\mathbb{Q}_3}, \log|3|_{\mathbb{Q}_3}),\ (\log|2|_{\mathbb{R}}, \log|3|_{\mathbb{R}})\rangle \\ &= \langle (-\log 2, 0\rangle,\ (0,-\log 3),\ (\log 2, \log 3)\rangle.\end{aligned}$$

(4) Ledrappier's Example 6.8 has Lyapunov list

$$\begin{aligned}\mathcal{L}(\alpha_{R_2/\mathfrak{p}}) = \bigl\langle &(\log|t|_{\mathbb{F}_2(\!(t)\!)}, \log|1+t|_{\mathbb{F}_2(\!(t)\!)}),\ (\log|t|_{\mathbb{F}_2(\!(1+t)\!)}, \log|1+t|_{\mathbb{F}_2(\!(1+t)\!)}),\\ &(\log|t|_{\mathbb{F}_2(\!(t^{-1})\!)}, \log|1+t|_{\mathbb{F}_2(\!(t^{-1})\!)})\bigr\rangle \\ = \langle &(-\log 2, 0),\ (0,-\log 2),\ (\log 2,\log 2)\rangle.\end{aligned}$$

Suppose now that α_M is a Noetherian algebraic $\mathbb{Z}^d$-action with entropy rank one. Let

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M, \quad M_j/M_{j-1} \cong R_d/\mathfrak{q}_j \tag{8.1}$$

be a prime filtration of M. If N is a submodule of M, it is easy to see that $\operatorname{asc}(M) \subset \operatorname{asc}(N) \cup \operatorname{asc}(M/N)$. Thus $\operatorname{asc}(M) \subset \{\mathfrak{q}_1,\dots,\mathfrak{q}_r\}$. By Proposition 4.4, each $\alpha_{R_d/\mathfrak{q}_j}$ has entropy rank one. Thus we can define

$$\mathcal{L}(\alpha_M) = \bigcup_{j=1}^{r} \mathcal{L}(\alpha_{R_d/\mathfrak{q}_j}).$$

Although the list of $\mathfrak{q}_j$ appearing in (8.1) is not necessarily unique, those $\mathfrak{q}_j$ contributing nonempty lists to $\mathcal{L}(\alpha_M)$ always appear, and with the same multiplicity, in every prime filtration. This is a consequence of the following algebraic result.

Lemma 8.2. *Let M be a Noetherian R_d-module such that* $\operatorname{kdim} R_d/\mathfrak{p} \leqslant 1$ *for every* $\mathfrak{p} \in \operatorname{asc}(M)$. *Fix a minimal element* $\mathfrak{p}$ *of* $\operatorname{asc}(M)$. *Then the quotient* $R_d/\mathfrak{p}$ *appears, and with the same multiplicity, in every prime filtration* (8.1).

Proof. This is proved in [19], but there is a simpler argument using localization. Let $\mathfrak{p}$ be a minimal element in $\operatorname{asc}(M)$. Localizing (8.1) at $\mathfrak{p}$ and using standard identifications, we obtain

$$0 = (M_0)_{\mathfrak{p}} \subset (M_1)_{\mathfrak{p}} \subset \cdots \subset (M_r)_{\mathfrak{p}} = M_{\mathfrak{p}},$$

where

$$(M_j)_{\mathfrak{p}}/(M_{j-1})_{\mathfrak{p}} \cong (M_j/M_{j-1})_{\mathfrak{p}} \cong (R_d/\mathfrak{q}_j)_{\mathfrak{p}}.$$

Letting $\mathbb{K}(\mathfrak{p})$ denote the fraction field of $R_d/\mathfrak{p}$, minimality of $\mathfrak{p}$ shows that

$$(R_d/\mathfrak{q}_j)_{\mathfrak{p}} = \begin{cases} \mathbb{K}(\mathfrak{p}) \text{ if } \mathfrak{q}_j = \mathfrak{p}, \\ 0 \text{ if } \mathfrak{q}_j \neq \mathfrak{p}. \end{cases}$$

Hence the number of j for which $R_d/\mathfrak{q}_j \cong R_d/\mathfrak{p}$ equals $\dim_{\mathbb{K}(\mathfrak{p})} M \otimes_{R_d} \mathbb{K}(\mathfrak{p})$, and so is the same for every prime filtration of M. □

Proposition 8.3. *Let M be a Noetherian R_d-module such that* $\operatorname{kdim} R_d/\mathfrak{p} \leqslant 1$ *for every $\mathfrak{p} \in \operatorname{asc}(M)$. If*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_{r-1} \subset M_r = M, \qquad M_i/M_{i-1} \cong R_d/\mathfrak{p}_i,$$
$$0 = N_0 \subset N_1 \subset \cdots \subset N_{s-1} \subset N_s = M, \qquad N_j/N_{j-1} \cong R_d/\mathfrak{q}_j$$

are prime filtrations of M, then

$$\bigcup_{i=1}^{r} \mathcal{L}(\alpha_{R_d/\mathfrak{p}_i}) = \bigcup_{j=1}^{s} \mathcal{L}(\alpha_{R_d/\mathfrak{q}_j}).$$

Proof. The minimal prime ideals $\mathfrak{p}$ in $\operatorname{asc}(M)$ are the only ones for which $\mathcal{L}(\alpha_{R_d/\mathfrak{p}})$ is nonempty, so the result follows from Lemma 8.2. □

Remark 8.4. The product formula for global fields (5.2) shows that if $\alpha_{R_d/\mathfrak{p}}$ is a prime action of entropy rank one, then $\sum_{\mathbf{v}\in\mathcal{L}(\alpha_{R_d/\mathfrak{p}})} \mathbf{v} = \mathbf{0}$. Hence for a general Noetherian action of entropy rank one, the sum of its Lyapunov vectors is $\mathbf{0}$.

We can use Lyapunov vectors to compute entropy of elements of an action. The following result generalizes that classical formula that the entropy of a toral automorphism with Lyapunov exponents $\log|\lambda_j|$ is $\sum_j \max\{\log|\lambda_j|, 0\}$.

Proposition 8.5. *Let α be a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one, and let $\mathcal{L}(\alpha)$ be its list of Lyapunov vectors. Then for every $\mathbf{n} \in \mathbb{Z}^d$ we have that*

$$\mathsf{h}(\alpha^{\mathbf{n}}) = \sum_{\mathbf{v}\in\mathcal{L}(\alpha)} \max\{\mathbf{v}\cdot\mathbf{n}, 0\} \tag{8.2}$$

Proof. First consider a prime action $\alpha_{R_d/\mathfrak{p}}$, and let β be the corresponding diagonal action on $\Bbbk^{(1)} \times \cdots \times \Bbbk^{(m)}$. Then $\beta^{\mathbf{n}}$ is uniformly continuous, and Haar measure is homogeneous in the sense of Bowen [3]. It follows from [3, Prop. 7] that $\mathsf{h}(\beta^{\mathbf{n}})$ is given by the right side of (8.2), and therefore so is $\mathsf{h}(\alpha^{\mathbf{n}}_{R_d/\mathfrak{p}})$. If $\alpha = \alpha_M$ is a Noetherian action, then the addition formula for entropy (see [19] or [28, Thm. 14.1]), applied to a prime filtration for M, shows that (8.2) holds. □

Remark 8.6. For an arbitrary topological $\mathbb{Z}^d$-action Milnor defined its directional entropy in the direction of a vector $\mathbf{w} \in \mathbb{R}^d$ (see [22], and further investigations in [4, Sec. 6]). An easy modification of the previous proof shows that for a Noetherian algebraic $\mathbb{Z}^d$-action α of entropy rank one, its entropy in direction $\mathbf{w}$ is given by $\sum_{\mathbf{v}\in\mathcal{L}(\alpha)} \max\{\mathbf{v}\cdot\mathbf{w}, 0\}$, and is therefore a continuous function of the direction. For more on continuity of direction entropy for general actions Park's article [23].

9. Volume decrease in local fields

As a preliminary to computing fiber entropy in the next section, here we compute the rate of fiber volume decrease for a skew product whose fiber is a local field.

Let (Y, ν) be a measure space, and $T\colon Y \to Y$ be a measurable transformation preserving ν. Recall that a function $f\colon Y \to \mathbb{R}$ is called T-*ergodic* if the ergodic averages

$$\frac{f(y) + f(Ty) + \cdots + f(T^{n-1}y)}{n} \to \int_Y f\, d\nu \qquad \text{as } n \to \infty$$

for ν-almost every $y \in Y$.

Let $\Bbbk$ be a local field with Haar measure $\mu_\Bbbk$ and absolute value $|\cdot|_\Bbbk$. Thus $\mu_\Bbbk(aE) = |a|_\Bbbk \mu_\Bbbk(E)$ for every $a \in \Bbbk$ and compact $E \subset \Bbbk$. Let $g\colon Y \to \Bbbk^\times$ be measurable. The following result computes the rate of volume decrease in fibers for the skew product transformation of $Y \times \Bbbk$ defined by $(y, a) \mapsto (Ty, g(y)a)$.

Proposition 9.1. *Let $\Bbbk$ be a local field with Haar measure $\mu_\Bbbk$ and absolute value $|\cdot|_\Bbbk$. Let T be a measure-preserving transformation of (Y,ν), and let $g\colon Y \to \Bbbk^\times$ be measurable. Assume that $\log|g(y)|_\Bbbk$ is ν-integrable and T-ergodic. For $y \in Y$ and $\varepsilon > 0$ define*

$$D_N(\varepsilon, y) = \Big\{ a \in \Bbbk : \Big| a \prod_{j=0}^{n} g(T^j y)\Big|_\Bbbk < \varepsilon \text{ for } 0 \leqslant n \leqslant N-1 \Big\}.$$

Then for every $\varepsilon > 0$ and almost every $y \in Y$ we have that

$$\lim_{N\to\infty} -\frac{1}{N} \log \mu_\Bbbk\big(D_N(\varepsilon, y)\big) = \max\Big\{ \int_Y \log|g|_\Bbbk\, d\nu, 0\Big\}.$$

For the proof we require the following elementary result.

Lemma 9.2. *Let $\{a_n\}$ be a sequence of real numbers such that $a_n/n \to a$ as $n \to \infty$. Then*

$$\max_{1\leqslant n\leqslant N} \frac{a_n}{N} \to \max\{a, 0\} \quad \textit{as} \quad n \to \infty.$$

Proof. First suppose that $a > 0$. Fix $\varepsilon > 0$. There is an $M > 0$ such that $|a_n/n - a| < \varepsilon$ for all $n \geqslant M$. Hence $\max_{1\leqslant n\leqslant N} a_n/N \geqslant a_N/N \geqslant a - \varepsilon$ for $N \geqslant M$. Also, for $M \leqslant n \leqslant N$ we have that $a_n/N \leqslant a_n/n < a + \varepsilon$, and for N sufficiently large that $a_n/N < a + \varepsilon$ for $1 \leqslant n \leqslant M$. This completes the case $a > 0$.

Now suppose that $a \leqslant 0$. Fix $\varepsilon > 0$. Clearly $\max_{1\leqslant n\leqslant N} a_n/N \geqslant a_1/N > -\varepsilon$ for large enough N. Since $a \leqslant 0$, there is an M such that $a_n/n < \varepsilon$ for all $n \geqslant M$. If $M \leqslant i \leqslant N$ and $a_i > 0$, then $a_i/N \leqslant a_i/i < \varepsilon$, while if $a_n \leqslant 0$ then $a_n/N < \varepsilon$ trivially. Thus for all N sufficiently large we see that $\max_{1\leqslant n\leqslant N} a_n/N < \varepsilon$. □

Proof of Proposition 9.1. Let $B(r)$ denote the ball in $\Bbbk$ of radius r. Clearly

$$D_N(\varepsilon, y) = \bigcap_{n=0}^{N-1} B\Big(\varepsilon \prod_{j=0}^{n} |g(T^j y)^{-1}|_{\Bbbk}\Big) = B\Big(\min_{0\leqslant n\leqslant N-1} \varepsilon \prod_{j=0}^{n} |g(T^j y)^{-1}|_{\Bbbk}\Big).$$

Hence

$$\mu_{\Bbbk}(D_N(\varepsilon, y)) = \min_{0\leqslant n\leqslant N-1} \Big\{\mu_{\Bbbk}(B(\varepsilon)) \prod_{j=0}^{n} |g(T^j y)^{-1}|_{\Bbbk}\Big\},$$

and so

$$-\frac{1}{N} \log \mu_{\Bbbk}(D_N(\varepsilon, y)) = -\frac{1}{N} \log \mu_{\Bbbk}(B(\varepsilon)) + \max_{0\leqslant n\leqslant N-1} \frac{1}{N} \sum_{j=0}^{n} \log |g(T^j y)|_{\Bbbk}.$$

Since $\log |g(y)|_{\Bbbk}$ is assumed to be T-ergodic, we have for almost every y that

$$\frac{1}{n} \sum_{j=0}^{n-1} \log |g(T^j y)|_{\Bbbk} \to \int_Y \log |g|_{\Bbbk} \, d\nu.$$

Then Lemma 9.2 with the sequence $a_n = \sum_{j=0}^{n-1} \log |g(T^j y)|_{\Bbbk}$ shows that

$$-\frac{1}{N} \log \mu_{\Bbbk}(D_N(\varepsilon, y)) \to \max\Big\{\int_Y \log |g|_{\Bbbk} \, d\nu, 0\Big\}$$

for every $\varepsilon > 0$ and almost every $y \in Y$. □

10. Fiber entropies

Let α be an algebraic $\mathbb{Z}^d$-action of entropy rank one on a compact abelian group X. To define skew products with α, let T be a measure-preserving transformation of (Y, ν) and $\mathbf{s}\colon Y \to \mathbb{Z}^d$ be measurable. Define the skew product $T \times^{\mathbf{s}} \alpha$ on $Y \times X$ by $(T \times^{\mathbf{s}} \alpha)(y, x) = (Ty, \alpha^{\mathbf{s}(y)}(x))$.

Our goal in the next two sections is to compute the measure entropy $\mathsf{h}_{\nu\times\mu}(T \times^{\mathbf{s}} \alpha)$ and, in case T and $\mathbf{s}$ are continuous, the topological entropy $\mathsf{h}(T \times^{\mathbf{s}} \alpha)$.

According to a formula due to Abromov and Rohlin [1] and to Adler [2], the measure entropy $\mathsf{h}_{\nu\times\mu}(T \times^{\mathbf{s}} \alpha)$ equals $\mathsf{h}_{\nu}(T)$ plus the integral over Y of the measure fiber entropies of $T \times^{\mathbf{s}} \alpha$ on $\{y\} \times X$. In this section we first introduce various topological fiber entropies, and show that they coincide and equal the required measure fiber entropy. We then show how to compute topological (and therefore measure) fiber entropy for prime actions from the corresponding diagonal action on an adele, by controlling possible wrapping in the adele modulo the invariant lattice.

In building up a general Noetherian rank one action from prime actions, we are inevitably led to consider skew products with affine, rather than automorphism, fiber maps. To describe these, let $\tau\colon Y\to X$ be measurable. Define the affine skew product $T\times_{\tau}^{\mathbf{s}}\alpha$ by

$$(T\times_{\tau}^{\mathbf{s}}\alpha)(y,x)=\big(Ty,\alpha^{\mathbf{s}(y)}(x)+\tau(y)\big).$$

Then

$$(T\times_{\tau}^{\mathbf{s}}\alpha)^n(y,x)=\big(T^n y,\alpha^{\mathbf{s}_n(y)}(x)+\tau_n(y)\big),$$

where

$$\mathbf{s}_n(y)=\mathbf{s}(y)+\mathbf{s}(Ty)+\cdots+\mathbf{s}(T^{n-1}y)$$

and

$$\tau_n(y)=\sum_{j=1}^{n}\alpha^{\mathbf{s}_{n-j}(T^j y)}\tau(T^{j-1}y).$$

Thus

$$(T\times_{\tau}^{\mathbf{s}}\alpha)^n=T^n\times_{\tau_n}^{\mathbf{s}_n}\alpha.$$

Define the affine maps A_y^n on X by

$$A_y^n(x)=\alpha^{\mathbf{s}_n(y)}(x)+\tau_n(y). \tag{10.1}$$

Then iterates of $T\times_{\tau}^{\mathbf{s}}\alpha$ on a fiber $\{y\}\times X$ are effectively given by the maps A_y^n on X, and it is these we use to define fiber entropies. We abbreviate the notation for various fiber entropies $\mathsf{h}_*(T\times_{\tau}^{\mathbf{s}}\alpha,\{y\}\times X)$ to $\mathsf{h}_*(T\times_{\tau}^{\mathbf{s}}\alpha,y)$.

The following gives fiber analogues of standard definitions due originally to Bowen [3].

Definition 10.1. Let the affine maps $A_y^n\colon X\to X$ for the skew product $T\times_{\tau}^{\mathbf{s}}\alpha$ on $Y\times X$ be given by (10.1). Let ρ be a translation-invariant metric on X compatible with its topology.

(1) A set $E\subset X$ is (N,ε,y)-*spanning* for $T\times_{\tau}^{\mathbf{s}}\alpha$ if for every $x\in X$ there is an $x'\in E$ such that $\rho\big(A_y^n(x),A_y^n(x')\big)<\varepsilon$ for $0\leqslant n\leqslant N-1$. Let $r_N(\varepsilon,y)$ be the smallest cardinality of an (N,ε,y)-spanning set, and put

$$\mathsf{h}_{\mathsf{span}}(T\times_{\tau}^{\mathbf{s}}\alpha,y)=\lim_{\varepsilon\to0}\limsup_{N\to\infty}\frac{1}{N}\log r_N(\varepsilon,y).$$

(2) A set $F\subset X$ is (N,ε,y)-*separated* for $T\times_{\tau}^{\mathbf{s}}\alpha$ if for distinct points $x,x'\in F$ there is n with $0\leqslant n\leqslant N-1$ for which $\rho\big(A_y^n(x),A_y^n(x')\big)\geqslant\varepsilon$. Let $s_N(\varepsilon,y)$ be the largest cardinality of an (N,ε,y)-separated set, and put

$$\mathsf{h}_{\mathsf{sep}}(T\times_{\tau}^{\mathbf{s}}\alpha,y)=\lim_{\varepsilon\to0}\limsup_{N\to\infty}\frac{1}{N}\log s_N(\varepsilon,y).$$

(3) Let $B_X(\varepsilon) = \{x \in X : \rho(x, 0_X) < \varepsilon\}$, and put

$$D_N(\varepsilon, y) = \bigcap_{n=0}^{N-1} \alpha^{-\mathbf{s}_n(y)}(B_X(\varepsilon)).$$

Define the volume decrease fiber entropy for skew products with automorphisms by

$$\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y) = \lim_{\varepsilon \to 0} \limsup_{N \to \infty} -\frac{1}{N} \log \mu(D_N(\varepsilon, y)).$$

Similarly, if β is a diagonal action on a finite product $\mathbb{A}$ of local fields, and $B_{\mathbb{A}}(\varepsilon)$ is the ε-ball in $\mathbb{A}$, we define $\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \beta, y)$ as above, with $B_{\mathbb{A}}(\varepsilon)$ replacing $B_X(\varepsilon)$.

Lemma 10.2. *All of the topological fiber entropies in Definition 10.1 agree:*

$$\begin{aligned} &\mathsf{h}_{\mathsf{span}}(T \times^{\mathbf{s}}_{\tau} \alpha, y) = \mathsf{h}_{\mathsf{sep}}(T \times^{\mathbf{s}}_{\tau} \alpha, y) \\ &= \mathsf{h}_{\mathsf{span}}(T \times^{\mathbf{s}} \alpha, y) = \mathsf{h}_{\mathsf{sep}}(T \times^{\mathbf{s}} \alpha, y) = \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y). \end{aligned}$$

Proof. The equality of spanning set entropy for $T \times^{\mathbf{s}} \alpha$ and its affine counterpart $T \times^{\mathbf{s}}_{\tau} \alpha$ follows because the metric ρ is translation-invariant; similarly for separated set entropy.

If F is a maximal (N, ε, y)-separated for $T \times^{\mathbf{s}} \alpha$, it is also (N, ε, y)-spanning. Hence $r_N(\varepsilon, y) \leqslant s_N(\varepsilon, y)$, and so $\mathsf{h}_{\mathsf{span}}(T \times^{\mathbf{s}} \alpha, y) \leqslant \mathsf{h}_{\mathsf{sep}}(T \times^{\mathbf{s}} \alpha, y)$. Furthermore, the sets $x + D_N(\varepsilon/2, y)$ for $x \in F$ are disjoint, so $s_N(\varepsilon, y)\mu(D_N(\varepsilon/2, y)) \leqslant 1$, proving that $\mathsf{h}_{\mathsf{sep}}(T \times^{\mathbf{s}} \alpha, y) \leqslant \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y)$. Finally, if E is (N, ε, y)-spanning, then $\bigcup_{x \in E}(x + D_N(\varepsilon, y)) = X$. Hence $r_N(\varepsilon, y)\mu(D_N(\varepsilon, y)) \geqslant 1$, so that $\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y) \leqslant \mathsf{h}_{\mathsf{span}}(T \times^{\mathbf{s}} \alpha, y)$. $\square$

Next we turn to measure fiber entropy. Let $\mathsf{H}_\mu(P)$ denote the usual entropy of a finite measurable partition P of X.

Definition 10.3. The *measure fiber entropy* of an affine skew product $T \times^{\mathbf{s}}_{\tau} \alpha$ is defined by

$$\mathsf{h}_\mu(T \times^{\mathbf{s}}_{\tau} \alpha, y) = \sup_P \limsup_{N \to \infty} \frac{1}{N} \mathsf{H}_\mu\Big(\bigvee_{n=0}^{N-1} (A^n_y)^{-1}(P) \Big),$$

where the supremum is taken over all finite measurable partitions of X.

The following proposition, which is a special case of results due to Abromov and Rohlin [1] and to Adler [2], computes the entropy of $T \times^{\mathbf{s}}_{\tau} \alpha$ in terms of the base and measure fiber entropies.

Proposition 10.4. *Let T be a measure-preserving transformation of (Y, ν), $\mathbf{s}$ be ν-integrable, and $\tau\colon Y \to X$ be measurable. Then*

$$\mathsf{h}_{\nu \times \mu}(T \times^{\mathbf{s}}_{\tau} \alpha) = \mathsf{h}_\nu(T) + \int_Y \mathsf{h}_\mu(T \times^{\mathbf{s}}_{\tau} \alpha, y)\, d\nu(y).$$

To make use of this result, we need to relate topological and measure fiber entropy. The following fact, whose proof is exactly the same as in the proof of Theorem 13.3 in [28], shows that they are within a universal constant of each other. We use this later to show that they in fact agree.

Lemma 10.5. *The topological and measure fiber entropies satisfy*

$$\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y) \leqslant \mathsf{h}_{\mu}(T \times^{\mathbf{s}} \alpha, y) \leqslant \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha) + 1 + \log 2.$$

The volume decrease fiber entropy is easy to compute for diagonal actions on products of local fields.

Lemma 10.6. *Let $\mathbb{A} = \Bbbk^{(1)} \times \cdots \times \Bbbk^{(m)}$ be a product of local fields, and β be a diagonal $\mathbb{Z}^d$-action on $\mathbb{A}$ with Lyapunov list $\langle \mathbf{v}^{(1)}, \dots, \mathbf{v}^{(m)} \rangle$. Let T be a measure-preserving transformation of (Y, ν), and $\mathbf{s}\colon Y \to \mathbb{Z}^d$ be ν-integrable and T-ergodic with average value $\nu(\mathbf{s}) \in \mathbb{R}^d$. Then for almost every $y \in Y$ we have that*

$$\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \beta, y) = \sum_{j=1}^{m} \max\{\nu(\mathbf{s}) \cdot \mathbf{v}^{(j)}, 0\}. \tag{10.2}$$

Proof. Clearly we are free to choose a compatible metric on $\mathbb{A}$ when computing $\mathsf{h}_{\mathsf{vol}}$, and we use

$$\rho\big((a^{(j)}), (b^{(j)})\big) = \max_{1 \leqslant j \leqslant m} |a^{(j)} - b^{(j)}|_{\Bbbk^{(j)}}. \tag{10.3}$$

Then $B_{\mathbb{A}}(\varepsilon) = B_{\Bbbk^{(1)}}(\varepsilon) \times \cdots \times B_{\Bbbk^{(m)}}(\varepsilon)$. We can now apply Proposition 9.1 to each factor $\Bbbk^{(j)}$ separately, resulting in a contribution of $\max\{\nu(\mathbf{s}) \cdot \mathbf{v}^{(j)}, 0\}$. Adding these together completes the proof. □

Suppose that $\alpha = \alpha_{R_d/\mathfrak{p}}$ is a prime action on $X = X_{R_d/\mathfrak{p}}$ of entropy rank one. Let β be the corresponding diagonal action on the adele $\mathbb{A} = \mathbb{A}_{R_d/\mathfrak{p}}$, with β-invariant cocompact discrete subgroup $\Lambda \subset \mathbb{A}$ such that $\mathbb{A}/\Lambda \cong X$. If the skewing function $\mathbf{s}$ is assumed to be bounded, then the local isomorphism between X and $\mathbb{A}$ shows that $\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y) = \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \beta, y)$ for every $y \in Y$. This is effectively Bowen's calculation of the entropy of a toral automorphism from the entropy of the covering linear map [3, Cor. 16]. However, when $\mathbf{s}$ is unbounded the intersections to compute $\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y)$ can be much more complicated than those for $\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \beta, y)$ owing to wrapping phenomena in X for sets $\alpha^{-\mathbf{s}(T^j y)}(B_X(\varepsilon))$ when $\mathbf{s}(T^j y)$ is very large. Marcus and Newhouse [21] control this for $\mathbb{Z}$-actions by inducing on a subset of Y defined by a first exit time to reduce to the case of bounded $\mathbf{s}$; however, this technique is not available for $\mathbb{Z}^d$-actions.

Proposition 10.7. *Let α be a prime action of entropy rank one, and β be the corresponding diagonal action. Let T be a measure-preserving transformation of (Y, ν), and $\mathbf{s}\colon Y \to \mathbb{Z}^d$ be ν-integrable. Then*

$$\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y) = \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \beta, y)$$

for ν-almost every $y \in Y$.

Proof. Let $\mathbb{A} = \Bbbk^{(1)} \times \cdots \times \Bbbk^{(m)}$ and β be the diagonal action defined by

$$\beta^{\mathbf{n}}(a^{(1)}, \dots, a^{(m)}) = (\mathbf{c}_1^{\mathbf{n}} a^{(1)}, \dots, \mathbf{c}_m^{\mathbf{n}} a^{(m)}),$$

where $\mathbf{c}_j \in (\Bbbk^{(j)})^d$. Let Λ be the β-invariant lattice in $\mathbb{A}$ such that $X \cong \mathbb{A}/\Lambda$. We use the metric on $\mathbb{A}$ in (10.3), and its quotient metric on X. Thus the quotient map $\phi\colon \mathbb{A} \to \mathbb{A}/\Lambda = X$ is a local isometry. We normalize Haar measure on $\mathbb{A}$ so that ϕ is also locally measure-preserving, and let μ denote Haar measure on both $\mathbb{A}$ and X.

Define

$$D_N^X(\varepsilon, y) = \bigcap_{n=0}^{N-1} \alpha^{-\mathbf{s}_n(y)}(B_X(\varepsilon)) \quad \text{and} \quad D_N^{\mathbb{A}}(\varepsilon, y) = \bigcap_{n=0}^{N-1} \beta^{-\mathbf{s}_n(y)}(B_{\mathbb{A}}(\varepsilon)).$$

Since ϕ is a local isometry, for sufficiently small ε we have that $\phi(D_N^{\mathbb{A}}(\varepsilon, y)) \subset D_N^X(\varepsilon, y)$. Hence $\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y) \leqslant \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \beta, y)$ for every $y \in Y$.

To prove the reverse inequality, we first claim that there is a constant θ such that if $r_j \geqslant 1$ and Q is the rectangle $B_{\Bbbk^{(1)}}(r_1) \times \cdots \times B_{\Bbbk^{(m)}}(r_m)$, then $|Q \cap \Lambda| \leqslant \theta\mu(Q)$. To see this, observe that since Λ is discrete there is $0 < \eta < 1/2$ such that $\rho(a, b) > 2\eta$ for distinct $a, b \in \Lambda$. Hence the sets $B_{\mathbb{A}}(\eta) + a$ are disjoint for $a \in \Lambda$. Furthermore, since $r_j \geqslant 1$ and $\eta < 1/2$, there is a $\gamma > 0$ such that if $a \in Q$ then $\mu(Q \cap (B_{\mathbb{A}}(\eta) + a)) \geqslant \gamma$. Hence $|Q \cap \Lambda|\gamma \leqslant \mu(Q)$, and so we can take $\theta = 1/\gamma$ to verify our claim.

Next, for $\mathbf{n} \in \mathbb{Z}^d$ and $\varepsilon > 0$, define $f_\varepsilon(\mathbf{n})$ to be the number of lattice points $a \in \Lambda$ for which $B_{\mathbb{A}}(\varepsilon) \cap \beta^{-\mathbf{n}}(B_{\mathbb{A}}(\varepsilon) + a) \neq \varnothing$. Clearly, for fixed $\mathbf{n}$ the function $f_\varepsilon(\mathbf{n})$ decreases as ε decreases, and $f_\varepsilon(\mathbf{n}) \to 1$ as $\varepsilon \to 0$.

If $a \in \Lambda$ is such that $B_{\mathbb{A}}(1) \cap \beta^{-\mathbf{n}}(B_{\mathbb{A}}(1) + a) \neq \varnothing$, then

$$a \in \beta^{\mathbf{n}}(B_{\mathbb{A}}(1)) + B_{\mathbb{A}}(1) \subset Q = \prod_{j=1}^{m} B_{\Bbbk^{(j)}}(1 + \|\mathbf{c}_j^{\mathbf{n}}\|_{\Bbbk^{(j)}}),$$

where $\|\cdot\|_{\Bbbk^{(j)}}$ is the sup norm on $\big(\Bbbk^{(j)}\big)^d$. Using obvious estimates on the measures of balls in $\Bbbk^{(j)}$ together with the inequality $|Q \cap \Lambda| \leqslant \theta\mu(Q)$ from the previous paragraph, we see that there are constants $C > 0$ and $\lambda > 1$ such that

$$f_\varepsilon(\mathbf{n}) \leqslant f_1(\mathbf{n}) \leqslant \theta\mu(Q) \leqslant C\lambda^{\|\mathbf{n}\|}. \tag{10.4}$$

Next, observe that for ε small enough, $\alpha^{-\mathbf{s}(y)}(B_X(\varepsilon)) \cap B_X(\varepsilon)$ is made up of at most $f_\varepsilon(\mathbf{s}(y))$ pieces, one for each lattice point $a \in \Lambda$ for which $B_{\mathbb{A}}(\varepsilon) \cap \beta^{-\mathbf{s}(y)}(B_{\mathbb{A}}(\varepsilon) + a) \neq \varnothing$, and each piece is contained in a translate of $\phi(B_{\mathbb{A}}(2\varepsilon) \cap \beta^{-\mathbf{s}(y)} B_{\mathbb{A}}(2\varepsilon))$. Continuing inductively, we see that

$$D_N^X(\varepsilon, y) = B_X(\varepsilon) \cap \alpha^{-\mathbf{s}_1(y)}(B_X(\varepsilon)) \cap \cdots \cap \alpha^{-\mathbf{s}_{N-1}(y)}(B_X(\varepsilon))$$

is the union of at most

$$p_N(\varepsilon, y) = \prod_{n=1}^{N-1} f_\varepsilon(\mathbf{s}(T^n y))$$

pieces, each contained in a translate of $\phi(D_N^{\mathbb{A}}(2\varepsilon, y))$. Hence

$$\mu(D_N^X(\varepsilon, y)) \leqslant p_N(\varepsilon, y) \cdot \mu(D_N^{\mathbb{A}}(2\varepsilon, y)). \tag{10.5}$$

By (10.4), $0 \leqslant \log f_\varepsilon(\mathbf{s}(y)) \leqslant \log C + \|\mathbf{s}(y)\| \log \lambda$, so that $\log f_\varepsilon(\mathbf{s}(y))$ is ν-integrable on Y since $\mathbf{s}(y)$ is. Also,

$$\frac{1}{N} \log p_N(\varepsilon, y) = \frac{1}{N} \sum_{n=1}^{N-1} \log f_\varepsilon(\mathbf{s}(T^n y))$$

is an ergodic average of $\log f_\varepsilon(\mathbf{s}(y))$. By the ergodic theorem, there is an integrable function $g_\varepsilon \geqslant 0$ with $\int_Y g_\varepsilon(y)\, d\nu(y) = \int_Y \log f_\varepsilon(\mathbf{s}(y))\, d\nu(y)$ such that $N^{-1} \log p_N(\varepsilon, y) \to g_\varepsilon(y)$ as $N \to \infty$ for almost every y. Clearly g_ε decreases as ε decreases. Since $\log f_\varepsilon(\mathbf{s}(y)) \to 0$ as $\varepsilon \to 0$, it follows from the monotone convergence theorem that

$$\begin{aligned}\int_Y \lim_{\varepsilon\to 0} g_\varepsilon(y)\, d\nu(y) = \lim_{\varepsilon\to 0} \int_Y g_\varepsilon(y)\, d\nu(y) &= \lim_{\varepsilon\to 0} \int_Y \log f_\varepsilon(\mathbf{s}(y))\, d\nu(y) \\ = \int_Y \lim_{\varepsilon\to 0} \log f_\varepsilon(\mathbf{s}(y))\, d\nu(y) &= 0.\end{aligned}$$

Hence $\lim_{\varepsilon\to 0} g_\varepsilon(y) = 0$ for almost every y.

Finally, from (10.5) we see that

$$\begin{aligned}\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y) &\geqslant \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \beta, y) - \lim_{\varepsilon\to 0} \lim_{N\to\infty} \frac{1}{N} \log p_N(\varepsilon, y) \\ &= \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \beta, y) - \lim_{\varepsilon\to 0} g_\varepsilon(y) = \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \beta)\end{aligned}$$

for ν-almost every y, concluding the proof. □

To prove our entropy formula for actions built up from prime actions, we need the following simple inequality.

Lemma 10.8. *Let $K \subset X$ be a compact α-invariant subgroup, and denote the restriction of α to K by α_K and the resulting action on the quotient X/K by $\alpha_{X/K}$. Then*

$$\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y) \geqslant \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha_K, y) + \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha_{X/K}, y)$$

for every y.

Proof. Let $\pi\colon X \to X/K$ be the natural projection map. We may assume that the metric on X induces the metrics on K and on X/K.

Fix N and $\varepsilon > 0$. Define sets $D_N^X(\varepsilon, y)$, $D_N^K(\varepsilon, y)$, and $D_N^{X/K}(\varepsilon, y)$ using these metrics and the actions α, α_K, and $\alpha_{X/K}$, respectively. Clearly $\pi(D_N^X(\varepsilon, y)) \subset D_N^{X/K}(\varepsilon, y)$. If $x, x' \in D_N^X(\varepsilon, y)$ and $x - x' \in K$, then

$x - x' \in D_N^K(2\varepsilon, y)$. By Fubini's Theorem,

$$\begin{aligned}\mu_X\big(D_N^X(\varepsilon,y)\big) &= \int_{X/K} \mu_K\big(D_N^X(\varepsilon,y)-\overline{x}\big)\,d\mu_{X/K}(\overline{x})\\ &\leqslant \int_{D_N^{X/K}(\varepsilon,y)} \mu_K\big(D_N^K(2\varepsilon,y)\big)\,d\mu_{X/K}(\overline{x})\\ &\leqslant \mu_{X/K}\big(D_N^{X/K}(\varepsilon,y)\big)\mu_K\big(D_N^K(2\varepsilon,y)\big).\end{aligned}$$

The result now follows using Definition 10.1(3). □

11. Entropy of skew products

Our fiber entropy results of the previous section provide the basis for computing the measure and topological entropy of skew products with Noetherian algebraic $\mathbb{Z}^d$-actions of entropy rank one. The following result includes Theorem 2.2.

Theorem 11.1. *Let α be a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one and Lyapunov vector list $\mathcal{L}(\alpha)$. Let T be a measure-preserving transformation of (Y,ν). Suppose that $\mathbf{s}\colon Y \to \mathbb{Z}^d$ is T-ergodic and ν-integrable with average value $\nu(\mathbf{s})$, and that $\tau\colon Y \to X$ is measurable. Then the measure fiber entropy of $T \times_\tau^{\mathbf{s}} \alpha$ is given by*

$$\mathsf{h}_\mu(T \times_\tau^{\mathbf{s}} \alpha, y) = \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y) = \sum_{\mathbf{v}\in\mathcal{L}(\alpha)} \max\{\nu(\mathbf{s})\cdot\mathbf{v}, 0\} \tag{11.1}$$

for ν-almost every y. Hence the measure entropy of $T \times_\tau^{\mathbf{s}} \alpha$ is

$$\mathsf{h}_{\nu\times\mu}(T \times_\tau^{\mathbf{s}} \alpha) = \mathsf{h}_\nu(T) + \sum_{\mathbf{v}\in\mathcal{L}(\alpha)} \max\{\nu(\mathbf{s})\cdot\mathbf{v}, 0\}. \tag{11.2}$$

Proof. Consider first the case of a prime action $\alpha = \alpha_{R_d/\mathfrak{p}}$. Put

$$h = h(\mathbf{s}) = \sum_{\mathbf{v}\in\mathcal{L}(\alpha)} \max\{\nu(\mathbf{s})\cdot\mathbf{v}, 0\}.$$

By Lemmas 10.5 and 10.6 and Proposition 10.7,

$$h \leqslant \mathsf{h}_\mu(T \times_\tau^{\mathbf{s}} \alpha, y) \leqslant h + 1 + \log 2$$

for almost every y. By Proposition 10.4,

$$\mathsf{h}_\nu(T) + h \leqslant \mathsf{h}_{\nu\times\mu}(T \times_\tau^{\mathbf{s}} \alpha) \leqslant \mathsf{h}_\nu(T) + h + 1 + \log 2. \tag{11.3}$$

Now $(T \times_\tau^{\mathbf{s}} \alpha)^n = T^n \times_{\tau_n}^{\mathbf{s}_n} \alpha$. Since

$$\nu(\mathbf{s}_n) = \sum_{j=0}^{n-1} \nu(\mathbf{s}\circ T^j) = n\,\nu(\mathbf{s}),$$

then $h(\mathbf{s}_n) = n\,h(\mathbf{s}) = n\,h$. Applying (11.3) to $(T \times_\tau^{\mathbf{s}} \alpha)^n$ gives

$$\mathsf{h}_\nu(T^n) + h(\mathbf{s}_n) \leqslant \mathsf{h}_{\nu\times\mu}\big((T \times_\tau^{\mathbf{s}} \alpha)^n\big) \leqslant \mathsf{h}_\nu(T^n) + h(\mathbf{s}_n) + 1 + \log 2,$$

or

$$n\,\mathsf{h}_\nu(T) + n\,h \leqslant n\,\mathsf{h}_{\nu\times\mu}(T \times_\tau^{\mathbf{s}} \alpha) \leqslant n\,\mathsf{h}_\nu(T) + n\,h + 1 + \log 2.$$

Dividing by n and letting $n \to \infty$ proves (11.2) in this case.

By Lemma 10.5, $h = \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y) \leqslant \mathsf{h}_{\mu}(T \times^{\mathbf{s}}_{\tau} \alpha, y)$ for almost every y, and by the above $\int_Y \mathsf{h}_{\mu}(T \times^{\mathbf{s}}_{\tau} \alpha, y)\, d\nu(y) = h$, which establishes (11.1) in this case as well.

We prove the general Noetherian action case by induction on the length of a prime filtration, the above establishing the result for filtrations of length one.

Suppose that K is a compact α-invariant subgroup of X, that α_K is a prime action, and that (11.1) and (11.2) hold for α_K and for $\alpha_{X/K}$. We represent $T \times^{\mathbf{s}}_{\tau} \alpha$ as a succession of two skew products to which our results apply, as follows.

By [24, I.5.1], there is a Borel cross-section $\sigma \colon X/K \to X$ to the natural quotient map $\pi \colon X \to X/K$ such that $\pi \circ \sigma$ is the identity on X/K. This induces a measurable isomorphism $\phi \colon X \to (X/K) \times K$, given by $\phi(x) = (\overline{x}, b(x))$, where $\overline{x} = x + K \in X/K$ and $b(x) = x - \sigma(\overline{x})$. Under this isomorphism Haar measure μ on X corresponds to the product $\mu_{X/K} \times \mu_K$ on X/K and on K. Furthermore, $\alpha^{\mathbf{n}}$ is conjugated to the map $(\overline{x}, k) \mapsto \big(\alpha^{\mathbf{n}}_{X/K}(\overline{x}), \alpha^{\mathbf{n}}_K(k) + b(\alpha^{\mathbf{n}}\sigma(\overline{x}))\big)$ for $(\overline{x}, k) \in (X/K) \times K$.

Define $\overline{\tau} \colon Y \to X/K$ by $\overline{\tau}(y) = \overline{\tau(y)} = \tau(y) + K$. Consider the skew product $S = T \times^{\mathbf{s}}_{\overline{\tau}} \alpha_{X/K}$ on $Y' = Y \times (X/K)$. From our induction hypothesis, we know that

$$\begin{aligned} \mathsf{h}_{\nu\times\mu_{X/K}}(S) &= \mathsf{h}_{\nu\times\mu_{X/K}}(T \times^{\mathbf{s}}_{\overline{\tau}} \alpha_{X/K}) \\ &= \mathsf{h}_{\nu}(T) + \sum_{\mathbf{v}\in\mathcal{L}(\alpha_{X/K})} \max\{\nu(\mathbf{s})\cdot\mathbf{v}, 0\}. \end{aligned} \tag{11.4}$$

Consider next the skew product $S \times^{\mathbf{s}'}_{\tau'} \alpha_K$ on $Y' \times K$, where $\mathbf{s}'(y, \overline{x}) = \mathbf{s}(y)$ and $\tau'(y, \overline{x}) = b\big(\alpha^{\mathbf{s}(y)}\sigma(\overline{x}) + \tau(y)\big)$. Observe that since $\mathbf{s}'$ depends only on the first coordinate, and agrees with $\mathbf{s}$ there, it follows that $\mathbf{s}'$ is S-ergodic with respect to $\nu \times \mu_{X/K}$, and that $(\nu \times \mu_{X/K})(\mathbf{s}') = \nu(\mathbf{s})$. Since α_K is prime, our earlier work shows that

$$\mathsf{h}_{(\nu\times\mu_{X/K})\times\mu_K}(S \times^{\mathbf{s}'}_{\tau'} \alpha_K) = \mathsf{h}_{\nu\times\mu_{X/K}}(S) + \sum_{\mathbf{v}\in\mathcal{L}(\alpha_K)} \max\{\nu(\mathbf{s})\cdot\mathbf{v}, 0\}. \tag{11.5}$$

Now $\mathrm{Id}_Y \times \phi$ conjugates $T \times^{\mathbf{s}}_{\tau} \alpha$ to

$$(T \times^{\mathbf{s}}_{\overline{\tau}} \alpha_{X/K}) \times^{\mathbf{s}'}_{\tau'} \alpha_K = S \times^{\mathbf{s}'}_{\tau'} \alpha_K.$$

Putting together (11.4) and (11.5), and recalling that $\mathcal{L}(\alpha) = \mathcal{L}(\alpha_{X/K}) \cup \mathcal{L}(\alpha_K)$, we obtain (11.2) for α.

Let

$$h_K = \sum_{\mathbf{v}\in\mathcal{L}(\alpha_K)} \max\{\nu(\mathbf{s})\cdot\mathbf{v}, 0\} \quad \text{and} \quad h_{X/K} = \sum_{\mathbf{v}\in\mathcal{L}(\alpha_{X/K})} \max\{\nu(\mathbf{s})\cdot\mathbf{v}, 0\}.$$

By our induction hypothesis, we know that

$$\mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha_K, y) = h_K \quad \text{and} \quad \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha_{X/K}, y) = h_{X/K}$$

for ν-almost every y. By our calculation of $\mathsf{h}(T \times_\tau^{\mathbf{s}} \alpha)$ and Proposition 10.4, we have that

$$\int_Y \mathsf{h}_\mu(T \times_\tau^{\mathbf{s}} \alpha, y)\, d\nu(y) = h_K + h_{X/K}.$$

Finally, by Lemmas 10.5 and 10.8,

$$\begin{aligned}\mathsf{h}_\mu(T \times_\tau^{\mathbf{s}} \alpha, y) &\geqslant \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha, y)\\ &\geqslant \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha_K, y) + \mathsf{h}_{\mathsf{vol}}(T \times^{\mathbf{s}} \alpha_{X/K}) = h_K + h_{X/K}.\end{aligned}$$

It follows that $\mathsf{h}_\mu(T \times_\tau^{\mathbf{s}} \alpha, y) = h_K + h_{X/K}$ for almost every y, completing the proof. □

We next compute topological entropy for continuous skew products. For this, suppose that Y is a compact metric space, and that $T\colon Y \to Y$, $\mathbf{s}\colon Y \to \mathbb{Z}^d$, and $\tau\colon Y \to X$ are continuous. Thus $T \times_\tau^{\mathbf{s}} \alpha$ is a continuous transformation of the compact metric space $Y \times X$, whose topological entropy we denote by $\mathsf{h}(T \times_\tau^{\mathbf{s}} \alpha)$. The topological fiber entropies from Definition 10.1 we denote by $\mathsf{h}(T \times_\tau^{\mathbf{s}} \alpha, y)$. We let $\mathsf{P}(f, T)$ denote the topological pressure of a continuous real-valued function f on Y with respect to T. The following includes Theorem 2.3.

Theorem 11.2. *Let α be a Noetherian algebraic $\mathbb{Z}^d$-action of entropy rank one with Lyapunov vector list $\mathcal{L}(\alpha)$. Let Y be a compact metric space, and assume that $T\colon Y \to Y$, $\mathbf{s}\colon Y \to \mathbb{Z}^d$, and $\tau\colon Y \to X$ are continuous. For every $E \subset \mathcal{L}(\alpha)$ define $f_E(y) = \sum_{\mathbf{v}\in E} \mathbf{s}(y)\cdot \mathbf{v}$. Then*

$$h(T \times_\tau^{\mathbf{s}} \alpha) = \max_{E\subset\mathcal{L}(\alpha)} \mathsf{P}(f_E, T). \tag{11.6}$$

Proof. Let $\pi\colon Y \times X \to Y$ be projection to the first coordinate. Fix an arbitrary T-invariant ergodic measure ν on Y. The relative variational principle of Ledrappier and Walters [17] asserts that

$$\sup_{\lambda\in\pi^{-1}(\nu)} \mathsf{h}_\lambda(T \times_\tau^{\mathbf{s}} \alpha) = \mathsf{h}_\nu(T) + \int_Y \mathsf{h}(T \times_\tau^{\mathbf{s}} \alpha, y)\, d\nu(y),$$

where the supremum is taken over all measures λ invariant under $T \times_\tau^{\mathbf{s}} \alpha$ that project to ν. By Theorem 11.1

$$\begin{aligned}\mathsf{h}(T \times_\tau^{\mathbf{s}} \alpha, y) &= \sum_{\mathbf{v}\in\mathcal{L}(\alpha)} \max\{\nu(\mathbf{s})\cdot\mathbf{v}, 0\}\\ &= \sup_{E\subset\mathcal{L}(\alpha)} \sum_{\mathbf{v}\in E} \nu(\mathbf{s})\cdot\mathbf{v} = \sup_{E\subset\mathcal{L}(\alpha)} \int_Y f_E\, d\nu.\end{aligned}$$

By the usual variational principle, we can compute $\mathsf{h}(T \times_\tau^{\mathbf{s}} \alpha)$ as the supremum of $\mathsf{h}_\lambda(T \times_\tau^{\mathbf{s}} \alpha)$ over all ergodic measures λ on $Y \times X$. Such a measure projects under π to a T-invariant ergodic measure on Y. Hence

$$\begin{aligned}\mathsf{h}(T \times^{\mathbf{s}}_{\tau} \alpha) &= \sup_{\nu}\Big\{\sup_{\lambda\in\pi^{-1}(\nu)} \mathsf{h}_{\lambda}(T \times^{\mathbf{s}}_{\tau} \alpha)\Big\} = \sup_{\nu}\Big\{\mathsf{h}_{\nu}(T) + \sup_{E\subset\mathcal{L}(\alpha)} \int_Y f_E\, d\nu\Big\} \\ &= \sup_{E\subset\mathcal{L}(\alpha)} \sup_{\nu} \Big\{\mathsf{h}_{\nu}(T) + \int_Y f_E\, d\nu\Big\} = \sup_{E\subset\mathcal{L}(\alpha)} \mathsf{P}(f_E, T),\end{aligned}$$

where in the last line we use the variational principle for the pressure of f_E with respect to T. □

For example, if T is a shift of finite type, then s depends on only finitely many coordinates, and each of the pressures $\mathsf{P}(f_E, T)$ can be computed explicitly.

Example 11.3. Let α be a $\mathbb{Z}^2$-action of entropy rank one with Lyapunov vectors (v_1, w_1), $\dots$, (v_m, w_m). Let $Y_2 = \{1, 2\}^{\mathbb{Z}}$ and T_2 be the 2-shift on Y_2. Define $\mathsf{s}\colon Y_2 \to \mathbb{Z}^2$ be $\mathsf{s}(y) = \mathbf{e}_{y_0}$. An easy calculation of pressure shows that

$$\mathsf{h}(T_2 \times^{\mathsf{s}} \alpha) = \max_{E\subset\{1,\dots,m\}} \log\Big[\exp\Big(\sum_{j\in E} v_j\Big) + \exp\Big(\sum_{j\in E} w_j\Big)\Big]. \tag{11.7}$$

Note that taking $E = \varnothing$ gives $\log 2$ on the right side, corresponding to the fact that $\mathsf{h}(T_2 \times^{\mathsf{s}} \alpha)$ must be at least as large as the entropy $'h(T_2) = \log 2$ of the base.

12. Relational entropy for commuting group automorphisms

In [8] Friedland studies a general notion of entropy for relations, defined as follows. Let Z be a compact metric space and $\mathcal{R}$ be a closed subset of $Z \times Z$, or *relation on* Z. Put

$$\mathcal{X}(\mathcal{R}) = \{x \in Z^{\mathbb{N}} : (x_i, x_{i+1}) \in \mathcal{R} \text{ for all } i \in \mathbb{N}\}.$$

Let $\sigma_{\mathcal{X}(\mathcal{R})}$ denote the one-sided shift on $\mathcal{X}(\mathcal{R})$. Define the *relational entropy* $\mathsf{h}_{\mathsf{rel}}(\mathcal{R})$ of $\mathcal{R}$ to be the topological entropy $\mathsf{h}(\sigma_{\mathcal{X}(\mathcal{R})})$.

As Friedland notes, if $T\colon Z \to Z$ is continuous and $\mathcal{R}_T = \{(z, Tz) : z \in Z\}$ is the graph of T, then $\mathsf{h}_{\mathsf{rel}}(\mathcal{R}_T)$ reduces to the usual topological entropy $\mathsf{h}(T)$ of T. In this sense relational entropy generalizes topological entropy.

Using these ideas, Geller and Pollicott [9] introduced the *relational entropy* $\mathsf{e}(S, T)$ of a pair S, T of commuting transformations. They put $\mathcal{R}_{S,T} = \mathcal{R}_S \cup \mathcal{R}_T$, the union of the graphs of S and of T, and defined $\mathsf{e}(S, T) = \mathsf{h}_{\mathsf{rel}}(\mathcal{R}_{S,T})$. This definition has an obvious extension to $\mathsf{e}(T_1, \dots, T_d)$ for d commuting maps T_j by using the relation $\mathcal{R}_{T_1,\dots,T_d} = \mathcal{R}_{T_1} \cup \dots \cup \mathcal{R}_{T_d}$.

One of their main results is that if $Z = \mathbb{T}$, S is multiplication by p, and T is multiplication by $q \neq p$, then $\mathsf{e}(S, T) = \log(p + q)$, verifying a conjecture of Friedland. They considered pairs of transformations that are commuting automorphisms of a compact abelian group, and conjectured a formula for $\mathsf{e}(A, B)$ where A and B are commuting toral automorphisms.

Let $A_1, \ldots, A_d$ be commuting automorphisms of a compact abelian group X with Haar measure μ. Denote by α the algebraic $\mathbb{Z}^d$-action they generate via $\alpha^{\mathbf{e}_j} = A_j$. Assume that the A_j are essentially distinct, namely that they satisfy the condition $\mu(\{x \in X : A_i(x) = A_j(x)\}) = 0$ for all $i \neq j$, corresponding to the condition $p \neq q$ above. Then we can compute $\mathsf{e}(A_1, \ldots, A_d)$ using our previous results on skew products, in this case using the full d-shift as base. The following result contains Theorem 2.4 as the case $d = 2$.

Theorem 12.1. *Let $A_1, \ldots, A_d$ be commuting automorphisms of a compact abelian group X with Haar measure μ. Assume that the algebraic $\mathbb{Z}^d$-action they generate is Noetherian, and that $\mu(\{x \in X : A_i(x) = A_j(x)\}) = 0$ for all $i \neq j$. Let $Y_d = \{1, \ldots, d\}^{\mathbb{Z}}$ and T_d be the d-shift on Y_d. Define $\mathbf{s}\colon Y_d \to \mathbb{Z}^d$ by $\mathbf{s}(y) = \mathbf{e}_{y_0}$. Then*

$$\mathsf{e}(A_1, \ldots, A_d) = \mathsf{h}(T_d \times^{\mathbf{s}} \alpha),$$

where the right side is computed according to Theorem 11.2.

Proof. Define $\phi\colon Y_d \times^{\mathbf{s}} X \to \mathcal{X}(\mathcal{R}_{A_1,\ldots,A_d})$ by

$$\phi(y, x) = \bigl(x, A_{y_0}(x), A_{y_1}A_{y_0}(x), A_{y_2}A_{y_1}A_{y_0}(x), \ldots\bigr).$$

Clearly ϕ is continuous, surjective, and intertwines $T_d \times_{\mathbf{s}} \alpha$ with the shift on $\mathcal{X}(\mathcal{R}_{A_1,\ldots,A_d})$. Therefore $\mathsf{e}(A_1, \ldots, A_d) \leqslant \mathsf{h}(T_d \times^{\mathbf{s}} \alpha)$.

To prove the reverse inequality, recall that since T_d is a shift of finite type, $T_d \times^{\mathbf{s}} \alpha$ has a measure of maximal entropy of the form $\nu \times \mu$. We claim that ϕ is one-to-one on a set of full $\nu \times \mu$ measure. For suppose that $\phi(y, x) = \phi(y', x')$ with $(y, x) \neq (y', x')$. By definition of ϕ we have $x = x'$. Choose n minimal so that $y_n \neq y'_n$. Then

$$A_{y_n}(A_{y_{n-1}} \ldots A_{y_0}(x)) = A_{y'_n}(A_{y_{n-1}} \ldots A_{y_0}(x)),$$

so that

$$x \in A_{y_0}^{-1} A_{y_1}^{-1} \ldots A_{y_{n-1}}^{-1} (\ker A_{y_n}^{-1} A_{y'_n}).$$

But $\mu(\ker A_{y_n}^{-1} A_{y'_n}) = 0$ by hypothesis, and so x lies is a countable union of μ-null sets, verifying our claim. Therefore

$$\begin{aligned} \mathsf{h}(T_d \times^{\mathbf{s}} \alpha) = \mathsf{h}_{\nu\times\mu}(T_d \times^{\mathbf{s}} \alpha) &= h_{\phi_*(\nu\times\mu)}(\sigma_{\mathcal{X}(\mathcal{R}_{A_1,\ldots,A_d})}) \\ &\leqslant \mathsf{h}(\sigma_{\mathcal{X}(\mathcal{R}_{A_1,\ldots,A_d})}) = \mathsf{e}(A_1, \ldots, A_d), \end{aligned}$$

completing the proof. □

We remark that some condition about distinctness of the A_j is needed. For example, if $A_1 = A_2 = A$, then $\mathsf{e}(A, A) = \mathsf{h}(A)$, while $\mathsf{h}(T_2 \times^{\mathbf{s}} \alpha) = \mathsf{h}(A) + \log 2$. The discrepancy arises here because the map ϕ of the proof is no longer essentially one-to-one.

Examples 12.2. (1) Let $Z = \mathbb{T}$, A be multiplication by p, and B be multiplication by $q \neq p$. Here the local fields are $\mathbb{Q}_p$, $\mathbb{Q}_q$, and $\mathbb{Q}_\infty = \mathbb{R}$, with corresponding Lyapunov vectors for the generated $\mathbb{Z}^2$-action being

$(-\log p, 0)$, $(0, -\log q)$, and $(\log p, \log q)$. Theorem 12.1 then shows that $\mathsf{e}(A, B) = \log(p + q)$, agreeing with Theorem 2 of [9].

(2) Let A and B be commuting automorphisms of $\mathbb{T}^r$, and suppose for simplicity that all eigenvalues are real, say they are ξ_j and η_j on the jth eigenspace. Then the Lyapunov vectors for the $\mathbb{Z}^2$-action they generate are $(\log|\xi_j|, \log|\eta_j|)$, and so by Example 11.3 we see that

$$\mathsf{e}(A, B) = \max_{E \subset \{1,\dots,r\}} \log\Big(\prod_{j \in E} |\xi_j| + \prod_{j \in E} |\eta_j|\Big).$$

This shows that the formula conjectured by Geller and Pollicott in [9, 5(3)] is not correct.

(3) Let α be the Ledrappier $\mathbb{Z}^2$ action from Example 6.8, and $A = \alpha^{\mathbf{e}_1}$, $B = \alpha^{\mathbf{e}_2}$. Here the Lyapunov vectors are $(\log 2, 0)$, $(0, \log 2)$, and $(-\log 2, -\log 2)$. Hence by Theorem 12.1, we find that $\mathsf{e}(A, B) = \log 4$, not the value $\log(2 + \sqrt{2})$ reported in [9, Sec. 3]. Unfortunately, this means that their claim that e can be used to differentiate between Ledrappier-type examples defined using shapes with the same convex hull is not correct. In particular, Theorem 3 of [9] is false.

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